ON A CLASS OF MODEL HILBERT SPACES

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Dedicated with great pleasure to Jerry Goldstein
on the occasion of his 70th birthday

ABSTRACT. A detailed description of the model Hilbert space $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ is
provided, where $\mathcal{K}$ represents a complex, separable Hilbert space, and Σ de-
notes a bounded operator-valued measure. In particular, we show that several
alternative approaches to such a construction in the literature are equivalent.
   These spaces are of fundamental importance in the context of perturba-
tion theory of self-adjoint extensions of symmetric operators, and the spectral
theory of ordinary differential operators with operator-valued coefficients.

1. **Introduction.** The principal purpose of this note is to recall and elaborate
on the construction of the model Hilbert space $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ and related Banach
spaces $L^p(\mathbb{R}; wd\Sigma; \mathcal{K})$, $p \geq 1$. Here $\mathcal{K}$ represents a complex, separable Hilbert
space, Σ denotes a bounded operator-valued measure, and w is an appropriate
scalar nonnegative weight function. This model Hilbert space is known to play a
fundamental role in various applications such as, perturbation theory of self-adjoint
operators, the theory of self-adjoint extensions of symmetric operators, and the
spectral theory of ordinary differential operators with operator-valued coefficients
(cf. the end of Section 2).

In Section 2 we describe in detail the construction of $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ following the
approach used in [24]. We will present a slight generalization to the effect that we
now explicitly permit that the bounded operator $T = \Sigma(\mathbb{R})$ in $\mathcal{K}$ has a nontrivial null
space. In the last part of Section 2 we will show that our construction is equivalent
to alternative constructions employed by Berezanskii [7, Sect. VII.2.3] and another
approach originally due to Gel'fand–Kostyuchenko [22] and Berezanskii [7, Ch. V].

It is a somewhat curious fact that in the alternative construction due to Berezan-
skii [7, Sect. VII.2.3] one has a choice in the order in which one takes a certain

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completion and a quotient with respect to a semi-inner product. In fact, different authors frequently chose one or the other of these two different routes without commenting on the equivalence of these two possibilities. Thus, we prove their equivalence in Appendix A.

We now briefly comment on the notation used in this paper: Throughout, $\mathcal{H}$ and $\mathcal{K}$ denote separable, complex Hilbert spaces, the inner product and norm in $\mathcal{H}$ are denoted by $(\cdot, \cdot)_{\mathcal{H}}$ (linear in the second argument) and $\|\cdot\|_{\mathcal{H}}$, respectively. The identity operator in $\mathcal{H}$ is written as $I_{\mathcal{H}}$. We denote by $\mathcal{B}(\mathcal{H})$ the Banach space of linear bounded operators in $\mathcal{H}$. The domain, range, kernel (null space) of a linear operator will be denoted by $\mathrm{dom}(\cdot)$, $\mathrm{ran}(\cdot)$, $\mathrm{ker}(\cdot)$, respectively. The closure of a closable operator S is denoted by $\overline{S}$. The Borel σ-algebra on $\mathbb{R}$ is denoted by $\mathfrak{B}(\mathbb{R})$.

2. **Direct Integrals and the Construction of the Model Hilbert Space.**
In this section we describe in detail the construction of the model Hilbert space $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ (and related Banach spaces $L^p(\mathbb{R}; w d\Sigma; \mathcal{K})$, $p \geq 1$, w an appropriate scalar nonnegative weight function) following (and extending) a method first described in [24].

As general background literature for the topic to follow, we refer to the theory of direct integrals of Hilbert spaces as presented, for instance, in [5, Ch. 4], [11, Ch. 7], [19, Ch. II], [55, Ch. XII].

Throughout this section we make the following assumptions:

Hypothesis 2.1. *Let μ denote a σ-finite Borel measure on $\mathbb{R}$, $\mathfrak{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$, and suppose that $\mathcal{K}$ and $\mathcal{K}_\lambda$, $\lambda \in \mathbb{R}$, denote separable, complex Hilbert spaces such that the dimension function $\mathbb{R} \ni \lambda \mapsto \dim(\mathcal{K}_\lambda) \in \mathbb{N} \cup \{\infty\}$ is μ-measurable.*

Assuming Hypothesis 2.1, let $\mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}})$ be the vector space associated with the Cartesian product $\prod_{\lambda \in \mathbb{R}} \mathcal{K}_\lambda$ equipped with the obvious linear structure. Elements of $\mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}})$ are maps

$$f \in \mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}}), \quad \mathbb{R} \ni \lambda \mapsto f(\lambda) \in \mathcal{K}_\lambda, \tag{1}$$

in particular, we identify $f = \{f(\lambda)\}_{\lambda \in \mathbb{R}}$.

Definition 2.2. Assume Hypothesis 2.1. A *measurable family of Hilbert spaces* $\mathcal{M}$ modeled on μ and $\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}}$ is a linear subspace $\mathcal{M} \subset \mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}})$ such that $f \in \mathcal{M}$ if and only if the map $\mathbb{R} \ni \lambda \mapsto (f(\lambda), g(\lambda))_{\mathcal{K}_\lambda} \in \mathbb{C}$ is μ-measurable for all $g \in \mathcal{M}$. Moreover, $\mathcal{M}$ is said to be *generated* by some subset $\mathcal{F}$, $\mathcal{F} \subset \mathcal{M}$, if for every $g \in \mathcal{M}$ we can find a sequence of functions $h_n \in \mathrm{lin.span}\{\chi_B f \in \mathcal{S}(\{\mathcal{K}_\lambda\}) \,|\, B \in \mathfrak{B}(\mathbb{R}), f \in \mathcal{F}\}$ with $\lim_{n \to \infty} \|g(\lambda) - h_n(\lambda)\|_{\mathcal{K}_\lambda} = 0$ μ-a.e.

The definition of $\mathcal{M}$ was chosen with its maximality in mind and we refer to Lemma 2.4 and for more details in this respect. An explicit construction of an example of $\mathcal{M}$ will be given in Theorem 2.8.

Remark 2.3. The following properties are proved in a standard manner:
(i) If $f \in \mathcal{M}$, $g \in \mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}})$ and $g = f$ μ-a.e. then $g \in \mathcal{M}$.
(ii) If $\{f_n\}_{n \in \mathbb{N}} \in \mathcal{M}$, $g \in \mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}})$ and $f_n(\lambda) \to g(\lambda)$ as $n \to \infty$ μ-a.e. (i.e., $\lim_{n \to \infty} \|f_n(\lambda) - g(\lambda)\|_{\mathcal{K}_\lambda} = 0$ μ-a.e.) then $g \in \mathcal{M}$.
(iii) If ϕ is a scalar-valued μ-measurable function and $f \in \mathcal{M}$ then $\phi f \in \mathcal{M}$.
(iv) If $f \in \mathcal{M}$ then $\mathbb{R} \ni \lambda \mapsto \|f(\lambda)\|_{\mathcal{K}_\lambda} \in [0, \infty)$ is μ-measurable.

Lemma 2.4 ([24]). *Assume Hypothesis 2.1. Suppose that $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda\in\mathbb{R}})$ is such that*

(α) $\mathbb{R} \ni \lambda \mapsto (f_m(\lambda), f_n(\lambda))_{\mathcal{K}_\lambda} \in \mathbb{C}$ is μ-measurable for all $m, n \in \mathbb{N}$.

(β) For μ-a.e. $\lambda \in \mathbb{R}$, $\overline{\mathrm{lin.span}\{f_n(\lambda)\}} = \mathcal{K}_\lambda$.

(In particular, any orthonormal basis $\{e_n(\lambda)\}_{n\in\mathbb{N}}$ in $\mathcal{K}_\lambda$ will satisfy (α) and (β).)
Then setting

$$\mathcal{M} = \{g \in \mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda\in\mathbb{R}}) \,|\, (f_n(\lambda), g(\lambda))_{\mathcal{K}_\lambda} \text{ is } \mu\text{-measurable for all } n \in \mathbb{N}\}, \qquad (2)$$

one has the following facts:

(i) $\mathcal{M}$ is a measurable family of Hilbert spaces.

(ii) $\mathcal{M}$ is generated by $\{f_n\}_{n\in\mathbb{N}}$.

(iii) $\mathcal{M}$ is the unique measurable family of Hilbert spaces containing the sequence $\{f_n\}_{n\in\mathbb{N}}$.

(iv) If $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ is any sequence satisfying (β) then $\mathcal{M}$ is generated by $\{g_n\}_{n\in\mathbb{N}}$.

Next, let w be a μ-measurable function, $w > 0$ μ-a.e., and consider the space

$$\dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M}) = \left\{ f \in \mathcal{M} \,\middle|\, \int_{\mathbb{R}} w(\lambda)d\mu(\lambda) \|f(\lambda)\|_{\mathcal{K}_\lambda}^2 < \infty \right\} \qquad (3)$$

with its obvious linear structure. On $\dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})$ one defines a semi-inner product $(\cdot, \cdot)_{\dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})}$ (and hence a semi-norm $\|\cdot\|_{\dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})}$) by

$$(f, g)_{\dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})} = \int_{\mathbb{R}} w(\lambda)d\mu(\lambda)\, (f(\lambda), g(\lambda))_{\mathcal{K}_\lambda}, \quad f, g \in \dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M}). \qquad (4)$$

That (4) defines a semi-inner product immediately follows from the corresponding properties of $(\cdot, \cdot)_{\mathcal{K}_\lambda}$ and the linearity of the integral. Next, one defines the equivalence relation $\sim$, for elements $f, g \in \dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})$ by

$$f \sim g \text{ if and only if } f = g \quad \mu\text{-a.e.} \qquad (5)$$

and hence introduces the set of equivalence classes of $\dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})$ denoted by

$$L^2(\mathbb{R}; wd\mu; \mathcal{M}) = \dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})/\sim. \qquad (6)$$

In particular, introducing the subspace of null functions

$$\begin{aligned}\mathcal{N}(\mathbb{R}; wd\mu; \mathcal{M}) &= \left\{ f \in \dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M}) \,\middle|\, \|f(\lambda)\|_{\mathcal{K}_\lambda} = 0 \text{ for } \mu\text{-a.e. } \lambda \in \mathbb{R} \right\} \\ &= \left\{ f \in \dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M}) \,\middle|\, \|f\|_{\dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})} = 0 \right\}, \qquad (7)\end{aligned}$$

$L^2(\mathbb{R}; wd\mu; \mathcal{M})$ is precisely the quotient space $\dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})/\mathcal{N}(\mathbb{R}; wd\mu; \mathcal{M})$. Denoting the equivalence class of $f \in \dot{L}^2(\mathbb{R}; wd\mu; \mathcal{M})$ temporarily by $[f]$, the semi-inner product on $L^2(\mathbb{R}; wd\mu; \mathcal{M})$

$$([f], [g])_{L^2(\mathbb{R}; wd\mu; \mathcal{M})} = \int_{\mathbb{R}} w(\lambda)d\mu(\lambda)\, (f(\lambda), g(\lambda))_{\mathcal{K}_\lambda} \qquad (8)$$

is well-defined (i.e., independent of the chosen representatives of the equivalence classes) and actually an inner product. Thus, $L^2(\mathbb{R}; wd\mu; \mathcal{M})$ is a normed space and by the usual abuse of notation we denote its elements in the following again by f, g, etc. Moreover, $L^2(\mathbb{R}; wd\mu; \mathcal{M})$ is also complete:

Theorem 2.5. *Assume Hypothesis 2.1. Then the normed space $L^2(\mathbb{R}; wd\mu; \mathcal{M})$ is complete and hence a Hilbert space. In addition, $L^2(\mathbb{R}; wd\mu; \mathcal{M})$ is separable.*

That $L^2(\mathbb{R}; wd\mu; \mathcal{M})$ is complete was shown in [5, Subsect. 4.1.2], [11, Sect. 7.1], and more recently, in [24]. Separability of $L^2(\mathbb{R}; wd\mu; \mathcal{M})$ is proved in [11, Sect. 7.1] (see also [5, Subsect. 4.3.2]).

Remark 2.6. Clearly, the analogous construction then defines the Banach spaces $L^p(\mathbb{R}; wd\mu; \mathcal{M})$, $p \geq 1$.

Thus, $L^2(\mathbb{R}; wd\mu; \mathcal{M})$ corresponds precisely to the direct integral of the Hilbert spaces $\mathcal{K}_\lambda$ with respect to the measure $wd\mu$ (see, e.g., [5, Ch. 4], [11, Ch. 7], [19, Ch. II], [55, Ch. XII]) and is frequently denoted by $\int_{\mathbb{R}}^{\oplus} w(\lambda) d\mu(\lambda) \mathcal{K}_\lambda$.

Having reviewed the construction of $L^2(\mathbb{R}; wd\mu; \mathcal{M}) = \int_{\mathbb{R}}^{\oplus} w(\lambda) d\mu(\lambda) \mathcal{K}_\lambda$ in connection with a scalar measure $wd\mu$, we now turn to the case of operator-valued measures and recall the following definition (we refer, for instance, to [5, Sects. 1.2, 3.1, 5.1], [7, Sect. VII.2.3], [11, Ch. 6], [18, Ch. I], [20, Ch. X], [40] for vector-valued and operator-valued measures):

Definition 2.7. Let $\mathcal{H}$ be a separable, complex Hilbert space. A map $\Sigma : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$, with $\mathfrak{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$, is called a *bounded, nonnegative, operator-valued measure* if the following conditions (i) and (ii) hold:
(i) $\Sigma(\emptyset) = 0$ and $0 \leq \Sigma(B) \in \mathcal{B}(\mathcal{H})$ for all $B \in \mathfrak{B}(\mathbb{R})$.
(ii) $\Sigma(\cdot)$ is strongly countably additive (i.e., with respect to the strong operator topology in $\mathcal{H}$), that is,

$$\Sigma(B) = \underset{N \to \infty}{\text{s-lim}} \sum_{j=1}^{N} \Sigma(B_j) \tag{9}$$

whenever $B = \bigcup_{j \in \mathbb{N}} B_j$, with $B_k \cap B_\ell = \emptyset$ for $k \neq \ell$, $B_k \in \mathfrak{B}(\mathbb{R})$, $k, \ell \in \mathbb{N}$.

Moreover, $\Sigma(\cdot)$ is called an (*operator-valued*) *spectral measure* (or an *orthogonal operator-valued measure*) if additionally the following condition (iii) holds:
(iii) $\Sigma(\cdot)$ is projection-valued (i.e., $\Sigma(B)^2 = \Sigma(B)$, $B \in \mathfrak{B}(\mathbb{R})$) and $\Sigma(\mathbb{R}) = I_{\mathcal{H}}$.

In the following, let $\Sigma : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{K})$ be a bounded nonnegative measure, that is, Σ satisfies requirements (i) and (ii) in Definition 2.7. Denoting $T = \Sigma(\mathbb{R})$, one has

$$0 \leq \Sigma(B) \leq T \in \mathcal{B}(\mathcal{K}), \quad B \in \mathfrak{B}(\mathbb{R}), \tag{10}$$

and hence

$$\left\| \Sigma(B)^{1/2} \xi \right\|_{\mathcal{K}} \leq \left\| T^{1/2} \xi \right\|_{\mathcal{K}}, \quad \xi \in \mathcal{K}, \tag{11}$$

shows that

$$\ker(T) = \ker\left(T^{1/2}\right) \subseteq \ker\left(\Sigma(B)^{1/2}\right) = \ker(\Sigma(B)), \quad B \in \mathfrak{B}(\mathbb{R}). \tag{12}$$

We will use the orthogonal decomposition

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1, \quad \mathcal{K}_0 = \ker(T), \quad \mathcal{K}_1 = \ker(T)^{\perp} = \overline{\text{ran}(T)}, \tag{13}$$

and identify $f_0 = (f_0 \quad 0)^{\top} \in \mathcal{K}_0$ and $f_1 = (0 \quad f_1)^{\top} \in \mathcal{K}_1$. In particular, with $f = (f_0 \quad f_1)^{\top}$, one has $\|f\|_{\mathcal{K}}^2 = \|f_0\|_{\mathcal{K}_0}^2 + \|f_1\|_{\mathcal{K}_1}^2$. Then T permits the 2×2 block operator representation

$$T = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix}, \quad \text{with } 0 \leq T_1 \in \mathcal{B}(\mathcal{K}_1), \ \ker(T_1) = \{0\}, \tag{14}$$

with respect to the decomposition (13). By (12) one concludes that $\Sigma(B)$, $B \in \mathfrak{B}(\mathbb{R})$, is necessarily of the form

$$\Sigma(B) = \begin{pmatrix} 0 & D^* \\ D & \Sigma_1(B) \end{pmatrix}, \text{ for some } 0 \leq \Sigma_1(B) \in \mathcal{B}(\mathcal{K}_1), \ D \in \mathcal{B}(\mathcal{K}_0, \mathcal{K}_1), \quad (15)$$

with respect to the decomposition (13). The computation

$$0 = \Sigma(B) \begin{pmatrix} f_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & D^* \\ D & \Sigma_1(B) \end{pmatrix} \begin{pmatrix} f_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Df_0 \end{pmatrix}, \quad f_0 \in \mathcal{K}_0, \quad (16)$$

yields $D = 0$ as $f_0 \in \mathcal{K}_0$ was arbitrary. Thus, $\Sigma(B)$, $B \in \mathfrak{B}(\mathbb{R})$, is actually also of diagonal form

$$\Sigma(B) = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_1(B) \end{pmatrix}, \text{ for some } 0 \leq \Sigma_1(B) \in \mathcal{B}(\mathcal{K}_1), \quad (17)$$

with respect to the decomposition (13).

Moreover, let μ be a control measure for Σ (equivalently, for Σ_1), that is,

$$\mu(B) = 0 \text{ if and only if } \Sigma(B) = 0 \text{ for all } B \in \mathfrak{B}(\mathbb{R}). \quad (18)$$

(E.g., $\mu(B) = \sum_{n \in \mathcal{I}} 2^{-n}(e_n, \Sigma(B)e_n)_{\mathcal{K}}$, $B \in \mathfrak{B}(\mathbb{R})$, with $\{e_n\}_{n \in \mathcal{I}}$ a complete orthonormal system in $\mathcal{K}$, $\mathcal{I} \subseteq \mathbb{N}$, an appropriate index set.)

The following theorem was first stated in [24] under the implicit assumption that $\Sigma(\mathbb{R}) = T = I_{\mathcal{K}}$. In this paper we now treat the general case $T \in \mathcal{B}(\mathcal{K})$, in particular, we explicitly permit the existence of a nontrivial kernel of T:

Theorem 2.8. *Let $\mathcal{K}$ be a separable, complex Hilbert space, $\Sigma : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{K})$ a bounded, nonnegative operator-valued measure, and μ a control measure for Σ. Then there are separable, complex Hilbert spaces $\mathcal{K}_\lambda$, $\lambda \in \mathbb{R}$, a measurable family of Hilbert spaces $\mathcal{M}_\Sigma$ modelled on μ and $\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}}$, and a bounded linear map $\underline{\Lambda} \in \mathcal{B}\big(\mathcal{K}, L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma)\big)$, satisfying*

$$\|\underline{\Lambda}\|_{\mathcal{B}(\mathcal{K}, L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma))} = \big\|T^{1/2}\big\|_{\mathcal{B}(\mathcal{K})}, \quad (19)$$

and

$$\ker(\underline{\Lambda}) = \ker(T), \quad (20)$$

so that the following assertions (i)–(iii) hold:
(i) For all $B \in \mathfrak{B}(\mathbb{R})$, $\xi, \eta \in \mathcal{K}$,

$$(\eta, \Sigma(B)\xi)_{\mathcal{K}} = \int_B d\mu(\lambda) \left((\underline{\Lambda}\eta)(\lambda), (\underline{\Lambda}\xi)(\lambda)\right)_{\mathcal{K}_\lambda}, \quad (21)$$

in particular,

$$(\eta, T\xi)_{\mathcal{K}} = \int_{\mathbb{R}} d\mu(\lambda) \left((\underline{\Lambda}\eta)(\lambda), (\underline{\Lambda}\xi)(\lambda)\right)_{\mathcal{K}_\lambda}. \quad (22)$$

(ii) Let $\mathcal{I} = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$, or $\mathcal{I} = \mathbb{N}$. $\underline{\Lambda}(\{e_n\}_{n \in \mathcal{I}})$ generates $\mathcal{M}_\Sigma$, where $\{e_n\}_{n \in \mathcal{I}}$ denotes any sequence of linearly independent elements in $\mathcal{K}$ with the property $\overline{\text{lin.span}}\{e_n\}_{n \in \mathcal{I}} = \mathcal{K}$. In particular, $\underline{\Lambda}(\mathcal{K})$ generates $\mathcal{M}_\Sigma$.
(iii) For all $B \in \mathfrak{B}(\mathbb{R})$ and $\xi \in \mathcal{K}$,

$$\underline{\Lambda}(S(B)\xi) = \{\chi_B(\lambda)(\underline{\Lambda}\xi)(\lambda)\}_{\lambda \in \mathbb{R}}, \quad (23)$$

where (cf. (14) and (17))

$$S(B) = \begin{pmatrix} I_{\mathcal{K}_0} & 0 \\ 0 & T_1^{-1/2}\Sigma_1(B)^{1/2} \end{pmatrix}, \quad S(\mathbb{R}) = I_{\mathcal{K}}, \quad (24)$$

with respect to the decomposition (13).

Proof. Since the current version of this theorem extends the earlier one in [24], we now repeat it for the convenience of the reader. Moreover, we will shed additional light on the proof of (23), correcting an oversight in this connection in [24]. Introducing

$$\mathcal{V} = \text{lin.span}\{e_n \in \mathcal{K} \mid n \in \mathcal{I}\}, \quad \overline{\mathcal{V}} = \mathcal{K}, \tag{25}$$

the Radon–Nikodym theorem implies that there exist μ-measurable $\phi_{m,n}$ such that

$$\int_B d\mu(\lambda)\phi_{m,n}(\lambda) = (e_m, \Sigma(B)e_n)_{\mathcal{K}}, \quad B \in \mathfrak{B}(\mathbb{R}), \ m, n \in \mathcal{I}. \tag{26}$$

Next, suppose $v = \sum_{n=1}^{N} \alpha_n e_n \in \mathcal{V}$, $\alpha_n \in \mathbb{C}$, $n = 1, \dots, N$, $N \in \mathcal{I}$. Then

$$0 \leq (v, \Sigma(B)v)_{\mathcal{K}} = \int_B d\mu(\lambda) \sum_{m,n=1}^{N} \phi_{m,n}(\lambda)\overline{\alpha_m}\alpha_n, \quad B \in \mathfrak{B}(\mathbb{R}). \tag{27}$$

By considering only rational linear combinations (i.e., $\alpha_n \in \mathbb{Q} + i\,\mathbb{Q}$) we can deduce the existence of a set $E \in \mathfrak{B}(\mathbb{R})$ with $\mu(\mathbb{R}\backslash E) = 0$ such that for $\lambda \in E$,

$$\sum_{m,n} \phi_{m,n}(\lambda)\overline{\alpha_m}\alpha_n \geq 0 \text{ for all finite sequences } \{\alpha_n\} \subset \mathbb{C}. \tag{28}$$

Hence we can define a semi-inner product $(\cdot, \cdot)_\lambda$ on $\mathcal{V}$,

$$(v, w)_\lambda = \sum_{m,n} \phi_{m,n}(\lambda)\overline{\alpha_m}\beta_n, \quad \lambda \in E, \tag{29}$$

for all $v = \sum_n \alpha_n e_n$, $w = \sum_n \beta_n e_n \in \mathcal{V}$.
Next, let $\mathcal{K}_\lambda$ be the completion of $\mathcal{V}/\mathcal{N}_\lambda$ with respect to $\|\cdot\|_\lambda$, where

$$\mathcal{N}_\lambda = \{\xi \in \mathcal{V} \mid (\xi, \xi)_\lambda = 0\}, \quad \lambda \in E, \tag{30}$$

and define (for convenience) $\mathcal{K}_\lambda = \mathcal{K}$ for $\lambda \in \mathbb{R}\backslash E$. Consider $\mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}})$, then each $v \in \mathcal{V}$ defines an element $\underline{v} = \{\underline{v}(\lambda)\}_{\lambda \in \mathbb{R}} \in \mathcal{S}(\{\mathcal{K}_\lambda\}_{\lambda \in \mathbb{R}})$ by

$$\underline{v}(\lambda) = v \text{ for all } \lambda \in \mathbb{R}. \tag{31}$$

Again we identify an element $v \in \mathcal{V}$ with an element in $\mathcal{V}/\mathcal{N}_\lambda \subseteq \mathcal{K}_\lambda$. Applying Lemma 2.4, the collection $\{\underline{e}_n\}_{n \in \mathcal{I}}$ then generates a measurable family of Hilbert spaces $\mathcal{M}_\Sigma$. If $v = \sum_{n=1}^{N} \alpha_n e_n \in \mathcal{V}$, $\alpha_n \in \mathbb{C}$, $n = 1, \dots, N$, $N \in \mathcal{I}$, then

$$\|\underline{v}\|^2_{L^2(\mathbb{R};d\mu;\mathcal{M}_\Sigma)} = \int_{\mathbb{R}} d\mu(\lambda) (\underline{v}(\lambda), \underline{v}(\lambda))_\lambda = \int_{\mathbb{R}} d\mu(\lambda) \sum_{m,n=1}^{N} \phi_{m,n}(\lambda)\overline{\alpha_m}\alpha_n$$

$$= (v, \Sigma(\mathbb{R})v)_{\mathcal{K}} = (v, Tv)_{\mathcal{K}} = \|T^{1/2}v\|^2_{\mathcal{K}}. \tag{32}$$

Hence we can define

$$\underline{\dot{\Lambda}} : \begin{cases} \mathcal{V} \to L^2(\mathbb{R};d\mu;\mathcal{M}_\Sigma), \\ v \mapsto \underline{\dot{\Lambda}}v = \underline{v} = \{\underline{v}(\lambda) = v\}_{\lambda \in \mathbb{R}}, \end{cases} \tag{33}$$

and denote by $\underline{\Lambda} \in \mathcal{B}(\mathcal{K}, L^2(\mathbb{R};d\mu;\mathcal{M}_\Sigma))$, with

$$\|\underline{\Lambda}\|_{\mathcal{B}(\mathcal{K},L^2(\mathbb{R};d\mu;\mathcal{M}_\Sigma))} = \|T^{1/2}\|_{\mathcal{B}(\mathcal{K})}, \tag{34}$$

the closure of $\underline{\dot{\Lambda}}$. In particular, one obtains

$$\|\underline{\Lambda}\xi\|^2_{L^2(\mathbb{R};d\mu;\mathcal{M}_\Sigma)} = \int_{\mathbb{R}} d\mu(\lambda) \|(\underline{\Lambda}\xi)(\lambda)\|^2_{\mathcal{K}_\lambda} = \|T^{1/2}\xi\|^2_{\mathcal{K}}, \quad \xi \in \mathcal{K}, \tag{35}$$

and hence
$$\ker(\underline{\Lambda}) = \ker(T) = \mathcal{K}_0. \tag{36}$$

Then properties (i) and (ii) hold and we proceed to illustrating property (iii): Introduce the operator
$$\widetilde{S}(B) = \begin{pmatrix} I_{\mathcal{K}_0} & 0 \\ 0 & \Sigma_1(B)^{1/2}T_1^{-1/2} \end{pmatrix}, \quad \mathrm{dom}\,\big(\widetilde{S}(B)\big) = \mathcal{K}_0 \oplus \mathrm{dom}\,\big(T_1^{-1/2}\big), \quad B \in \mathfrak{B}(\mathbb{R}), \tag{37}$$

in $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$. Since
$$\begin{aligned} \mathcal{K} &= \ker(T) \oplus \overline{\mathrm{ran}(T)} \\ &= \ker(T^{1/2}) \oplus \overline{\mathrm{ran}(T^{1/2})} \\ &= \ker(T^{1/2}) \oplus \overline{\mathrm{ran}(T_1^{1/2})} \end{aligned} \tag{38}$$

$\mathrm{dom}\,\big(T_1^{-1/2}\big) = \mathrm{ran}\,\big(T_1^{1/2}\big)$ is dense in $\mathcal{K}_1$. (Alternatively, this follows from the fact that $\ker\big(T_1^{1/2}\big) = \{0\}$ and $0 \leq T_1^{1/2}$ is self-adjoint.) Thus $\widetilde{S}(B)$ is densely defined in $\mathcal{K}$. Applying (11), the computation
$$\begin{aligned} \big\|\Sigma_1(B)^{1/2}T_1^{-1/2}f\big\|_{\mathcal{K}_1} &= \big\|\Sigma(B)^{1/2}\big(0 \ \ T_1^{-1/2}f\big)\big)^\top\big\|_{\mathcal{K}} \\ &\leq \big\|T^{1/2}\big(0 \ \ T_1^{-1/2}f\big)^\top\big\|_{\mathcal{K}} = \left\|\begin{pmatrix} 0 & 0 \\ 0 & T_1^{1/2} \end{pmatrix}\big(0 \ \ T_1^{-1/2}f\big)^\top\right\|_{\mathcal{K}} \\ &= \|f\|_{\mathcal{K}_1}, \quad f \in \mathrm{dom}\,\big(T_1^{-1/2}\big), \end{aligned} \tag{39}$$

then shows that $\Sigma_1(B)^{1/2}T_1^{-1/2}$ has a bounded extension (its closure) to all of $\mathcal{K}_1$ with
$$\left\|\overline{\Sigma_1(B)^{1/2}T_1^{-1/2}}\right\|_{\mathcal{B}(\mathcal{K}_1)} \leq 1, \quad B \in \mathfrak{B}(\mathbb{R}). \tag{40}$$

Hence, also $\widetilde{S}(B)$ has a bounded extension (its closure) to all of $\mathcal{K}$ with
$$\left\|\overline{\widetilde{S}(B)}\right\|_{\mathcal{B}(\mathcal{K})} \leq 1, \quad B \in \mathfrak{B}(\mathbb{R}). \tag{41}$$

Moreover, one computes
$$\widetilde{S}(B)T^{1/2} = \begin{pmatrix} I_{\mathcal{K}_0} & 0 \\ 0 & \Sigma_1(B)^{1/2}T_1^{-1/2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & T_1^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_1(B)^{1/2} \end{pmatrix} = \Sigma(B)^{1/2}. \tag{42}$$

In particular, this also yields
$$\overline{\widetilde{S}(B)}T^{1/2} = \widetilde{S}(B)T^{1/2} = \Sigma(B)^{1/2}, \quad B \in \mathfrak{B}(\mathbb{R}). \tag{43}$$

Next, we introduce
$$S(B) = \big(\widetilde{S}(B)\big)^* = \left(\overline{\widetilde{S}(B)}\right)^* \in \mathcal{B}(\mathcal{K}), \quad B \in \mathfrak{B}(\mathbb{R}), \tag{44}$$

and note that
$$S(\mathbb{R}) = I_{\mathcal{K}}. \tag{45}$$

Then one obtains
$$S(B) = \begin{pmatrix} I_{\mathcal{K}_0} & 0 \\ 0 & \Sigma_1(B)^{1/2}T_1^{-1/2} \end{pmatrix}^* = \begin{pmatrix} I_{\mathcal{K}_0} & 0 \\ 0 & T_1^{-1/2}\Sigma_1(B)^{1/2} \end{pmatrix}, \quad B \in \mathfrak{B}(\mathbb{R}), \tag{46}$$

since
$$\big[\Sigma_1(B)^{1/2}T_1^{-1/2}\big]^* = T_1^{-1/2}\Sigma_1(B)^{1/2}, \quad B \in \mathfrak{B}(\mathbb{R}). \tag{47}$$

as $\Sigma_1(B)^{1/2} \in \mathcal{B}(\mathcal{K}_1)$ and $T_1^{-1/2}$ is densely defined (in fact, self-adjoint) in $\mathcal{K}_1$ (cf. [56, Theorem 4.19 (b)]). By the fact that $\widetilde{S(B)} \in \mathcal{B}(\mathcal{K})$ and hence $S(B) \in \mathcal{B}(\mathcal{K})$, one concludes that $T_1^{-1/2}\Sigma_1(B)^{1/2} \in \mathcal{B}(\mathcal{K}_1)$, that is, $\text{ran}\left(\Sigma_1(B)^{1/2}\right) \subseteq \text{ran}\left(T_1^{1/2}\right)$.

Thus, taking adjoints in (43) one obtains

$$T^{1/2}S(B) = \Sigma(B)^{1/2}, \quad B \in \mathfrak{B}(\mathbb{R}). \tag{48}$$

Let $\xi \in \mathcal{K}$, then combinng (21), (22), and (48) yields

$$\begin{aligned}
(\eta, \Sigma(B)\xi)_\mathcal{K} &= (\Sigma(B)^{1/2}\eta, \Sigma(B)^{1/2}\xi)_\mathcal{K} \\
&= (T^{1/2}S(B)\eta, T^{1/2}S(B)\xi)_\mathcal{K} \\
&= (S(B)\eta, TS(B)\xi)_\mathcal{K} \\
&= \int_\mathbb{R} d\mu(\lambda) \left((\underline{\Lambda}S(B)\eta)(\lambda), (\underline{\Lambda}S(B)\xi)(\lambda)\right)_{\mathcal{K}_\lambda} \\
&= \int_B d\mu(\lambda) \left((\underline{\Lambda}\eta)(\lambda), (\underline{\Lambda}\xi)(\lambda)\right)_{\mathcal{K}_\lambda} \\
&= \int_\mathbb{R} d\mu(\lambda) \left(\chi_B(\lambda)(\underline{\Lambda}\eta)(\lambda), \chi_B(\lambda)(\underline{\Lambda}\xi)(\lambda)\right)_{\mathcal{K}_\lambda}, \quad B \in \mathfrak{B}(\mathbb{R}), \tag{49}
\end{aligned}$$

implying (23). $\qquad\square$

Implicitly in the proof of Theorem 2.8 is a special case of the following result, which appears to be of independent interest. It may well be known, but since we could not quickly find it in the literature we include its short proof for the convenience of the reader:

Lemma 2.9. *Let $\mathcal{H}$ be a complex, separable Hilbert space, F, G self-adjoint operators in $\mathcal{H}$, and $0 \le F \le G$. Then*

$$\text{ran}(F^\alpha) \subseteq \text{ran}(G^\alpha), \quad \alpha \in (0, 1/2]. \tag{50}$$

In particular, if in addition $F, G \in \mathcal{B}(\mathcal{H})$ and $\ker(G) = \{0\}$, then

$$G^{-\alpha}F^\alpha \in \mathcal{B}(\mathcal{H}), \quad \alpha \in (0, 1/2]. \tag{51}$$

Proof. The hypothesis $0 \le F \le G$ implies

$$\|F^{1/2}x\|_\mathcal{H} \le \|G^{1/2}x\|_\mathcal{H}, \quad x \in \text{dom}(G^{1/2}) \subseteq \text{dom}(F^{1/2}), \tag{52}$$

and hence one concludes as before in the context of bounded operators (cf. (12)) that

$$\ker(G) \subseteq \ker(F). \tag{53}$$

Thus, in analogy to (13), (14), and (17), we again decompose

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \quad \mathcal{H}_0 = \ker(G), \ \mathcal{H}_1 = \ker(G)^\perp = \overline{\text{ran}(G)}, \tag{54}$$

and hence obtain the 2×2 block operator representations

$$F = \begin{pmatrix} 0 & 0 \\ 0 & F_1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 \\ 0 & G_1 \end{pmatrix}, \quad \ker(G_1) = \{0\}, \tag{55}$$

with respect to the decomposition (54), with self-adjoint operators F_1, G_1 in $\mathcal{H}_1$ satisfying $0 \le F_1 \le G_1$. In particular,

$$\|F_1^{1/2}x\|_{\mathcal{H}_1} \le \|G_1^{1/2}x\|_{\mathcal{H}_1}, \quad x \in \text{dom}(G_1^{1/2}) \subseteq \text{dom}(F_1^{1/2}), \tag{56}$$

and Heinz's inequality (cf. [33, Satz 3], [34, Theorem 2]) (56) implies that

$$\|F_1^\alpha x\|_\mathcal{H} \le \|G_1^\alpha x\|_\mathcal{H}, \quad x \in \text{dom}(G_1^\alpha) \subseteq \text{dom}(F_1^\alpha), \ \alpha \in (0, 1/2]. \tag{57}$$

Then for any $y \in \mathrm{dom}(F_1^\alpha)$ and $x \in \mathrm{ran}(G_1^\alpha) = \mathrm{dom}(G_1^{-\alpha})$, one computes using self-adjointness of F_1^α and (57)

$$|(F_1^\alpha y, G_1^{-\alpha} x)_{\mathcal{H}_1}| = |(y, F_1^\alpha G_1^{-\alpha} x)_{\mathcal{H}_1}| \leq \|y\|_{\mathcal{H}_1} \|F_1^\alpha G_1^{-\alpha} x\|_{\mathcal{H}_1}$$
$$\leq \|y\|_{\mathcal{H}_1} \|G_1^\alpha G_1^{-\alpha} x\|_{\mathcal{H}_1} = \|y\|_{\mathcal{H}_1} \|x\|_{\mathcal{H}_1}. \tag{58}$$

This implies $F_1^\alpha y \in \mathrm{dom}((G_1^{-\alpha})^*) = \mathrm{dom}(G_1^{-\alpha}) = \mathrm{ran}(G_1^\alpha) = \mathrm{ran}(G^\alpha)$ and hence (50) since also $\mathrm{ran}(F_1^\alpha) = \mathrm{ran}(F^\alpha)$. The fact (51) then follows from the closed graph theorem. $\qquad\square$

Next, we recall that the construction in Theorem 2.8 is essentially unique:

Theorem 2.10 ([24]). *Suppose $\mathcal{K}_\lambda'$, $\lambda \in \mathbb{R}$ is a family of separable complex Hilbert spaces, $\mathcal{M}'$ is a measurable family of Hilbert spaces modelled on μ and $\{\mathcal{K}_\lambda'\}$, and $\underline{\Delta}' \in \mathcal{B}(\mathcal{K}, L^2(\mathbb{R}; d\mu; \mathcal{M}'))$ is a map satisfying (i), (ii), and (iii) of Theorem 2.8. Then for μ-a.e. $\lambda \in \mathbb{R}$ there is a unitary operator $U_\lambda : \mathcal{K}_\lambda \to \mathcal{K}_\lambda'$ such that $f = \{f(\lambda)\}_{\lambda \in \mathbb{R}} \in \mathcal{M}_\Sigma$ if and only if $\{U_\lambda f(\lambda)\}_{\lambda \in \mathbb{R}} \in \mathcal{M}'$ and for all $\xi \in \mathcal{K}$,*

$$(\underline{\Delta}'\xi)(\lambda) = U_\lambda(\underline{\Delta}\xi)(\lambda) \quad \mu\text{-a.e.} \tag{59}$$

Remark 2.11. (i) Without going into further details, we note that $\mathcal{M}_\Sigma$ depends of course on the control measure μ. However, a change in μ merely effects a change in density and so $\mathcal{M}_\Sigma$ can essentially be viewed as μ-independent.
(ii) With $0 < w$ a μ-measurable weight function, one can also consider the Hilbert space $L^2(\mathbb{R}; wd\mu; \mathcal{M}_\Sigma)$. In view of our comment in item (i) concerning the mild dependence on the control measure μ of $\mathcal{M}_\Sigma$, one typically puts more emphasis on the operator-valued measure Σ and hence uses the more suggestive notation $L^2(\mathbb{R}; wd\Sigma; \mathcal{K})$ instead of the more precise $L^2(\mathbb{R}; wd\mu; \mathcal{M}_\Sigma)$ in this case.

Next, let

$$\mathcal{V} = \mathrm{lin.span}\{e_n \in \mathcal{K} \,|\, n \in \mathcal{I}\}, \quad \overline{\mathcal{V}} = \mathcal{K}, \tag{60}$$

and define

$$\underline{\mathcal{V}}_\Sigma = \mathrm{lin.span}\{\chi_B(\cdot)\, \underline{e}_n \in L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma) \,\big|\, B \in \mathfrak{B}(\mathbb{R}), \, n \in \mathcal{I}\}. \tag{61}$$

The fact that $\{\underline{e}_n\}_{n \in \mathcal{I}}$ generates $\mathcal{M}_\Sigma$ then implies that $\underline{\mathcal{V}}_\Sigma$ is dense in the Hilbert space $L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma)$, that is,

$$\overline{\underline{\mathcal{V}}_\Sigma} = L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma). \tag{62}$$

Since the operator-valued distribution function $\Sigma(\cdot)$ has at most countably many discontinuities on $\mathbb{R}$, denoting by $\mathfrak{S}_\Sigma$ the corresponding set of discontinuities of $\Sigma(\cdot)$, introducing the set of intervals

$$\mathcal{B}_\Sigma = \{(\alpha, \beta] \subset \mathbb{R} \,|\, \alpha, \beta \in \mathbb{R}\backslash\mathfrak{S}_\Sigma\}, \tag{63}$$

the minimal σ-algebra generated by $\mathcal{B}_\Sigma$ coincides with the Borel algebra $\mathfrak{B}(\mathbb{R})$. Hence one can introduce

$$\widetilde{\underline{\mathcal{V}}}_\Sigma = \mathrm{lin.span}\{\chi_{(\alpha,\beta]}(\cdot)\, \underline{e}_n \in L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma) \,\big|\, \alpha, \beta \in \mathbb{R}\backslash\mathfrak{S}_\Sigma, \, n \in \mathcal{I}\}, \tag{64}$$

which still retains the density property in (62), that is,

$$\overline{\widetilde{\underline{\mathcal{V}}}_\Sigma} = L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma). \tag{65}$$

In the following we briefly describe an alternative construction of $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ used by Berezanskii [7, Sect. VII.2.3] in order to identify the two constructions.

Introduce

$$C_{0,0}(\mathbb{R};\mathcal{K}) = \left\{ \underline{u} : \mathbb{R} \to \mathcal{K} \,\middle|\, \underline{u}(\cdot) \text{ is strongly continuous in } \mathcal{K}, \, \mathrm{supp}(u) \text{ is compact,} \right.$$
$$\left. \bigcup_{\lambda \in \mathbb{R}} \mathrm{ran}(\underline{u}(\lambda)) \subseteq \mathcal{K}_{\underline{u}}, \, \dim(\mathcal{K}_{\underline{u}}) < \infty \right\} \tag{66}$$

On $C_{0,0}(\mathbb{R};\mathcal{K})$ one can introduce the semi-inner product

$$(\underline{u}, \underline{v})_{L^2(\mathbb{R};d\Sigma;\mathcal{K})} = \int_{\mathbb{R}} d(\underline{u}(\lambda), \Sigma(\lambda)\underline{v}(\lambda))_{\mathcal{K}}, \quad \underline{u}, \underline{v} \in C_{0,0}(\mathbb{R};\mathcal{K}), \tag{67}$$

where the integral on the right-hand side of (67) is well-defined in the Riemann–Stieltjes sense. Introducing the kernel of this semi-inner product by

$$\mathcal{N} = \{ \underline{u} \in C_{0,0}(\mathbb{R};\mathcal{K}) \,|\, (\underline{u}, \underline{u})_{L^2(\mathbb{R};d\Sigma;\mathcal{K})} = 0 \}, \tag{68}$$

Berezanskii [7, Sect. VII.2.3] obtains the separable Hilbert space $L^2(\mathbb{R};d\Sigma;\mathcal{K})$ as the completion of $C_{0,0}(\mathbb{R};\mathcal{K})/\mathcal{N}$ with respect to the inner product in (67) as

$$L^2(\widehat{\mathbb{R};d\Sigma};\mathcal{K}) = \overline{C_{0,0}(\mathbb{R};\mathcal{K})/\mathcal{N}}. \tag{69}$$

In particular,

$$([\underline{u}], [\underline{v}])_{L^2(\widehat{\mathbb{R};d\Sigma};\mathcal{K})} = \int_{\mathbb{R}} d(\underline{u}(\lambda), \Sigma(\lambda)\underline{v}(\lambda))_{\mathcal{K}}, \quad \underline{u}, \underline{v} \in C_{0,0}(\mathbb{R};\mathcal{K}), \tag{70}$$

and (cf. also [40, Corollary 2.6]) (70) extends to piecewise continuous $\mathcal{K}$-valued functions with compact support as long as the discontinuities of $\underline{u}$ and $\underline{v}$ are disjoint from the set $\mathfrak{S}_\Sigma$ (the set of discontinuities of $\Sigma(\cdot)$).

Since Kats' work in the case of a finite-dimensional Hilbert space $\mathcal{K}$ (cf. [35], [36] and also Fuhrman [21, Sect. II.6] and Rosenberg [51]), and especially in the work of Malamud and Malamud [40], who studied the general case $\dim(\mathcal{K}) \leq \infty$, it has become customary to interchange the order of taking the quotient with respect to the semi-inner product and completion in this process of constructing $L^2(\widehat{\mathbb{R};d\Sigma};\mathcal{K})$. More precisely, in this context one first completes $C_{0,0}(\mathbb{R},\mathcal{K})$ with respect to the semi-inner product (67) to obtain a semi-Hilbert space

$$L^2(\widetilde{\mathbb{R};d\Sigma};\mathcal{K}) = \overline{C_{0,0}(\mathbb{R};\mathcal{K})}, \tag{71}$$

and then takes the quotient with respect to the kernel of the underlying semi-inner product, as described in method (**I**) of Appendix A. Berezanskii's approach in [7, Sect. VII.2.3] corresponds to method (**II**) discussed in Appendix A. The equivalence of these two methods is not stated in these sources and hence we spelled this out explicitly in Lemma A.1 in Appendix A.

Next we will indicate that Berezanskii's construction of $L^2(\widehat{\mathbb{R};d\Sigma};\mathcal{K})$ (and hence the corresponding construction by Kats (if $\dim(\mathcal{K}) < \infty$) and by Malamud and Malamud (if $\dim(\mathcal{K}) \leq \infty$) is equivalent to the one in [24] and hence to that outlined in Theorem 2.8:

Theorem 2.12. *The spaces $L^2(\widehat{\mathbb{R};d\Sigma};\mathcal{K})$ and $L^2(\mathbb{R};d\mu;\mathcal{M}_\Sigma)$ are isometrically isomorphic.*

Proof. We first recall that the set $\widetilde{\mathcal{V}}_\Sigma$ in (64) is dense in $L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma)$. On the other hand, it was shown in the proof of Theorem 2.14 in [40] that $\widetilde{\mathcal{V}}_\Sigma$ is also dense in $L^2(\widehat{\mathbb{R}; d\Sigma}; \mathcal{K})$. The fact

$$
\begin{aligned}
\|\chi_{(\alpha,\beta]} \underline{e_n}\|^2_{L^2(\widehat{\mathbb{R};d\Sigma};\mathcal{K})} &= \int_{\mathbb{R}} d(\chi_{(\alpha,\beta]}(\lambda) e_n, \Sigma(\lambda)\chi_{(\alpha,\beta]}(\lambda) e_n)_\mathcal{K} \\
&= \int_{(\alpha,\beta]} d(e_n, \Sigma(\lambda)e_n)_\mathcal{K} \\
&= (e_n, \Sigma((\alpha,\beta]) e_n)_\mathcal{K} \\
&= \int_{\mathbb{R}} d\mu(\lambda) \|((\underline{\Delta}(\chi_{(\alpha,\beta]}(\lambda)e_n)(\lambda)\|^2_{\mathcal{K}_\lambda} \\
&= \|\chi_{(\alpha,\beta]} \underline{e_n}\|^2_{L^2(\mathbb{R};d\mu;\mathcal{M}_\Sigma)}, \quad \alpha,\beta \in \mathbb{R}\backslash\mathfrak{S}_\Sigma,\ n \in \mathcal{I}, \quad (72)
\end{aligned}
$$

then establishes a densely defined isometry between the Hilbert spaces $L^2(\widehat{\mathbb{R}; d\Sigma}; \mathcal{K})$ and $L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma)$ which extends by continuity to a unitary map. $\square$

As a result, dropping the additional "hat" on the left-hand side of (69), and hence just using the notation $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ for both Hilbert space constructions is consistent.

We continue this section by yet another approach originally due to Gel'fand and Kostyuchenko [22] and Berezanskii [7, Ch. V]. In this context we also refer to Berezankii [8, Sect. 2.2], Berezansky, Sheftel, and Us [9, Ch. 15], Birman and Entina [10], Gel'fand and Shilov [23, Ch. IV], and M. Malamud and S. Malamud [39], [40]: Introducing an operator $K \in \mathcal{B}_2(\mathcal{H})$ with $\ker(K) = \ker(K^*) = \{0\}$, one has the existence of the weakly μ-measurable nonnegative operator-valued function $\Psi_K(\cdot)$ with values in $\mathcal{B}_1(\mathcal{H})$, such that

$$
\begin{aligned}
(f, \Sigma(B)g)_\mathcal{H} &= \int_B d\mu(t) \left(\Psi_K(t)^{1/2}K^{-1}f, \Psi_K(t)^{1/2}K^{-1}g\right)_\mathcal{H}, \\
&\quad f,g \in \operatorname{dom}\left(K^{-1}\right),\ B \in \mathfrak{B}(\mathbb{R}),\ B \text{ bounded},
\end{aligned} \quad (73)
$$

with

$$
\Psi_K(\cdot) = \frac{dK^*\Sigma K}{d\mu}(\cdot) \quad \mu\text{-a.e.} \quad (74)
$$

In fact, the derivative $\Psi_K(\cdot)$ exists in the $\mathcal{B}_1(\mathcal{H})$-norm (cf. [10] and [39], [40]). Introducing the semi-Hilbert space $\widetilde{\mathcal{H}}_t$, $t \in \mathbb{R}$, as the completion of $\operatorname{dom}\left(K^{-1}\right)$ with respect to the semi-inner product

$$
(f,g)_{\widetilde{\mathcal{H}}_t} = \left(\Psi_K(t)^{1/2}K^{-1}f, \Psi_K(t)^{1/2}K^{-1}g\right)_\mathcal{H}, \quad f,g \in \operatorname{dom}\left(K^{-1}\right),\ t \in \mathbb{R}, \quad (75)
$$

factoring $\widetilde{\mathcal{H}}_t$ by the kernel of the corresponding semi-norm $\ker(\|\cdot\|_{\widetilde{\mathcal{H}}_t})$ then yields the Hilbert space $\mathcal{H}_t = \widetilde{\mathcal{H}}_t/\ker(\|\cdot\|_{\widetilde{\mathcal{H}}_t})$, $t \in \mathbb{R}$. One can show (cf. [39], [40]) that

$$
L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \text{ and } \int_{\mathbb{R}}^{\oplus} d\mu(t)\,\mathcal{H}_t \text{ are isometrically isomorphic}, \quad (76)
$$

yielding yet another construction of $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$.

We conclude this section by sketching some applications to the perturbation theory of self-adjoint operators and to the theory of self-adjoint extensions of symmetric operators, following [24]. We will also briefly comment on work in preparation concerning the spectral theory of ordinary differential operators with operator-valued coefficients.

(I) Self-adjoint perturbations of self-adjoint operators.

    We start by recalling the following result:

Lemma 2.13 ([24]). *Suppose $\mathcal{K}$, $\mathcal{H}$ are separable complex Hilbert spaces, $K \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $\{E(B)\}_{B \in \mathfrak{B}(\mathbb{R})}$ is a family of orthogonal projections in $\mathcal{H}$, and assume that*

$$\overline{\mathrm{lin.span}\{E(B)Ke_n \in \mathcal{H} \mid B \in \mathfrak{B}(\mathbb{R}), \, n \in \mathcal{I}\}} = \mathcal{H}, \tag{77}$$

with $\{e_n\}_{n \in \mathcal{I}}$, $\mathcal{I} \subset \mathbb{N}$ a complete orthonormal system in $\mathcal{K}$. Define

$$\Sigma : \Sigma \to \mathcal{B}(\mathcal{K}), \quad \Sigma(B) = K^* E(B) K, \tag{78}$$

and introduce

$$\dot{U} : \underline{\mathcal{V}}_\Sigma \to \mathcal{H},$$

$$\underline{\mathcal{V}}_\Sigma \ni \sum_{m=1}^M \sum_{n=1}^N \alpha_{m,n} \chi_{B_m} \underline{e}_n \mapsto \dot{U}\left(\sum_{m=1}^M \sum_{n=1}^N \alpha_{m,n} \chi_{B_m} \underline{e}_n\right) \tag{79}$$

$$= \sum_{m=1}^M \sum_{n=1}^N \alpha_{m,n} E(B_m) K e_n \in \mathcal{H},$$

$$\alpha_{m,n} \in \mathbb{C}, \, m = 1, \ldots, M, \, n = 1, \ldots, N, \, M, N \in \mathcal{I}.$$

Then $\dot{U}$ extends to a unitary operator $U : L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma) \to \mathcal{H}$.

    Next, let H_0 a self-adjoint (possibly unbounded) operator in $\mathcal{H}$, L a bounded self-adjoint operator in $\mathcal{K}$, and $K : \mathcal{K} \to \mathcal{H}$ a bounded operator.

    Define the self-adjoint operator H_L in $\mathcal{H}$,

$$H_L = H_0 + KLK^*, \quad \mathrm{dom}(H_L) = \mathrm{dom}(H_0). \tag{80}$$

Given the perturbation H_L of H_0, we introduce the associated operator-valued Herglotz function in $\mathcal{K}$,

$$M_L(z) = K^*(H_L - z)^{-1} K, \quad z \in \mathbb{C} \backslash \mathbb{R}. \tag{81}$$

    Next, let $\{E_0(\lambda)\}_{\lambda \in \mathbb{R}}$ be the family of strongly right-continuous orthogonal spectral projections of H_0 in $\mathcal{H}$ and suppose that $K\mathcal{K} \subseteq \mathcal{H}$ is a generating subspace for H_0, that is, one of the following (equivalent) equations holds:

$$\mathcal{H} = \overline{\mathrm{lin.span}\{(H_0 - z)^{-1} K e_n \in \mathcal{H} \mid n \in \mathcal{I}, \, z \in \mathbb{C} \backslash \mathbb{R}\}} \tag{82}$$

$$= \overline{\mathrm{lin.span}\{E_0(\lambda) K e_n \in \mathcal{H} \mid n \in \mathcal{I}, \, \lambda \in \mathbb{R}\}}, \tag{83}$$

where $\{e_n\}_{n \in \mathcal{I}}$, $\mathcal{I} \subseteq \mathbb{N}$ an appropriate index set, represents a complete orthonormal system in $\mathcal{K}$.

    Denoting by $\{E_L(\lambda)\}_{\lambda \in \mathbb{R}}$ the family of strongly right-continuous orthogonal spectral projections of H_L in $\mathcal{H}$ one introduces

$$\Omega_L(\lambda) = K^* E_L(\lambda) K, \quad \lambda \in \mathbb{R}, \tag{84}$$

and hence verifies

$$M_L(z) = K^*(H_L - z)^{-1} K = K^* \int_{\mathbb{R}} dE_L(\lambda)(\lambda - z)^{-1} K$$

$$= \int_{\mathbb{R}} d\Omega_L(\lambda)(\lambda - z)^{-1}, \quad z \in \mathbb{C} \backslash \mathbb{R}, \tag{85}$$

where the operator Stieltjes integral (85) converges in the norm of $\mathcal{B}(\mathcal{K})$ (cf. Theorems I.4.2 and I.4.8 in [13]). Since s-$\lim_{z \to i\infty} z(H_L - z)^{-1} = -I_{\mathcal{H}}$, (84) implies

$$\Omega_L(\mathbb{R}) = K^* K. \tag{86}$$

Moreover, since s-$\lim_{\lambda \downarrow -\infty} E_L(\lambda) = 0$, s-$\lim_{\lambda \uparrow \infty} E_L(\lambda) = I_{\mathcal{H}}$, one infers

$$\operatorname*{s-lim}_{\lambda \downarrow -\infty} \Omega_L(\lambda) = 0, \quad \operatorname*{s-lim}_{\lambda \uparrow \infty} \Omega_L(\lambda) = K^* K \tag{87}$$

and $\{\Omega_L(\lambda)\}_{\lambda \in \mathbb{R}} \subset \mathcal{B}(\mathcal{K})$ is a family of uniformly bounded, nonnegative, nondecreasing, strongly right-continuous operators from $\mathcal{K}$ into itself. Let μ_L be a σ−finite control measure on $\mathbb{R}$ defined, for instance, by

$$\mu_L(\lambda) = \sum_{n \in \mathcal{I}} 2^{-n}(e_n, \Omega_L(\lambda) e_n)_{\mathcal{K}}, \quad \lambda \in \mathbb{R}, \tag{88}$$

where $\{e_n\}_{n \in \mathcal{I}}$ denotes a complete orthonormal system in $\mathcal{K}$, and then introduce $L^2(\mathbb{R}; d\mu_L; \mathcal{M}_{\Omega_L}) \equiv L^2(\mathbb{R}, d\Omega_L; \mathcal{K})$ as in Remark 2.11 (ii), replacing the pair (Σ, μ) by (Ω_L, μ_L), etc. Abbreviating $\widehat{\mathcal{H}}_L = L^2(\mathbb{R}; d\Omega_L; \mathcal{K})$, we introduce the unitary operator $U_L : \widehat{\mathcal{H}}_L \to \mathcal{H}$, as the operator U in Lemma 2.13 and define $\widehat{H}_L$ in $\widehat{\mathcal{H}}_L$ by

$$(\widehat{H}_L \hat{f})(\lambda) = \lambda \hat{f}(\lambda), \quad \hat{f} \in \operatorname{dom}(\widehat{H}_L) = L^2(\mathbb{R}; (1 + \lambda^2) d\Omega_L; \mathcal{K}). \tag{89}$$

The following result yields a spectral representation (diagonalization) of H_L:

Theorem 2.14 ([24])**.** *The operator H_L in $\mathcal{H}$ is unitarily equivalent to $\widehat{H}_L$ in $\widehat{\mathcal{H}}_L$,*

$$H_L = U_L \widehat{H}_L U_L^{-1}. \tag{90}$$

The family of strongly right-continuous orthogonal spectral projections $\{\widehat{E}_L(\lambda)\}_{\lambda \in \mathbb{R}}$ of $\widehat{H}_L$ in $\widehat{\mathcal{H}}_L$ is given by

$$(\widehat{E}_L(\lambda) \hat{f})(\nu) = \theta(\lambda - \nu) \hat{f}(\nu) \text{ for } \Omega_L - a.e. \ \nu \in \mathbb{R}, \quad \hat{f} \in \widehat{\mathcal{H}}_L, \tag{91}$$

where $\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$

For a variety of additional results in this context we refer to [24].

(II) Self-adjoint extensions of symmetric operators.

We start by developing the following analog of Lemma 2.13: Suppose $\mathcal{N}$ is a separable complex Hilbert space and $\widetilde{\Sigma} : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{N})$ a positive measure. Assume

$$\widetilde{\Sigma}(\mathbb{R}) = \widetilde{T} \geq 0, \quad \widetilde{T} \in \mathcal{B}(\mathcal{N}), \tag{92}$$

and let $\tilde{\mu}$ be a control measure for $\widetilde{\Sigma}$. Moreover, let $\{u_n\}_{n \in \mathcal{I}}$, $\mathcal{I} \subseteq \mathbb{N}$ be a sequence of linearly independent elements in $\mathcal{N}$ with the property $\overline{\operatorname{lin.span}\{u_n\}_{n \in \mathcal{I}}} = \mathcal{N}$. As discussed in Theorem 2.8, this yields a measurable family of Hilbert spaces $\mathcal{M}_{\widetilde{\Sigma}}$ modelled on $\tilde{\mu}$ and $\{\mathcal{N}_\lambda\}_{\lambda \in \mathbb{R}}$ and a bounded map $\underline{\Lambda} \in \mathcal{B}(\mathcal{N}, L^2(\mathbb{R}; d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}))$, $\|\underline{\Lambda}\|_{\mathcal{B}(\mathcal{N}, L^2(\mathbb{R}; d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}))} = \|\widetilde{T}^{1/2}\|_{\mathcal{B}(\mathcal{N})}$, such that $\underline{\Lambda}(\{u_n\}_{n \in \mathcal{I}})$ generates $\mathcal{M}_{\widetilde{\Sigma}}$ and

$$\underline{\Lambda} : \mathcal{V} \to L^2(\mathbb{R}; d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}), \quad v \mapsto \underline{\Lambda} v = \underline{v} = \{\underline{v}(\lambda) = v\}_{\lambda \in \mathbb{R}}, \tag{93}$$

where

$$\mathcal{V} = \operatorname{lin.span}\{u_n\}_{n \in \mathcal{I}}. \tag{94}$$

Each $v \in \mathcal{V}$ defines an element

$$\underline{\underline{v}} = \{\underline{\underline{v}}(\lambda) = (\lambda - i)^{-1} v\}_{\lambda \in \mathbb{R}} \in \mathcal{S}(\{\mathcal{N}_\lambda\}_{\lambda \in \mathbb{R}}), \tag{95}$$

and introducing the weight function

$$w_1(\lambda) = 1 + \lambda^2, \quad \lambda \in \mathbb{R}, \tag{96}$$

and Hilbert space $L^2(\mathbb{R}; w_1 d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}})$, one computes

$$\|\underline{v}\|^2_{L^2(\mathbb{R}; d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}})} = \int_{\mathbb{R}} d\tilde{\mu}(\lambda) \|\underline{v}(\lambda)\|^2_{\mathcal{N}_\lambda} = \left(v, \widetilde{T}v\right)_{\mathcal{N}} = \left\|\big(\widetilde{T}\big)^{1/2} v\right\|^2_{\mathcal{N}}. \tag{97}$$

Thus, the linear map

$$\underline{\underline{\dot{\Lambda}}} : \mathcal{V} \to L^2(\mathbb{R}; w_1 d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}), \quad v \mapsto \underline{\dot{\Lambda}}v = \underline{v} = \left\{\underline{v}(\lambda) = (\lambda - i)^{-1} v\right\}_{\lambda \in \mathbb{R}} \tag{98}$$

extends to $\underline{\underline{\Lambda}} \in \mathcal{B}(\mathcal{N}, L^2(\mathbb{R}; w_1 d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}))$, $\|\underline{\underline{\Lambda}}\|_{\mathcal{B}(\mathcal{N}, L^2(\mathbb{R}; w_1 d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}))} = \left\|\big(\widetilde{T}\big)^{1/2}\right\|_{\mathcal{B}(\mathcal{N})}$. Introducing

$$\underline{\underline{\mathcal{V}}}_{\widetilde{\Sigma}} = \text{lin.span}\big\{\chi_B \underline{u_n} \in L^2(\mathbb{R}; w_1 d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}) \,|\, B \in \mathfrak{B}(\mathbb{R}), \, n \in \mathcal{I}\big\}, \tag{99}$$

one infers that $\underline{\underline{\mathcal{V}}}_{\widetilde{\Sigma}}$ is dense in $L^2(\mathbb{R}; w_1 d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}})$, that is,

$$\overline{\underline{\underline{V}}_{\widetilde{\Sigma}}} = L^2(\mathbb{R}; w_1 d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}). \tag{100}$$

Lemma 2.15 ([24]). *Suppose $\mathcal{H}$ is a separable complex Hilbert space, $\mathcal{N}$ a closed linear subspace of $\mathcal{H}$, $P_{\mathcal{N}}$ the orthogonal projection in $\mathcal{H}$ onto $\mathcal{N}$, $\{E(B)\}$, $B \in \mathfrak{B}(\mathbb{R})$ a family of orthogonal projections in $\mathcal{H}$, and assume*

$$\overline{\text{lin.span}\{E(B)u_n \in \mathcal{H} \,|\, B \in \mathfrak{B}(\mathbb{R}), \, n \in \mathcal{I}\}} = \mathcal{H}, \tag{101}$$

with $\{u_n\}_{n \in \mathcal{I}}$, $\mathcal{I} \subseteq \mathbb{N}$ a complete orthonormal system in $\mathcal{N}$. Define

$$\widetilde{\Sigma} : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{N}), \quad \widetilde{\Sigma}(B) = P_{\mathcal{N}} E(B) P_{\mathcal{N}}\big|_{\mathcal{N}}, \tag{102}$$

and introduce

$$\dot{\widetilde{U}} : \underline{\underline{\mathcal{V}}}_{\Sigma} \to \mathcal{H},$$

$$\underline{\underline{\mathcal{V}}}_{\Sigma} \ni \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_{m,n} \chi_{B_m} \underline{u_n} \mapsto \dot{\widetilde{U}}\bigg(\sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_{m,n} \chi_{B_m} \underline{u_n}\bigg) \tag{103}$$

$$= \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_{m,n} E(B_m) u_n \in \mathcal{H},$$

$$\alpha_{m,n} \in \mathbb{C}, \, m = 1, \dots, M, \, n = 1, \dots, N, \, M, N \in \mathcal{I}.$$

Then $\dot{\widetilde{U}}$ extends to a unitary operator $\widetilde{U} : L^2(\mathbb{R}; w_1 d\tilde{\mu}; \mathcal{M}_{\widetilde{\Sigma}}) \to \mathcal{H}$.

Next, let $\dot{H} : \text{dom}(\dot{H}) \to \mathcal{H}$, $\overline{\text{dom}(\dot{H})} = \mathcal{H}$ be a densely defined closed symmetric linear operator with equal deficiency indices $\text{def}(\dot{H}) = (k, k), k \in \mathbb{N} \cup \{\infty\}$. The deficiency subspaces $\mathcal{N}_\pm$ of $\dot{H}$ are given by

$$\mathcal{N}_\pm = \ker(\dot{H}^* \mp i), \quad \dim_{\mathbb{C}}(\mathcal{N}_\pm) = k. \tag{104}$$

In addition, let H be any self-adjoint extension H of $\dot{H}$ in $\mathcal{H}$, $\mathcal{N}$ a closed linear subspace of $\mathcal{N}_+$, $\mathcal{N} \subseteq \mathcal{N}_+$, and introduce the Weyl–Titchmarsh operator $M_{H,\mathcal{N}}(\cdot) \in \mathcal{B}(\mathcal{N})$ associated with the pair $(H, \mathcal{N})$ by

$$M_{H,\mathcal{N}}(z) = P_{\mathcal{N}}(zH + I_{\mathcal{H}})(H - z)^{-1} P_{\mathcal{N}}\big|_{\mathcal{N}}$$
$$= zI_{\mathcal{N}} + (1 + z^2) P_{\mathcal{N}}(H - z)^{-1} P_{\mathcal{N}}\big|_{\mathcal{N}}, \quad z \in \mathbb{C} \backslash \mathbb{R}, \tag{105}$$

with $I_\mathcal{N}$ the identity operator in $\mathcal{N}$ and $P_\mathcal{N}$ the orthogonal projection in $\mathcal{H}$ onto $\mathcal{N}$. The Herglotz property of $M_{H,\mathcal{N}}(\cdot)$ (i.e., $\mathrm{Im}(M_{H,\mathcal{N}}(z)) \geq 0$ for all $z \in \mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$) then yields (cf., e.g., [30, Appendix A])

$$M_{H,\mathcal{N}}(z) = \int_\mathbb{R} d\Omega_{H,\mathcal{N}}(\lambda)\left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\right], \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{106}$$

where

$$\Omega_{H,\mathcal{N}}(\lambda) = (1 + \lambda^2)\big(P_\mathcal{N} E_H(\lambda) P_\mathcal{N}\big|_\mathcal{N}\big), \tag{107}$$

$$\int_\mathbb{R} d\Omega_{H,\mathcal{N}}(\lambda)(1 + \lambda^2)^{-1} = I_\mathcal{N}, \tag{108}$$

$$\int_\mathbb{R} d(\xi, \Omega_{H,\mathcal{N}}(\lambda)\xi)_\mathcal{N} = \infty \text{ for all } \xi \in \mathcal{N}\backslash\{0\}, \tag{109}$$

and the Herglotz–Nevanlinna representation (106) is valid in the strong operator topology of $\mathcal{N}$.

Next we will prepare some material that eventually will lead to a model for the pair $(\dot{H}, H)$. Let $\{u_n\}_{n \in \mathcal{I}}$, $\mathcal{I} \subseteq \mathbb{N}$ be a complete orthonormal system in $\mathcal{N}$, $\{\widetilde{\Omega}(\lambda)\}_{\lambda \in \mathbb{R}}$ a family of strongly right-continuous nondecreasing $\mathcal{B}(\mathcal{N})$-valued functions normalized by

$$\widetilde{\Omega}(\mathbb{R}) = I_\mathcal{N}, \tag{110}$$

with the property

$$\int_\mathbb{R} d(\xi, \widetilde{\Omega}(\lambda)\xi)_\mathcal{N}(1 + \lambda^2) = \infty \text{ for all } \xi \in \mathcal{N}\backslash\{0\}. \tag{111}$$

Introducing the control measure $\tilde{\mu}(B) = \sum_{n \in \mathcal{I}} 2^{-n}(u_n, \widetilde{\Omega}(B)u_n)_\mathcal{N}$, $B \in \mathfrak{B}(\mathbb{R})$, and $\underline{\Lambda}$ as in Theorem 2.8, we may define $L^p(\mathbb{R}; w d\widetilde{\Omega}; \mathcal{N})$, $p \geq 1$, $w \geq 0$ an appropriate weight function. Of special importance in this section are weight functions of the type $w_r(\lambda) = (1 + \lambda^2)^r$, $r \in \mathbb{R}$, $\lambda \in \mathbb{R}$. In particular, introducing

$$\Omega(B) = \int_B (1 + \lambda^2)d\tilde{\mu}(\lambda)\frac{d\widetilde{\Omega}}{d\tilde{\mu}}(\lambda), \quad B \in \mathfrak{B}(\mathbb{R}), \tag{112}$$

we abbreviate $\widehat{\mathcal{H}} = L^2(\mathbb{R}; d\Omega; \mathcal{N})$ and define the self-adjoint operator $\widehat{H}$ in $\widehat{\mathcal{H}}$,

$$(\widehat{H}\hat{f})(\lambda) = \lambda\hat{f}(\lambda), \quad \hat{f} \in \mathrm{dom}(\widehat{H}) = L^2(\mathbb{R}; (1 + \lambda^2)d\Omega; \mathcal{N}), \tag{113}$$

with corresponding family of strongly right-continuous orthogonal spectral projections

$$(E_{\hat{H}}(\lambda)\hat{f})(\nu) = \theta(\lambda - \nu)\hat{f}(\nu) \text{ for } \Omega - \text{a.e. } \nu \in \mathbb{R}, \quad \hat{f} \in \widehat{\mathcal{H}}. \tag{114}$$

Associated with $\widehat{H}$ we consider the linear operator $\widehat{\dot{H}}$ in $\widehat{\mathcal{H}}$ defined as the following restriction of $\widehat{H}$

$$\mathrm{dom}(\widehat{\dot{H}}) = \left\{\hat{f} \in \mathrm{dom}(\widehat{H}) \,\middle|\, \int_\mathbb{R} (1 + \lambda^2)d\tilde{\mu}(\lambda)(\underline{\xi}, \hat{f}(\lambda))_{\mathcal{N}_\lambda} = 0 \text{ for all } \underline{\xi} \in \underline{\Lambda}(\mathcal{N})\right\},$$

$$\widehat{\dot{H}} = \widehat{H}\big|_{\mathrm{dom}(\hat{\dot{H}})}. \tag{115}$$

Here we used the notation introduced in the proof of Theorem 2.8,

$$\underline{\xi} = \underline{\Lambda}\xi = \{\underline{\xi}(\lambda) = \xi\}_{\lambda \in \mathbb{R}}. \tag{116}$$

Moreover, introducing the scale of Hilbert spaces $\widehat{\mathcal{H}}_{2r} = L^2(\mathbb{R}; (1 + \lambda^2)^r d\Omega); \mathcal{N}$, $r \in \mathbb{R}$, $\widehat{\mathcal{H}}_0 = \widehat{\mathcal{H}}$, we consider the unitary operator R from $\widehat{\mathcal{H}}_2$ to $\widehat{\mathcal{H}}_{-2}$,

$$R : \widehat{\mathcal{H}}_2 \to \widehat{\mathcal{H}}_{-2}, \quad \hat{f} \mapsto (1 + \lambda^2)\hat{f}, \tag{117}$$

$$(\hat{f}, \hat{g})_{\widehat{\mathcal{H}}_2} = (\hat{f}, R\hat{g})_{\widehat{\mathcal{H}}} = (R\hat{f}, \hat{g})_{\widehat{\mathcal{H}}} = (R\hat{f}, R\hat{g})_{\widehat{\mathcal{H}}_{-2}}, \quad \hat{f}, \hat{g} \in \widehat{\mathcal{H}}_2, \tag{118}$$

$$(\hat{u}, \underline{v})_{\widehat{\mathcal{H}}_{-2}} = (\hat{u}, R^{-1}\underline{v})_{\widehat{\mathcal{H}}_2} = (R^{-1}\hat{u}, \underline{v})_{\widehat{\mathcal{H}}} = (R^{-1}\hat{u}, R^{-1}\underline{v})_{\widehat{\mathcal{H}}_2}, \quad \hat{u}, \underline{v} \in \widehat{\mathcal{H}}_{-2}. \tag{119}$$

In particular,

$$\underline{\Delta}(\mathcal{N}) \subset \widehat{\mathcal{H}}, \quad \underline{\Delta}(\mathcal{N}) \subset \widehat{\mathcal{H}}_{-2}, \quad \underline{\xi} \in \underline{\Delta}(\mathcal{N})\backslash\{0\} \Rightarrow \underline{\xi} \notin \widehat{\mathcal{H}}. \tag{120}$$

Theorem 2.16 ([24])**.** *The operator $\widehat{\dot{H}}$ in* (115) *is densely defined symmetric and closed in $\widehat{\mathcal{H}}$. Its deficiency indices are given by*

$$\operatorname{def}(\widehat{\dot{H}}) = (k, k), \quad k = \dim_{\mathbb{C}}(\mathcal{N}) \in \mathbb{N} \cup \{\infty\}, \tag{121}$$

and

$$\ker(\widehat{\dot{H}}^* - z) = \overline{\operatorname{lin.span}\{\{(\lambda - z)^{-1}e_n\}_{\lambda \in \mathbb{R}} \in \widehat{\mathcal{H}} \,|\, n \in \mathcal{I}\}}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{122}$$

Next, we $\mathcal{H}$ decomposes into the direct orthogonal sum

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp, \quad \ker(\dot{H}^* - i) \subset \mathcal{H}_0, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{123}$$

where $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$ are invariant subspaces for all self-adjoint extensions of $\dot{H}$, that is,

$$(H - z)^{-1}\mathcal{H}_0 \subseteq \mathcal{H}_0, \quad (H - z)^{-1}\mathcal{H}_0^\perp \subseteq \mathcal{H}_0^\perp, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{124}$$

for all self-adjoint extensions H of $\dot{H}$ in $\mathcal{H}$. In the following we call a densely defined closed symmetric operator $\dot{H}$ with deficiency indices (k, k), $k \in \mathbb{N} \cup \{\infty\}$ *prime* if $\mathcal{H}_0^\perp = \{0\}$ in the decomposition (123).

Given these preliminaries we can now recall the model for the pair $(\dot{H}, H)$:

Theorem 2.17 ([24])**.** *Assume H is a self-adjoint extension of $\dot{H}$ in $\mathcal{H}$ and let $\{E_H(\lambda)\}_{\lambda \in \mathbb{R}}$ be the associated family of strongly right-continuous orthogonal spectral projections of H and define the unitary operator $\widetilde{U} : \widehat{\mathcal{H}} = L^2(\mathbb{R}; d\Omega_{H,\mathcal{N}_+}; \mathcal{N}_+) \to \mathcal{H}$ as the operator $\widetilde{U}$ in Lemma 2.15, where*

$$\Omega_{H,\mathcal{N}_+}(\lambda) = (1 + \lambda^2)\big(P_{\mathcal{N}_+} E_H(\lambda) P_{\mathcal{N}_+}\big|_{\mathcal{N}_+}\big), \tag{125}$$

with $P_{\mathcal{N}_+}$ the orthogonal projection onto $\mathcal{N}_+ = \ker(\dot{H}^* - i)$. Then the pair $(\dot{H}, H)$ is unitarily equivalent to the pair $(\widehat{\dot{H}}, \widehat{H})$,

$$\dot{H} = \widetilde{U}\widehat{\dot{H}}\widetilde{U}^{-1}, \quad H = \widetilde{U}\widehat{H}\widetilde{U}^{-1}, \tag{126}$$

where $\widehat{H}$ and $\widehat{\dot{H}}$ are defined in (113)–(115)*, and $\mathcal{N}$ is identified with $\mathcal{N}_+$, etc. Moreover,*

$$\widetilde{U}\widehat{\mathcal{N}}_+ = \mathcal{N}_+, \tag{127}$$

where

$$\widehat{\mathcal{N}}_+ = \overline{\operatorname{lin.span}\{\underline{u}_{+,n} \in \widehat{\mathcal{H}} \,|\, \underline{u}_{+,n}(\lambda) = (\lambda - i)^{-1}u_{+,n}, \lambda \in \mathbb{R}, n \in \mathcal{I}\}}, \tag{128}$$

with $\{u_{+,n}\}_{n \in \mathcal{I}}$ a complete orthonormal system in $\mathcal{N}_+ = \ker(\dot{H}^* - i)$.

Again, for a variety of additional results in this context we refer to [24]. We also note that additional applications of operator-valued Herglotz functions (such as (85) and (106)) can be found, for instance, in [1]–[4], [6], [12], [14], [15]–[17], [24]–[28], [29], [30], [32], [37], [38], [41], [42], [43]–[46], [47]–[49], [53], [54],

(III) Spectral theory for Schrödinger operators with operator-valued potentials.

Assuming $(a, b) \subseteq \mathbb{R}$, and supposing that

$$
\begin{aligned}
& V : (a, b) \to \mathcal{B}(\mathcal{H}) \text{ is a weakly measurable operator-valued function,} \\
& \|V(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}((a, b); dx), \\
& V(x) = V(x)^* \text{ for a.e. } x \in (a, b),
\end{aligned} \tag{129}
$$

we recently developed Weyl–Titchmarsh theory for certain self-adjoint operators H_α in $L^2((a, b); dx; \mathcal{H})$ associated with the operator-valued differential expression $\tau = -(d^2/dx^2) + V(\cdot)$ (cf. [30]). These are suitable restrictions of the *maximal* operator $H_{\max}$ in $L^2((a, b); dx; \mathcal{H})$ defined by

$$
\begin{aligned}
& H_{\max} f = \tau f, \\
& f \in \text{dom}(H_{\max}) = \big\{ g \in L^2((a, b); dx; \mathcal{H}) \,\big|\, g \in W^{2,1}_{\text{loc}}((a, b); dx; \mathcal{H}); \\
& \hspace{6cm} \tau g \in L^2((a, b); dx; \mathcal{H}) \big\}.
\end{aligned} \tag{130}
$$

In particular, assuming in addition that a is a regular endpoint for τ and b is of limit-point type for τ, the operator H_α defined as the restriction of $H_{\max}$ by

$$
\text{dom}(H_\alpha) = \{ u \in \text{dom}(H_{\max}) \,|\, \sin(\alpha)u'(a) + \cos(\alpha)u(a) = 0 \}. \tag{131}
$$

where $\alpha = \alpha^* \in \mathcal{B}(\mathcal{H})$ is self-adjoint in $L^2((a, b); dx; \mathcal{H})$. Conversely, all self-adjoint restrictions of $H_{\max}$ arise in this manner.

Introducing $\theta_\alpha(z, \cdot, x_0,), \phi_\alpha(z, \cdot, x_0)$ as those $\mathcal{B}(\mathcal{H})$-valued solutions of $\tau Y = zY$ which satisfy the initial conditions

$$
\theta_\alpha(z, x_0, x_0) = \phi_\alpha'(z, x_0, x_0) = \cos(\alpha), \quad -\phi_\alpha(z, x_0, x_0) = \theta_\alpha'(z, x_0, x_0) = \sin(\alpha), \tag{132}
$$

one of the principal results in [30] establishes the existence of $\mathcal{B}(\mathcal{H})$-valued Weyl–Titchmarsh solutions $\psi_\alpha(z, \cdot)$ of $\tau Y = zY$ of the form

$$
\psi_\alpha(z, x) = \theta_\alpha(z, x, a) + \phi_\alpha(z, x, a)m_\alpha(z), \quad z \in \mathbb{C} \backslash \mathbb{R}, \ x \in [a, b), \tag{133}
$$

satisfying

$$
\int_a^b dx \, \|\psi_\alpha(z, x)f\|_{\mathcal{H}}^2 < \infty, \quad f \in \mathcal{H}, \ z \in \mathbb{C} \backslash \mathbb{R}. \tag{134}
$$

Moreover, the $\mathcal{B}(\mathcal{H})$-valued-valued Weyl–Titchmarsh function $m_\alpha(\cdot)$ is shown to be a Herglotz function in [30]. Consequently, $m_\alpha(\cdot)$ permits the Herglotz–Nevanlinna

representation (cf. [30, Appendix A])

$$m_\alpha(z) = C_\alpha + D_\alpha z + \int_{\mathbb{R}} d\Omega_\alpha(\lambda) \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right], \quad z \in \mathbb{C}_+, \qquad (135)$$

$$\widetilde{\Omega}_\alpha((-\infty, \lambda]) = \underset{\varepsilon \downarrow 0}{\text{s-lim}} \int_{-\infty}^{\lambda + \varepsilon} \frac{d\Omega_\alpha(t)}{1 + t^2}, \quad \lambda \in \mathbb{R}, \qquad (136)$$

$$\widetilde{\Omega}_\alpha(\mathbb{R}) = \text{Im}(m_\alpha(i)) = \int_{\mathbb{R}} \frac{d\Omega_\alpha(\lambda)}{1 + \lambda^2} \in \mathcal{B}(\mathcal{H}), \qquad (137)$$

$$C_\alpha = \text{Re}(m_\alpha(i)), \quad D_\alpha = \underset{\eta \uparrow \infty}{\text{s-lim}} \frac{1}{i\eta} m_\alpha(i\eta) \geq 0, \qquad (138)$$

valid in the strong sense in $\mathcal{H}$, where $\widetilde{\Omega}_\alpha(B) = \int_B (1 + \lambda^2)^{-1} d\Omega_\alpha(\lambda)$, $B \in \mathfrak{B}(\mathbb{R})$. The function $m_\alpha(\cdot)$ contains all the spectral information of H_α and is closely related to the Green's function of H_α as discussed in [30].

Introducing the Hilbert space $L^2(\mathbb{R}; d\Omega_\alpha; \mathcal{H})$, the spectral representation (resp., model representation) of H_α then aims at exhibiting the unitary equivalence of H_α with the operator of multiplication by the independent variable in $L^2(\mathbb{R}; d\Omega_\alpha; \mathcal{H})$. Under stronger hypotheses on V than those recorded in (129), for instance, continuity of $V(\cdot)$ in $\mathcal{B}(\mathcal{H})$, such a result has been shown by Rofe-Beketov [50] and Gorbačuk [32], and subsequently, under hypotheses close to those in (129), by Saito [52]. Our own approach to this circle of ideas is in preparation [31].

We conclude this section by noting that certain classes of unbounded operator-valued potentials V lead to applications to multi-dimensional Schrödinger operators in $L^2(\mathbb{R}^n; d^n x)$, $n \in \mathbb{N}$, $n \geq 2$. It is precisely the connection between multi-dimensional Schrödinger operators and one-dimensional Schrödinger operators with unbounded operator-valued potentials which originally motivated our interest in this area.

Appendix A. **Completion of Semi-Metric Spaces.**

In this appendix we establish the equivalence of two approaches to the procedure of completion of semi-metric spaces. For background material on semi-metric spaces we refer to [57, Ch. 9].

We start by recalling that a semi-metric space (S, ρ) is a set S with a semi-metric $\rho : S \times S \to [0, \infty)$, satisfying

$$\rho(x, x) = 0, \quad x \in S, \qquad (1)$$

$$\rho(x, y) = \rho(y, x), \quad x, y \in S, \qquad (2)$$

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y), \quad x, y, z \in S. \qquad (3)$$

We point out that a semi-metric space may have two distinct elements $x, y \in S$ with $\rho(x, y) = 0$. Nevertheless, one can introduce the notion of Cauchy sequences and limits in a semi-metric space (S, ρ) as usual. Of course, in this case convergent sequences may have several distinct limits. A semi-metric space (S, ρ) is called *complete* if every Cauchy sequence of points in S has a limit which also lies in S. Next, we discuss two approaches to completion of a semi-metric space.

(**I**) Given a semi-metric space (S, ρ), one introduces a semi-metric space $(\widetilde{S}_1, \widetilde{\rho}_1)$, where $\widetilde{S}_1$ is the set of all Cauchy sequence in S and

$$\widetilde{\rho}_1(\widetilde{x}, \widetilde{y}) = \lim_{n \to \infty} \rho(x(n), y(n)), \quad \widetilde{x} = \{x(n)\}_{n \in \mathbb{N}}, \widetilde{y} = \{y(n)\}_{n \in \mathbb{N}} \in \widetilde{S}_1. \qquad (4)$$

It follows from the triangle inequality (3) for ρ that

$$|\rho(x(n), y(n)) - \rho(x(m), y(m))| \leq \rho(x(n), x(m)) + \rho(y(n), y(m)), \qquad (5)$$

and hence for all $\widetilde{x}, \widetilde{y} \in \widetilde{S}_1$ the sequence $\{\rho(x(n), y(n))\}_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$. Thus, the limit in (4) exists, and using (4) one verifies that $\widetilde{\rho}_1$ is a semi-metric on $\widetilde{S}_1$. Moreover, it has been shown in [57, p. 176] that $(\widetilde{S}_1, \widetilde{\rho}_1)$ is a complete semi-metric space and that (S, ρ) is isometric to a dense subset in $(\widetilde{S}_1, \widetilde{\rho}_1)$. Introducing an equivalence relation $\sim$ on $\widetilde{S}_1$ by

$$\widetilde{x} \sim \widetilde{y} \text{ whenever } \widetilde{\rho}_1(\widetilde{x}, \widetilde{y}) = 0, \quad \widetilde{x}, \widetilde{y} \in \widetilde{S}_1, \qquad (6)$$

one defines the set of equivalence classes

$$S_1 = \widetilde{S}_1 / \sim = \{[\widetilde{x}]_{\widetilde{\rho}_1} \mid \widetilde{x} \in \widetilde{S}_1\} \qquad (7)$$

and a metric on S_1 by

$$\rho_1([\widetilde{x}]_{\widetilde{\rho}_1}, [\widetilde{y}]_{\widetilde{\rho}_1}) = \widetilde{\rho}_1(\widetilde{x}, \widetilde{y}), \quad [\widetilde{x}]_{\widetilde{\rho}_1}, [\widetilde{y}]_{\widetilde{\rho}_1} \in S_1. \qquad (8)$$

It follows from the triangle inequality for $\widetilde{\rho}_1$ that

$$\widetilde{\rho}_1(\widetilde{x}_1, \widetilde{y}_1) = \widetilde{\rho}_1(\widetilde{x}_2, \widetilde{y}_2) \text{ whenever } \widetilde{x}_1 \sim \widetilde{x}_2 \text{ and } \widetilde{y}_1 \sim \widetilde{y}_2, \qquad (9)$$

and hence ρ_1 is a well-defined metric on S_1. Thus, (S_1, ρ_1) is a complete metric space, and (S, ρ) is isometric to a dense subset of (S_1, ρ_1).

Now, we consider a different approach:

(**II**) Given a semi-metric space (S, ρ), define an equivalence relation $\sim$ on S by

$$x \sim y \text{ whenever } \rho(x, y) = 0, \quad x, y \in S, \qquad (10)$$

and the set of equivalence classes

$$M = S / \sim = \{[x]_\rho \mid x \in S\}. \qquad (11)$$

It follows from the triangle inequality for ρ that for all $x_1, x_2, y_1, y_2 \in S$

$$\rho(x_1, y_1) = \rho(x_2, y_2) \text{ whenever } x_1 \sim x_2 \text{ and } y_1 \sim y_2. \qquad (12)$$

Hence, the function

$$d([x]_\rho, [y]_\rho) = \rho(x, y), \quad [x]_\rho, [y]_\rho \in M, \qquad (13)$$

is a well-defined metric on M. One completes the metric space (M, d) by introducing the set $\widehat{S}_2$ of all Cauchy sequences of points in M and a semi-metric

$$\widehat{\rho}_2(\widehat{x}, \widehat{y}) = \lim_{n \to \infty} d([x]_\rho(n), [y]_\rho(n)), \quad \widehat{x} = \{[x]_\rho(n)\}_{n \in \mathbb{N}}, \widehat{y} = \{[y]_\rho(n)\}_{n \in \mathbb{N}} \in \widehat{S}_2. \qquad (14)$$

Thus, $(\widehat{S}_2, \widehat{\rho}_2)$ is a complete semi-metric space. Introducing an equivalence relation $\sim$ on $\widehat{S}_2$ by

$$\widehat{x} \sim \widehat{y} \text{ whenever } \widehat{\rho}_2(\widehat{x}, \widehat{y}) = 0, \quad \widehat{x}, \widehat{y} \in \widehat{S}_2, \qquad (15)$$

one defines the set of equivalence classes

$$S_2 = \widehat{S}_2 / \sim = \{[\widehat{x}]_{\widehat{\rho}_2} \mid \widehat{x} \in \widehat{S}_2\} \qquad (16)$$

and a metric on S_2 by

$$\rho_2([\widehat{x}]_{\widehat{\rho}_2}, [\widehat{y}]_{\widehat{\rho}_2}) = \widehat{\rho}_2(\widehat{x}, \widehat{y}), \quad [\widehat{x}]_{\widehat{\rho}_2}, [\widehat{y}]_{\widehat{\rho}_2} \in S_2. \qquad (17)$$

Again, it follows from the triangle inequality for $\widehat{\rho}_2$ that ρ_2 is a well-defined metric on S_2. Thus, (S_2, ρ_2) is a complete metric space, and (M, d) and hence (S, ρ) are isometric to a dense subset of (S_2, ρ_2).

The main result of this appendix is the following isometry lemma.

Lemma A.1. *The metric spaces (S_j, ρ_j), $j = 1, 2$, introduced in steps (I) and (II) above are isometric (i.e., there exists an isometric bijection $J : S_1 \to S_2$).*

Proof. To establish an isometric bijection $T : S_1 \to S_2$ one proceeds as follows: For every element $\widetilde{x} = \{x(n)\}_{n \in \mathbb{N}} \in \widetilde{S}_1$ one defines $\widehat{x} \in \widehat{S}_2$ by $\widehat{x} = \{[x(n)]_\rho\}_{n \in \mathbb{N}}$. Then

$$\begin{aligned}
\rho_1([\widetilde{x}]_{\widetilde{\rho}_1}, [\widetilde{y}]_{\widetilde{\rho}_1}) &= \widetilde{\rho}_1(\widetilde{x}, \widetilde{y}) = \lim_{n \to \infty} \rho(x(n), y(n)) \\
&= \lim_{n \to \infty} d([x]_\rho(n), [y]_\rho(n)) = \widehat{\rho}_2(\widehat{x}, \widehat{y}) = \rho_2([\widehat{x}]_{\widehat{\rho}_2}, [\widehat{y}]_{\widehat{\rho}_2}).
\end{aligned} \tag{18}$$

Since ρ_2 is a metric, it follows from (18) that for every $\widetilde{x}, \widetilde{y} \in \widetilde{S}_1$ with $[\widetilde{x}]_{\widetilde{\rho}_1} = [\widetilde{y}]_{\widetilde{\rho}_1}$ the above construction yields $[\widehat{x}]_{\widehat{\rho}_2} = [\widehat{y}]_{\widehat{\rho}_2}$. Thus, there is no ambiguity in defining an isometry

$$J : \begin{cases} S_1 \to S_2 \\ [\widetilde{x}]_{\widetilde{\rho}_1} \mapsto J([\widetilde{x}]_{\widetilde{\rho}_1}) = [\widehat{x}]_{\widehat{\rho}_2} \end{cases} \tag{19}$$

(i.e., J is well-defined). It follows from the above constructions that the domain of J is all of S_1 and the range is all of S_2. Moreover, since ρ_1 is a metric, (18) implies that J is one-to-one and hence a bijection. $\qquad\square$

Remark A.2. In the case of (S, ρ) being a semi-normed vector space, that is,

$$\rho(x, y) = \|x - y\|, \quad x, y \in S, \tag{20}$$

the spaces (S_1, ρ_1) and (S_2, ρ_2) become Banach spaces, and in this case

$$S_1 = \widetilde{S}_1 / \ker(\widetilde{\rho}_1), \quad M = S / \ker(\rho), \quad S_2 = \widehat{S}_2 / \ker(\widehat{\rho}_2), \tag{21}$$

where $\ker(\widetilde{\rho}_1)$, $\ker(\widehat{\rho}_2)$, and $\ker(\rho)$ denote the linear subspaces of elements of zero norm,

$$\ker(\rho) = \{x \in S \mid \|x\| = 0\}, \tag{22}$$

and similarly for $\ker(\widetilde{\rho}_1)$ and $\ker(\widehat{\rho}_2)$. In addition, the analog of the isometry J in Lemma A.1 now also becomes a linear map in the context of normed spaces.

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