The Spectral problem for the dispersionless Camassa-Holm equation

C. Bennewitz, B. M. Brown and R. Weikard

Abstract. We present a spectral and inverse spectral theory for the zero dispersion spectral problem associated with the Camassa-Holm equation. This is an alternative approach to that in [10] by Eckhardt and Teschl.

Mathematics Subject Classification (2010). Primary 37K15, 34B40 ; Secondary 35Q35, 34L05.

Keywords. Camassa-Holm equation, isospectral problem, inverse spectral theory.

1. Background

The Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x + 2\varkappa u_x = 2u_x u_{xx} + uu_{xxx}$$

was suggested as a model for shallow water waves by Camassa and Holm [5], although originally found by Fuchssteiner and Fokas [12]. Here $\varkappa$ is a constant related to dispersion. The equation has scaling properties such that one needs only study the cases $\varkappa = 1$ and the zero dispersion case $\varkappa = 0$.

There are compelling reasons to study the equation. Like the KdV equation it is an integrable system but, unlike the KdV equation, among its solutions are *breaking waves* (see Camassa and Holm [5] and Constantin [8]). These are solutions with smooth initial data that stay bounded, but where the wave front becomes vertical in finite time, so that the derivative blows up. A model for water waves displaying wave breaking was long sought after.

Since the CH equation is an integrable system it has an associated spectral problem, which is

$$-f'' + \tfrac{1}{4}f = \lambda w f, \tag{1.1}$$

where $w = u - u_{xx} + \varkappa$. At least two cases are particularly important, namely the periodic case and the case of decay at infinity. We only deal with the latter case here (see, *e.g*, Constantin and Escher [9]), so in the zero dispersion case we should have w small at infinity. For the periodic case see Constantin and McKean [6].

In the zero dispersion case the solitons (here called *peakons*) give rise to w which is a Dirac measure, so one should clearly at least allow w to be a measure[1]. It is also important that one does not assume that w has a fixed sign, since no wave breaking will then take place (see Jiang, Ni and Zhou [15]).

In [3] we discussed scattering and inverse scattering in the case $\varkappa \neq 0$, which is the important case for shallow water waves. We did not discuss the zero dispersion case $\varkappa = 0$, which is relevant in some other situations, but this case was treated by Eckhardt and Teschl in [10], based on the results of Eckhardt [11].

The approach of [10] was based on the fact that in the zero dispersion case it is possible to define a Titchmarsh-Weyl type m-function for the whole line spectral problem. This approach does not work if $\varkappa \neq 0$. The fundamental reason behind this is that for corresponding half-line problems one gets a discrete spectrum in the zero dispersion case, but there is always a half-line of continuous spectrum if $\varkappa \neq 0$. More conceptually, the continuous spectrum is of multiplicity 2 which excludes the existence of a scalar m-function. For the inverse theory Eckhardt [11] uses de Branges' theory of Hilbert spaces of entire functions. Our approach is different and analogous to that in our paper [3].

It should be noted that the methods of this note combined with those of [3] allow one to prove a uniqueness theorem for inverse scattering in the case $\varkappa \neq 0$ for the case when w is a measure, extending the results of [3] where it was assumed that $w \in L^1_{\text{loc}}$. These results do not appear to be accessible using de Branges' theory.

2. A Hilbert space

Instead of (1.1) we shall analyze the slightly more general spectral problem

$$-f'' + qf = \lambda w f, \tag{2.1}$$

where q is a positive measure not identically zero, since this presents few additional difficulties. A solution of (2.1), or more generally of $-f''+qf = \lambda wf+g$, where g is a given measure, is a continuous function f satisfying the equation in the sense of distributions. Since $(\lambda w - q)f + g$ is then a measure it follows that a solution is locally absolutely continuous with a derivative of locally bounded variation. It is known that a unique solution exists with prescribed values of f and, say, its left derivative at a given point (this result may be found for example in Bennewitz [1, Chapter 1]), and we will occasionally use this. It follows that the solution space of the homogeneous equation is of dimension 2.

[1] In this paper we use the word *measure* for a distribution of order 0.

We will also have occasion to talk about the Wronskian $[f_1, f_2] = f_1 f_2' - f_1' f_2$ of two solutions f_1 and f_2 of (2.1). The main property is that such a Wronskian is constant, which easily follows on differentiation and use of the equation. Note that the regularity of solutions is such that the product rule applies when differentiating the Wronskian in the sense of distributions. The unique solvability of the initial value problem shows that f_1 and f_2 are linearly dependent precisely if $[f_1, f_2] = 0$.

We shall consider (2.1) in a Hilbert space $\mathcal{H}_1$ with scalar product

$$\langle f, g\rangle = \int_{\mathbb{R}} (f'\overline{g'} + qf\overline{g}).$$

Thus we are viewing (2.1) as a 'left definite' equation. The space $\mathcal{H}_1$ consists of those locally absolutely continuous functions f which have derivative in $L^2(\mathbb{R})$ and for which $\int_{\mathbb{R}} q|f|^2 < \infty$, so it certainly contains the test functions $C_0^\infty(\mathbb{R})$. Some properties of the space $\mathcal{H}_1$ will be crucial for us.

Lemma 2.1. *Non-trivial solutions of $-u'' + qu = 0$ have at most one zero, and there is no non-trivial solution in $\mathcal{H}_1$.*

Proof. The real and imaginary parts of a solution u are also solutions and in $\mathcal{H}_1$ if u is, so it is enough to consider real-valued solutions. From the equation it is clear that such a solution is convex in any interval where it is positive, concave where it is negative.

The set of zeros of a real-valued non-trivial solution u is a closed set with no interior by the uniqueness of the initial value problem. Since u is continuous it keeps a fixed sign in any component of the complement. Convexity of $|u|$ in each component shows that any such component is unbounded, so u has at most one zero.

Since $|u|$ is convex and non-negative u' can only be in L^2 if u is constant. But this would imply $q = 0$, so the second claim follows. □

As we shall see there are, however, non-trivial solutions with $|u'|^2 + q|u|^2$ integrable on a half-line. We shall also need the following lemma (*cf.* Lemmas 2.1 and 2.2 of [3]).

Lemma 2.2. *Functions with square integrable (distributional) derivative for large $|x|$ are $o(\sqrt{|x|})$ as $x \to \pm\infty$ and point evaluations are bounded linear forms on $\mathcal{H}_1$. Furthermore, $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{H}_1$,*

Proof. The first two claims are proved in [3, Lemma 2.1]). The final claim follows since clearly $C_0^\infty(\mathbb{R}) \subset \mathcal{H}_1$ and if $u \in \mathcal{H}_1$ is orthogonal to $C_0^\infty(\mathbb{R})$ an integration by parts shows that $\int u(-\varphi'' + q\varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R})$ so that u is a distributional solution of $-u'' + qu = 0$. By Lemma 2.1 it is therefore identically 0. □

We also need the following result.

Lemma 2.3. *For any $\lambda \in \mathbb{C}$ there can be at most one linearly independent solution of $-f'' + qf = \lambda wf$ with f' in L^2 near infinity. Similarly for f' in L^2 near $-\infty$.*

This means that (2.1) is in the 'limit-point case' at $\pm\infty$, with a terminology borrowed from the right definite case. The lemma is a consequence of general facts about left definite equations (see our paper [2]), but we will give a simple direct proof.

Proof. Suppose there are two linearly independent solutions f, g with f', g' in L^2 near ∞. We may assume the Wronskian $fg' - f'g = 1$. Now by Lemma 2.2 $f(x)/\sqrt{x}$ and $g(x)/\sqrt{x}$ are bounded for large x. It follows that $(fg' - f'g)/\sqrt{x} = 1/\sqrt{x}$ is in L^2 for large x, which is a contradiction.

Similar calculations may be made for x near $-\infty$. $\square$

Let $E(x)$ be the norm of the linear form $\mathcal{H}_1 \ni f \mapsto f(x)$. We can easily find an expression for $E(x)$, since the Riesz representation theorem tells us that there is an element $g_0(x, \cdot) \in \mathcal{H}_1$ such that $f(x) = \langle f, g_0(x, \cdot)\rangle$. Thus $|f(x)| \le \|g_0(x, \cdot)\|\|f\|$, with equality for $f = g_0(x, \cdot)$ so that

$$E(x) = \|g_0(x, \cdot)\| = \sqrt{g_0(x, x)}.$$

If $\varphi \in C_0^\infty$ we have $\langle \varphi, g_0(x, \cdot)\rangle = \varphi(x)$, which after an integration by parts means

$$\int_{\mathbb{R}} (-\varphi'' + q\varphi)\overline{g_0(x, \cdot)} = \varphi(x)$$

so (in a distributional sense) $g_0(x, \cdot)$ is a solution of $-f'' + qf = \delta_x$, where δ_x is the Dirac measure at x. Since $g_0(x, y) = \langle g_0(x, \cdot), g_0(y, \cdot)\rangle$ we have a symmetry $g_0(x, y) = \overline{g_0(y, x)}$. Now g_0 is realvalued since $\operatorname{Im} g_0(x, \cdot)$ satisfies $-f'' + qf = 0$ and therefore vanishes according to Lemma 2.1. We may thus write

$$g_0(x, y) = F_+(\max(x, y))F_-(\min(x, y))$$

where $F_\pm$ are real-valued solutions of $-f'' + qf = 0$ small enough at $\pm\infty$ for $g_0(x, \cdot)$ to be in $\mathcal{H}_1$ and by Lemma 2.3 this determines $F_\pm$ up to real multiples. The equation satisfied by $g_0(x, \cdot)$ shows that the Wronskian $[F_+, F_-] = F_+F_-' - F_+'F_- = 1$. In particular, $E(x)$ is locally absolutely continuous. At any specified point of $\mathbb{R}$ there are elements of $\mathcal{H}_1$ that do not vanish, so that $E > 0$ and $F_\pm$ never vanish. Since $g_0(x, x) > 0$ we may therefore assume both to be strictly positive. Note that this still does not determine $F_\pm$ uniquely since multiplying F_+ and dividing F_- by the same positive constant does not change g_0.

However, $|F_\pm'|^2 + q|F_\pm|^2$ has finite integral near $\pm\infty$, although not, according to Lemma 2.3, over $\mathbb{R}$. If we can solve the equation $-f'' + qf = 0$ we can therefore determine $E(x)$. For example, if $q = 1/4$ we have $g_0(x, y) = \exp(-|x - y|/2)$ and $E(x) \equiv 1$.

We shall need some additional properties of $F_\pm$ and make the following definition.

Definition 2.4. Define $K = F_-/F_+$.

We have the following proposition.

Proposition 2.5.

- *$F_\pm$ are both convex,*
- $\lim_\infty F'_+ = \lim_{-\infty} F'_- = 0$*,*
- *$F'_\pm$ as well as F_- are non-decreasing while F_+ is non-increasing,*
- *$F_+(x) \to \infty$ as $x \to -\infty$ and $F_-(x) \to \infty$ as $x \to \infty$,*
- $\lim_{-\infty} F_-$ *and* $\lim_\infty F_+$ *are finite,*
- *$1/F_+ \in L^2$ near $-\infty$ while $1/F_- \in L^2$ near ∞,*
- *The function K is strictly increasing with range $\mathbb{R}_+$ and of class C^1 with a C^1 inverse, and $K' = 1/F_+^2$.*

Proof. The convexity of $F_\pm$ follows from positivity and the differential equation they satisfy. Thus $F'_\pm$ has finite or infinite limits at $\pm\infty$, and since $F'_\pm$ is in L^2 near $\pm\infty$ we have $\lim_{-\infty} F'_- = \lim_\infty F'_+ = 0$ so $F'_- \geq 0$ while $F'_+ \leq 0$. It follows that $\lim_\infty F_+$ and $\lim_{-\infty} F_-$ are finite.

Neither of $F_\pm$ is constant so it follows that $\lim_\infty F_+ = \lim_{-\infty} F_- + = +\infty$ and that the range of K is $\mathbb{R}_+$. Furthermore $K' = [F_+, F_-]/F_+^2 = 1/F_+^2$ so K' is continuous and > 0. Thus K has an inverse of class C^1.

Since $K(x) = \int_{-\infty}^x 1/F_+^2$ we have $1/F_+$ in L^2 near $-\infty$, and differentiating $1/K$ we similarly obtain $1/F_-$ in L^2 near ∞. □

3. Spectral theory

In addition to the scalar product, the Hermitian form $w(f,g) = \int_\mathbb{R} f\overline{g}w$ plays a role in the spectral theory of (2.1). We denote the total variation measure of w by $|w|$, and make the following assumption in the rest of the paper.

Assumption 3.1. w is a real-valued, non-zero measure (distribution of order zero) and $E^2|w|$ is a finite measure.

We then note the following.

Proposition 3.2. *If $E^2|w|$ is a finite measure the form $w(f,g)$ is bounded in $\mathcal{H}_1$.*

Proof. We have $|w(f,g)| \leq \|f\|\|g\| \int_\mathbb{R} E^2|w|$. □

As we shall soon see, the assumption actually implies that the form $w(f,g)$ is compact in $\mathcal{H}_1$. Note that if $q = 1/4$, or any other constant > 0, then the assumption is simply that $|w|$ is finite. It may be proved that this is the case also if $q - q_0$ is a finite signed measure for some constant $q_0 > 0$, and that it is in all cases enough if $(1 + |x|)w(x)$ is finite.

Using Riesz' representation theorem Proposition 3.2 immediately shows that there is a bounded operator R_0 on $\mathcal{H}_1$ such that

$$\int_\mathbb{R} f\overline{g}w = \langle R_0 f, g\rangle, \tag{3.1}$$

where $\|R_0\| \leq \int_\mathbb{R} E^2|w|$. Since w is realvalued the operator R_0 is symmetric. We also have $R_0u(x) = \langle R_0u, g_0(x,\cdot)\rangle = \int_\mathbb{R} ug_0(x,\cdot)w$ so that R_0 is an integral operator.

It is clear that $R_0 u = 0$ precisely if $uw = 0$, so unless[2] $\operatorname{supp} w = \mathbb{R}$ the operator R_0 has a nontrivial nullspace. We need the following definition.

Definition 3.3. The orthogonal complement of the nullspace of R_0 is denoted by $\mathcal{H}$.

The restriction of R_0 to $\mathcal{H}$, which we also denote by R_0, is an operator on $\mathcal{H}$ with dense range since the orthogonal complement of the range of $R_0^* = R_0$ is the nullspace of R_0. Thus the restriction of R_0 to $\mathcal{H}$ has a selfadjoint inverse T densely defined in $\mathcal{H}$ and R_0 is the resolvent of T at 0.

Lemma 3.4. *$f \in \mathcal{D}_T$ and $Tf = g$ precisely if $f, g \in \mathcal{H}$ and (in the sense of distributions) $-f'' + qf = wg$.*

Proof. $Tf = g$ means that $f = R_0 g$ which in turn means that $\langle f, \overline{\varphi} \rangle = \int g\varphi w$ for $\varphi \in C_0^\infty$ which may be written $\int f(-\varphi''+q\varphi) = \int g\varphi w$ after an integration by parts. But this is the meaning of the equation $-f'' + qf = wg$.

The same calculation in reverse, using that according to Lemma 2.2 $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{H}_1$, proves the converse. □

The complement of $\operatorname{supp} w$ is a countable union of disjoint open intervals. We shall call any such interval a *gap* in $\operatorname{supp} w$. We obtain the following characterisation of the elements of $\mathcal{H}$.

Corollary 3.5.

- *The projection of $v \in \mathcal{H}_1$ onto $\mathcal{H}$ equals v in $\operatorname{supp} w$, and if (a, b) is a gap in the support of w the projection is determined in the gap as the solution of $-u''+qu = 0$ which equals v in the endpoints a and b if these are finite.*

  If $a = -\infty$ the restriction of the projection to the gap is the multiple of F_- which equals v in b, and if $b = \infty$ it is the multiple of F_+ which equals v in a.
- *The support of an element of $\mathcal{H}$ can not begin or end inside a gap in the support of w.*
- *The reproducing kernel $g_0(x, \cdot) \in \mathcal{H}$ if and only if $x \in \operatorname{supp} w$.*

Proof. The difference between v and its projection onto $\mathcal{H}$ can be non-zero only in gaps of $\operatorname{supp} w$. Clearly $\varphi w = 0$ for any $\varphi \in C_0^\infty(a, b)$ so that $C_0^\infty(a, b)$ is orthogonal to $\mathcal{H}$. It follows that an element of $\mathcal{H}$ satisfies the equation $-u'' + qu = 0$ in any gap of the support of w.

The first two items are immediate consequences of this, that non-trivial solutions of $-u'' + qu = 0$ have at most one zero according to Lemma 2.1, and of the fact that elements of $\mathcal{H}$ are continuous.

The third item is an immediate consequence of the first two. □

Theorem 3.6. *Under Assumption 3.1 the operator R_0 is compact with simple spectrum, so T has discrete spectrum.*

---

[2] We always use supports in the sense of distributions.

Proof. Suppose $f_j \rightharpoonup 0$ weakly in $\mathcal{H}$. Since point evaluations are bounded linear forms we have $f_j \to 0$ pointwise, and $\{f_j\}_1^\infty$ is bounded in $\mathcal{H}$, as is $\{R_0 f_j\}_1^\infty$. We have

$$\|R_0 f_j\|^2 = \int_{\mathbb{R}} R_0 f_j \overline{f_j} w.$$

Here the coefficient of w tends point-wise to 0 and is bounded by $\|R_0\|\|f_j\|^2 E^2$ which in turn is bounded by a multiple of E^2. It follows by dominated convergence that $\|R_0 f_j\| \to 0$. Thus R_0 is compact, and the spectrum is simple by Lemma 2.3. □

Actually, R_0 is of trace class as is proved by Eckhardt and Teschl in [10] for the case $q = 1/4$, but we will not need this.

4. Jost solutions

In one-dimensional scattering theory *Jost solutions* play a crucial part. In the case of the Schrödinger equation these are solutions asymptotically equal at ∞ respectively $-\infty$ to certain solutions of the model equation $-f'' = \lambda f$. In the present case the model equation would be one where $w \equiv 0$, *i.e.*, $-f''+qf = 0$. We shall therefore look for solutions $f_\pm(\cdot,\lambda)$ of $-f''+qf = \lambda w f$ which are asymptotic to $F_\pm$ at $\pm\infty$.

Let us write $f_+(x,\lambda) = g(x,\lambda)F_+(x)$, so we are looking for g which tends to 1 at ∞. We shall see that if, with $K = F_-/F_+$ as in Definition 2.4, there is a bounded solution to the integral equation

$$g(x,\lambda) = 1 - \lambda \int_x^\infty (K - K(x))F_+^2 g(\cdot,\lambda)\, w, \tag{4.1}$$

then it will have the desired properties. For $x \le t$ Proposition 2.5 shows that

$$0 \le (K(t) - K(x))F_+^2(t) \le F_-(t)F_+(t) = E^2(t),$$

so that (4.1) implies that

$$|g(x,\lambda)| \le 1 + |\lambda| \int_x^\infty |g|E^2|w|. \tag{4.2}$$

Therefore successive approximations in (4.1) starting with 0 will lead to a bounded solution (see Bennewitz [1, Chapter 1]). The convergence is uniform in x and locally so in λ, so our 'Jost solution' $f_+(x,\lambda)$ exists for all complex λ and is an entire function of λ, locally uniformly in x and real-valued for real λ. Differentiating (4.1) we obtain

$$g'(x,\lambda) = \lambda F_+(x)^{-2} \int_x^\infty F_+^2 g(\cdot,\lambda) w, \tag{4.3}$$

so $f_+' = g'F_+ + gF_+' = \lambda F_+^{-1}\int_x^\infty F_+^2 g w + gF_+'$. Differentiating again shows that f_+ satisfies (2.1). Since $F_+^2(t) = E^2(t)F_+(t)/F_-(t) \le E^2(t)F_+(x)/F_-(x)$ if $x \le t$ clearly f_+' is in L^2 near ∞, so since $g(\cdot,\lambda)$ is bounded the first term is $\mathcal{O}((F_-(x))^{-1}\int_x^\infty E^2|w|)$, and the second term is $\mathcal{O}(|F_+'|)$. By Lemma 2.3 there can be no linearly independent solution with derivative in L^2 near ∞.

Since g is bounded in fact $|f_+'|^2 + q|f_+|^2$ is integrable near ∞. Similar statements, with ∞ replaced by $-\infty$, are valid for f_-. We summarize as follows.

Lemma 4.1. *The solutions $f_\pm$ have the following properties:*

- $f_+(x,\lambda) \sim F_+(x)$ *as* $x \to \infty$ *and* $f_-(x,\lambda) \sim F_-(x)$ *as* $x \to -\infty$.
- $f_+'(x,\lambda) \to 0$ *as* $x \to \infty$ *and* $f_-'(x,\lambda) \to 0$ *as* $x \to -\infty$.
- *Any solution f of* (2.1) *for which $|f'|^2 + q|f|^2$ is integrable near ∞ is a multiple of f_+. Similarly, integrability near $-\infty$ implies that f is a multiple of f_-.*
- *λ_k is an eigenvalue precisely if $f_+(\cdot,\lambda_k)$ and $f_-(\cdot,\lambda_k)$ are linearly dependent, and all eigenfunctions with eigenvalue λ_k are multiples of $f_+(\cdot,\lambda_k)$.*

Thus λ is an eigenvalue precisely if $f_\pm(\cdot,\lambda)$ are linearly dependent, the eigenvalues are simple, and the eigenfunctions are multiples of $f_+(\cdot,\lambda)$. Clearly $f_+'(x,\lambda) \to 0$ as $x \to \infty$, but in general one can not expect that $f_+' \sim F_+'$. For $u \in \mathcal{H}$ and every eigenvalue λ_n we define the Fourier coefficients

$$u_\pm(\lambda_n) = \langle u, f_\pm(\cdot,\lambda_n)\rangle = \lambda_n \int_{\mathbb{R}} u f_\pm(\cdot,\lambda_n) w, \tag{4.4}$$

where the second equality follows from (3.1).

Applying Gronwall's inequality[3] to (4.2) gives

$$|g(x,\lambda)| \le \exp\Big(|\lambda| \int_x^\infty E^2|w|\Big),$$

$$|g'(x,\lambda)| \le E^{-2}(x)\Big(\exp\Big(|\lambda| \int_x^\infty E^2|w|\Big) - 1\Big),$$

where the second formula is easily obtained by inserting the first in (4.3). Thus $f_+(x,\cdot)$ and $f_+'(x,\cdot)$ are entire functions of exponential type $\int_x^\infty E^2|w|$ at most. This is easily sharpened to yield the following lemma.

Lemma 4.2. *As functions of λ and locally uniformly in x, the quantities $f_\pm(x,\lambda)$ and $f_\pm'(x,\lambda)$ are entire functions of zero exponential type*[4]*. In fact, $\lambda \mapsto f_+(x,\lambda)/F_+(x)$ is of zero exponential type uniformly for x in any interval bounded from below and $f_-(x,\lambda)/F_-(x)$ in any interval bounded from above. Also the Wronskian $[f_+, f_-]$ is an entire function of λ of zero exponential type.*

Proof. Consider first a solution f of (2.1) with initial data at some point a. Differentiating $H = |f'|^2 + |\lambda||f|^2$ we obtain

$$\begin{aligned} H' &= 2\,\mathrm{Re}((f'' + |\lambda| f)\overline{f'}) \\ &= 2\,\mathrm{Re}((q - \lambda w + |\lambda|) f \overline{f'}) \le \sqrt{|\lambda|}(|w| + 1 + |q|/|\lambda|)H. \end{aligned}$$

[3] A version of Gronwall's inequality valid when w is a measure may be found in [1, Lemma 1.3], and [1, Lemma 1.2] may be useful for the estimate of g'.

[4] Uniformity here means that one can for every $\varepsilon > 0$ find a constant C_ε so that the function may be estimated by $e^{\varepsilon|\lambda|}$ for $|\lambda| \ge C_\varepsilon$, independently of x.

By use of Gronwall's inequality this shows that

$$H(x) \le H(a) \exp\left(\sqrt{|\lambda|}\left|\int_a^x (|w| + 1 + |q|/|\lambda|)\right|\right)$$

where the second factor contributes a growth of order $1/2$ and type locally bounded in x.

If now the initial data of f are entire functions of λ of exponential type then so are f and f', and at most of the same type as the initial data. It follows that locally uniformly in x the functions f_+ and f_+' are entire of exponential type $\int_a^\infty E^2|w|$ for any a, and are thus of zero type. For f_+/F_+ the uniformity extends to intervals bounded from below.

Similar arguments may be carried out for f_- and f_-', which immediately implies the result for the Wronskian. □

We shall need the following definition.

Definition 4.3. Let $\mathcal{H}(a,b) = \{u \in \mathcal{H} : \operatorname{supp} u \subset [a,b]\}$.

Clearly $\mathcal{H}(a,b)$ is a closed subspace of $\mathcal{H}$.

Corollary 4.4. *For every $u \in \mathcal{H}(a,\infty)$ with $a \in \mathbb{R}$ the generalized Fourier transform $\hat{u}_+$ extends to an entire function of zero exponential type vanishing at* 0 *and defined by*

$$\hat{u}_+(\lambda) = \lambda \int_{\mathbb{R}} u f_+(\cdot,\lambda) w.$$

A similar statement is valid for $\hat{u}_-$ given any $u \in \mathcal{H}(-\infty,a)$.

5. Inverse spectral theory

We shall give a uniqueness theorem for the inverse spectral problem. In order to avoid the trivial non-uniqueness caused by the fact that translating the coefficients of the equation by an arbitrary amount does not change the spectral properties of the corresponding operator, we normalize $F_\pm$, and thus $f_\pm$, by requiring $F_+(0) = F_-(0)$. This means that $F_+(0) = F_-(0) = E(0)$.

We will need the following lemma.

Lemma 5.1. *The Wronskian $W(\lambda) = [f_-(\cdot,\lambda), f_+(\cdot,\lambda)]$ is determined by the eigenvalues of T and if λ_k is an eigenvalue, then*

$$\lambda_k W'(\lambda_k) = \langle f_-(\cdot,\lambda_k), f_+(\cdot,\lambda_k)\rangle. \tag{5.1}$$

Proof. For any x we have

$$W(\lambda) - W(\lambda_k) = [f_-(x,\lambda) - f_-(x,\lambda_k), f_+(x,\lambda) - f_+(x,\lambda_k)] \\ + [f_-(x,\lambda), f_+(x,\lambda_k)] + [f_-(x,\lambda_k), f_+(x,\lambda)]$$

since $W(\lambda_k) = 0$. Since $f_\pm(x,\cdot)$ and $f_\pm'(x,\cdot)$ are entire functions the first term is $\mathcal{O}(|\lambda-\lambda_k|^2)$ as $\lambda \to \lambda_k$.

The function $h(x) = [f_-(x,\lambda), f_+(x,\lambda_k)] \to 0$ as $x \to -\infty$ by Lemma 4.1 and since $f_\pm$ are proportional for $\lambda = \lambda_k$.

We have $h'(x) = (\lambda - \lambda_k) f_-(x, \lambda) f_+(x, \lambda_k) w$ so if w has no point mass at x,

$$\frac{[f_-(x,\lambda), f_+(x,\lambda_k)]}{\lambda - \lambda_k} \to \int_{-\infty}^{x} f_-(\cdot,\lambda_k) f_+(\cdot,\lambda_k) w$$

as $\lambda \to \lambda_k$, by Lemma 4.2. A similar calculation shows that interchanging λ and λ_k in the Wronskian the limit is the same integral, but taken over (x, ∞), so we obtain $W'(\lambda_k) = \int_{\mathbb{R}} f_-(\cdot, \lambda_k) f_+(\cdot, \lambda_k) w$. Now, if $v \in \mathcal{H}$, then

$$\langle f_-(\cdot,\lambda_k), v\rangle = \lambda_k \langle R_0 f_-(\cdot,\lambda_k), v\rangle = \lambda_k \int_{\mathbb{R}} f_-(\cdot,\lambda_k) v w,$$

so we obtain (5.1).

The zeros of the Wronskian are located precisely at the eigenvalues, and by (5.1) the zeros of the Wronskian are all simple, so that the corresponding canonical product is determined by the eigenvalues.

However, if two entire functions with the same canonical product are both of zero exponential type, then their quotient is also entire of zero exponential type according to Lemma A.1 and has no zeros. It is therefore constant. It follows that the Wronskian , which equals -1 for $\lambda = 0$, is determined by the eigenvalues. □

In addition to the eigenvalues we introduce, for each eigenvalue λ_n, the corresponding *matching constant* α_n defined by $f_+(\cdot, \lambda_n) = \alpha_n f_-(\cdot, \lambda_n)$. Together with the eigenvalues the matching constants will be our data for the inverse spectral theory. Instead of the matching constants one could use *normalization constants* $\|f_+(\cdot, \lambda_n)\|$ or $\|f_-(\cdot, \lambda_n)\|$. If λ_n is an eigenvalue, then by Lemma 5.1 the scalar product $\langle f_-(\cdot, \lambda_n), f_+(\cdot, \lambda_n)\rangle$ is determined by the Wronskian, in other words by the eigenvalues, and since

$$\langle f_-(\cdot,\lambda_n), f_+(\cdot,\lambda_n)\rangle = \alpha_n \|f_-(\cdot,\lambda_n)\|^2 = \alpha_n^{-1} \|f_+(\cdot,\lambda_n)\|^2$$

all three sets of data are equivalent if the eigenvalues are known. We therefore make the following definition.

Definition 5.2. By the *spectral data* of the operator T we mean the set of eigenvalues for T together with the corresponding matching constants and the two sets of normalization constants.

The spectral data of T are thus determined if the eigenvalues and for each eigenvalue either the matching constant or one of the normalization constants are known.

In our main result we will be concerned with two operators T and $\breve{T}$ of the type we have discussed. Connected with $\breve{T}$ there are then coefficients $\breve{q}, \breve{w}$ and solutions $\breve{F}_\pm, \breve{f}_\pm$, *etc.*

Theorem 5.3. *Suppose T and $\breve{T}$ have the same spectral data. Then there are continuous functions r, s defined on $\mathbb{R}$ such that r is strictly positive with a derivative of locally bounded variation, $s : \mathbb{R} \to \mathbb{R}$ is bijective and $s(x) = \int_0^x r^{-2}$. Moreover, $\breve{q} \circ s = r^3(-r'' + qr)$ and $\breve{w} \circ s = r^4 w$.*

Conversely, if the coefficients of T and $\breve{T}$ are connected in this way,then T and $\breve{T}$ have the same spectral data.

Given additional information one may even conclude that $T = \breve{T}$.

Corollary 5.4. *Suppose in addition to the operators T and $\breve{T}$ having the same spectral data that $\breve{q} = q$. Then $T = \breve{T}$.*

We postpone the proofs to the next section.

Remark 5.5. The spectral data of T, as we have defined them, are particularly appropriate for dealing with the Camassa-Holm equation, *i.e.* the case $q = 1/4$, since if $w = u - u_{xx}$ where u is a solution of the Camassa-Holm equation for $\varkappa = 0$, then as w evolves with time the eigenvalues are unchanged while the other spectral data evolve in the following simple way:

- $\alpha_k(t) = e^{t/2\lambda_k}\alpha_k(0)$,
- $\|f_-(\cdot,\lambda_k;t)\|^2 = e^{-t/2\lambda_k}\|f_-(\cdot,\lambda_k;0)\|^2$,
- $\|f_+(\cdot,\lambda_k;t)\|^2 = e^{t/2\lambda_k}\|f_+(\cdot,\lambda_k;0)\|^2$.

6. Proofs of Theorem 5.3 and Corollary 5.4.

We begin with the proof of the converse of Theorem 5.3, and then define $\varphi_\pm(\cdot,\lambda) = r\breve{f}_\pm(s(\cdot),\lambda)$. Using that $r^2 s' = 1$ one easily checks that $[\varphi_-,\varphi_+] = [\breve{f}_-,\breve{f}_+]$. If we can prove that $\varphi_\pm = f_\pm$ it follows that eigenvalues and matching constants agree for the two equations.

Now $\varphi_\pm(x,\lambda)/\varphi_\pm(x,0) = \breve{f}_\pm(s(x),\lambda)/\breve{F}_\pm(s(x)) \to 1$ as $x \to \pm\infty$ so we only need to prove that $\varphi_\pm$ solve the appropriate equation and that $\varphi_\pm(\cdot,0) = F_\pm$. The first property follows by an elementary computation, so it follows that $\varphi_\pm(\cdot,0) = A_\pm F_+ + B_\pm F_-$ for constants $A_\pm$ and $B_\pm$. We have

$$\frac{A_- + B_- K}{A_+ + B_+ K} = \frac{\varphi_-(\cdot,0)}{\varphi_+(\cdot,0)} = \breve{K}\circ s,$$

so the Möbius transform $t \mapsto \frac{A_- + B_- t}{A_+ + B_+ t}$ has fixpoints 0, 1 and ∞ so that $A_- = B_+ = 0$ and $B_- = A_+ \neq 0$. Thus $\varphi_\pm(\cdot,0) = AF_\pm$ for some constant A which is > 0 since $\varphi_\pm(\cdot,0)$ and $F_\pm$ are all positive. But $1 = [\breve{F}_-,\breve{F}_+] = [\varphi_-(\cdot,0),\varphi_+(\cdot,0)] = A^2$ so $A = 1$ and the proof is finished.

Keys for proving our inverse result are the connections between the support of an element of $\mathcal{H}$ and the growth of its generalized Fourier transform. Such results are usually associated with the names of Paley and Wiener. We could easily prove a theorem of Paley-Wiener type for our equation, analogous to what is done in our paper [3], but shall not quite need this.

Lemma 6.1. *Suppose $\delta > 0$, $a \in \operatorname{supp} w$ and $u \in \mathcal{H}(a,\infty)$. Then*

$$\hat{u}_+(\lambda)/\lambda f_+(a,\lambda) = \mathcal{O}(|\lambda/\operatorname{Im}\lambda|) \text{ as } \lambda\to\infty,$$

$$\hat{u}_+(\lambda)/\lambda f_+(a,\lambda) = o(1) \text{ as } \lambda\to\infty \text{ in } |\operatorname{Im}\lambda| \geq \delta|\operatorname{Re}\lambda|.$$

Similar estimates hold for $\hat{u}_-(\lambda)/\lambda f_-(a,\lambda)$ if $u \in \mathcal{H}(-\infty,a)$.

Proof. For $\operatorname{Im}\lambda \neq 0$ we have $f_+(x,\lambda) = \lambda f_+(a,\lambda)\frac{f_+(x,\lambda)}{\lambda f_+(a,\lambda)}$, where we denote the last factor by $\psi_{[a,\infty)}(x,\lambda)$, since this is the Weyl solution for the left definite Dirichlet problem (1.1) on $[a,\infty)$ (see our paper [2, Lemma 4.10]). Like in [2, Chapter 3] one may show that

$$\langle u, \overline{\psi_{[a,\infty)}(\cdot,\lambda)}\rangle = \int_{\mathbb{R}} \frac{\tilde{u}(t)}{t-\lambda}\, d\sigma(t)$$

with absolute convergence, where $\tilde{u}$ is the generalized Fourier transform of u associated with the Dirichlet problem on $[a,\infty)$ and $d\sigma$ the corresponding spectral measure. Thus

$$\hat{u}_+(\lambda) = \lambda f_+(a,\lambda) \int_{\mathbb{R}} \frac{\tilde{u}(t)}{t-\lambda}\, d\sigma(t),$$

so the statement for $\hat{u}_+$ follows by Lemma A.3. Similar calculations give the result for for $\hat{u}_-$. □

We shall also need the following lemma.

Lemma 6.2. *Suppose* $x \in \operatorname{supp} w$. *Then*

$$\frac{f_-(x,\lambda) f_+(x,\lambda)}{[f_-, f_+]} = \mathcal{O}(|\lambda/\operatorname{Im}\lambda|) \text{ as } \lambda \to \infty.$$

Proof. Let $m_\pm(\lambda) = \pm f_\pm'(x,\lambda)/(\lambda f_\pm(x,\lambda))$. These are the Titchmarsh-Weyl m-functions (see [2, Chapter 3]) for the left definite problem (2.1) with Dirichlet boundary condition at x for the intervals $[x,\infty)$ and $(-\infty,x]$ respectively, and are thus Nevanlinna functions[5]. Setting $m = -1/(m_- + m_+)$ also m is a Nevanlinna function and

$$\frac{f_-(x,\lambda) f_+(x,\lambda)}{[f_-, f_+]} = -m(\lambda)/\lambda.$$

As a Nevanlinna function m may be represented as

$$m(\lambda) = A + B\lambda + \int_{\mathbb{R}} \frac{1+t\lambda}{t-\lambda} \frac{d\rho(t)}{t^2+1},$$

where $A \in \mathbb{R}$, $B \geq 0$ and $d\rho(t)/(t^2+1)$ is a finite positive measure. Thus

$$m(\lambda)/\lambda = A/\lambda + B + \frac{1}{\lambda}\int_{\mathbb{R}} \frac{1}{t-\lambda}\frac{d\rho(t)}{t^2+1} + \int_{\mathbb{R}} \frac{1}{t-\lambda}\frac{t\, d\rho(t)}{t^2+1}.$$

The lemma therefore follows by use of Lemma A.3. □

We may expand every $u \in \mathcal{H}$ in a series $u(x) = \sum \hat{u}_\pm(\lambda_n)\frac{f_\pm(x,\lambda_n)}{\|f_\pm(\cdot,\lambda_n)\|^2}$ where $\{\hat{u}_\pm(\lambda_n)/\|f_\pm(\cdot,\lambda_n)\|\} \in \ell^2$. Conversely, any such series converges to an element of $\mathcal{H}$ and thus locally uniformly. Similarly for $\breve{u} \in \breve{\mathcal{H}}$. If the eigenvalues and normalization constants for T and $\breve{T}$ are the same we may therefore define a unitary map $\mathcal{U} : \mathcal{H} \to \breve{\mathcal{H}}$ by setting

$$\mathcal{U}u(s) = \breve{u}(s) = \sum \hat{u}_+(\lambda_n)\frac{\breve{f}_+(s,\lambda_n)}{\|\breve{f}_+(\cdot,\lambda_n)\|^2}.$$

[5]That is, functions m analytic in $\mathbb{C}\setminus\mathbb{R}$ with $\operatorname{Im}\lambda \operatorname{Im} m(\lambda) \geq 0$ and $\overline{m(\lambda)} = m(\overline{\lambda})$.

Note that expanding with respect to $\{f_-(\cdot,\lambda_n)\}$ and defining $\mathcal{U}$ by use of these expansions we obtain the same operator $\mathcal{U}$. The following proposition is an immediate consequence of the definition of $\mathcal{U}$.

Proposition 6.3. *Suppose that $\breve{u} = \mathcal{U}u$, $\breve{v} = \mathcal{U}v$, λ_k is an eigenvalue and $\hat{u}_\pm(\lambda_k) = \langle u, f_\pm(\cdot,\lambda_k)\rangle$. Then $\hat{u}_\pm(\lambda_k) = \langle \breve{u}, \breve{f}_\pm(\cdot,\lambda_k)\rangle$, $\mathcal{U}f_\pm(\cdot,\lambda_k) = \breve{f}_\pm(\cdot,\lambda_k)$ and u is in the domain of T with $Tu = v$ if and only if $\breve{u}$ is in the domain of $\breve{T}$ with $\breve{T}\breve{u} = \breve{v}$.*

Assume now that the generalized Fourier transform $\hat{u}_\pm$ of $u \in \mathcal{H}$, which is defined on all eigenvalues λ_n, has an entire extension and define the auxiliary function

$$A_\pm(u,x,\lambda) = R_\lambda u(x) + \frac{\hat{u}_\pm(\lambda) f_\mp(x,\lambda)}{\lambda[f_-(\cdot,\lambda), f_+(\cdot,\lambda)]},$$

where R_λ is the resolvent at λ of T. Similar auxiliary functions $\breve{A}_\pm$ may be defined related to $\breve{T}$.

The next lemma is crucial.

Lemma 6.4. *Suppose $x \in \operatorname{supp} w$ and $y \in \operatorname{supp} \breve{w}$. Also suppose $u \in \mathcal{H}(x,\infty)$ and $\breve{v} \in \breve{\mathcal{H}}(y,\infty)$ and let $\breve{u} = \mathcal{U}u$, $v = \mathcal{U}^{-1}\breve{v}$. Then either $\breve{u} \in \breve{\mathcal{H}}(y,\infty)$ or $v \in \mathcal{H}(x,\infty)$.*

Similarly, if $u \in \mathcal{H}(-\infty,x)$ and $\breve{v} \in \breve{\mathcal{H}}(-\infty,y)$, then $\breve{u} \in \breve{\mathcal{H}}(-\infty,y)$ or $v \in \mathcal{H}(-\infty,x)$.

Proof. By Corollary 4.4 u and $\breve{v}$ have generalized Fourier transforms $\hat{u}_+$ and $\hat{v}_+$ which have entire extensions of zero exponential type. These are also extensions of the generalized Fourier transforms of $\breve{u}$ respectively v. We have

$$A_+(v,x,\lambda) = R_\lambda v(x) + \frac{\hat{v}_+(\lambda)}{\lambda \breve{f}_+(y,\lambda)} \, \frac{\breve{f}_+(y,\lambda)}{f_+(x,\lambda)} \, \frac{f_+(x,\lambda) f_-(x,\lambda)}{[f_-, f_+]}.$$

The first term is $\mathcal{O}(\|R_\lambda v\|)$ and therefore $\mathcal{O}(|\operatorname{Im}\lambda|^{-1})$, and by Lemmas 6.1 and 6.2 respectively both the first and last factors in the second term are $\mathcal{O}(|\lambda/\operatorname{Im}\lambda|)$ as $\lambda \to \infty$ while the first factor tends to 0 in any double sector $|\operatorname{Im}\lambda| \geq \delta|\operatorname{Re}\lambda|$. Adding similar considerations for $\breve{A}_+$ we therefore obtain

$$A_+(v,x,\lambda) = (|\lambda|/|\operatorname{Im}\lambda|)^2 \mathcal{O}\Big(1 + \Big|\frac{\breve{f}_+(y,\lambda)}{f_+(x,\lambda)}\Big|\Big) \text{ as } \lambda \to \infty,$$

$$\breve{A}_+(\breve{u},y,\lambda) = (|\lambda|/|\operatorname{Im}\lambda|)^2 \mathcal{O}\Big(1 + \Big|\frac{f_+(x,\lambda)}{\breve{f}_+(y,\lambda)}\Big|\Big) \text{ as } \lambda \to \infty,$$

$$A_+(v,x,\lambda) = o\Big(1 + \Big|\frac{\breve{f}_+(y,\lambda)}{f_+(x,\lambda)}\Big|\Big) \text{ as } \lambda \to \infty \text{ in } |\operatorname{Im}\lambda| \geq \delta|\operatorname{Re}\lambda|,$$

$$\breve{A}_+(\breve{u},y,\lambda) = o\Big(1 + \Big|\frac{f_+(x,\lambda)}{\breve{f}_+(y,\lambda)}\Big|\Big) \text{ as } \lambda \to \infty \text{ in } |\operatorname{Im}\lambda| \geq \delta|\operatorname{Re}\lambda|.$$

Thus

$$\min(|A_+(v,x,\lambda)|, |\breve{A}_+(\breve{u},y,\lambda)|) = \mathcal{O}(|\lambda/\operatorname{Im}\lambda|^2) \text{ as } \lambda \to \infty,$$

$$\min(|A_+(v,x,\lambda)|, |\breve{A}_+(\breve{u},y,\lambda)|) = o(1) \text{ as } \lambda \to \infty \text{ in } |\operatorname{Im}\lambda| \geq \delta|\operatorname{Re}\lambda|,$$

By Lemma 4.2 and Theorem A.4 the functions $A_+(v,x,\cdot)$ and $\breve{A}_+(\breve{u},y,\cdot)$ are of zero exponential type, so by Lemma A.6 one of them vanishes.

If $A_+(v,x,\cdot) = 0$ Lemma A.5 shows that $A_+(v,z,\cdot) = 0$ for all $z \leq x$. Thus inserting $f(z) = A_+(v,z,\lambda)$ in $-f'' + (q - \lambda w)f$ shows that $wv = 0$ in $(-\infty, x]$, so that $v = 0$ in $(-\infty, x]$ except in gaps of $\operatorname{supp} w$. Since v vanishes at the endpoints of any gap with endpoints in $(-\infty, x]$ it follows by Corollary 3.5 that v vanishes in all such gaps. We conclude that $v \in \mathcal{H}(x,\infty)$. Similarly, if $\breve{A}_+(\breve{u},y,\cdot) = 0$ we conclude that $\breve{u} \in \breve{\mathcal{H}}(y,\infty)$.

Analogous considerations involving A_- and $\breve{A}_-$ prove the second statement. □

We next show how supports of elements of $\mathcal{H}$ are related to the supports of their images under $\mathcal{U}$. Note that $\dim\mathcal{H}$ equals the number of points in $\operatorname{supp} w$ if this is finite and is infinite otherwise.

Lemma 6.5. *Suppose* $\operatorname{supp} w$ *contains at least two points. Then so does* $\operatorname{supp}\breve{w}$ *and there are strictly increasing, bijective maps*

$$s_+ : \operatorname{supp} w \setminus \{\sup\operatorname{supp} w\} \to \operatorname{supp}\breve{w} \setminus \{\sup\operatorname{supp}\breve{w}\}$$
$$s_- : \operatorname{supp} w \setminus \{\inf\operatorname{supp} w\} \to \operatorname{supp}\breve{w} \setminus \{\inf\operatorname{supp}\breve{w}\}$$

such that $\breve{\mathcal{H}}(s_+(x),\infty) = \mathcal{U}\mathcal{H}(x,\infty)$ *and* $\breve{\mathcal{H}}(-\infty, s_-(x)) = \mathcal{U}\mathcal{H}(-\infty,x)$ *for all x in the domains of s_+ respectively s_-.*

Proof. Suppose $u \in \mathcal{H}(x,\infty)$ where $x \in \operatorname{supp} w \setminus \{\sup\operatorname{supp} w\}$. There is at least one such $u \neq 0$ (obtained by subtracting an appropriate multiple of $g_0(z,\cdot)$ from $g_0(x,\cdot)$ where $x < z \in \operatorname{supp} w$). Therefore $\breve{u} \notin \breve{\mathcal{H}}(y,\infty)$ for some $y \in \operatorname{supp}\breve{w}$. By Lemma 6.4 this means that $v \in \mathcal{H}(x,\infty)$ for every $\breve{v} \in \breve{\mathcal{H}}(y,\infty)$. Now let $s_+(x)$ be the infimum of all $y \in \operatorname{supp}\breve{w}$ for which the last statement is true.

If $s_+(x) = -\infty$ the support of $\breve{w}$ is unbounded from below so that the projection onto $\breve{\mathcal{H}}$ of a compactly supported element of $\breve{\mathcal{H}}_1$ has a support bounded from below. Such elements of $\breve{\mathcal{H}}$ are dense, and consequently $\breve{\mathcal{H}} \subset \mathcal{U}\mathcal{H}(x,\infty)$. However, this would contradict the fact that $\mathcal{U}$ is unitary. Thus $s_+(x)$ is finite, so $s_+(x) \in \operatorname{supp}\breve{w}$.

Note that if $s_+(x)$ is the left endpoint of a gap in $\operatorname{supp}\breve{w}$, then the infimum defining $s_+(x)$ is attained. Thus, if it is not there are points of $\operatorname{supp}\breve{w}$ to the right of and arbitrarily close to $s_+(x)$. But then we may approximate elements of $\breve{\mathcal{H}}(s_+(x),\infty)$ arbitrarily well (see [3, Lemma 6.8]) by elements of $\breve{\mathcal{H}}(y,\infty)$ for some $y > s_+(x)$. It follows that $\breve{\mathcal{H}}(s_+(x),\infty) \subset \mathcal{U}\mathcal{H}(x,\infty)$.

On the other hand, if $y = -\infty$ or $\operatorname{supp}\breve{w} \ni y < s_+(x)$ there exists $\breve{v} \in \breve{\mathcal{H}}(y,\infty)$ such that $\mathcal{U}^{-1}\breve{v} \notin \mathcal{H}(x,\infty)$ and thus, by Lemma 6.4, $\mathcal{U}\mathcal{H}(x,\infty) \subset \breve{\mathcal{H}}(y,\infty)$. Since this is true for all $y \in \operatorname{supp}\breve{w}$ with $y < s_+(x)$ we have in fact $\mathcal{U}\mathcal{H}(x,\infty) \subset \breve{\mathcal{H}}(s_+(x),\infty)$ unless $s_+(x)$ is the right endpoint of a gap in $\operatorname{supp}\breve{w}$. In the latter case we may choose $y \geq -\infty$ so that $(y, s_+(x))$ is a gap in $\operatorname{supp}\breve{w}$.

Thus $\breve{\mathcal{H}}(y,\infty)$ is a one-dimensional extension of $\breve{\mathcal{H}}(s_+(x),\infty)$, so if there exists $u \in \mathcal{H}(x,\infty)$ with $\operatorname{supp}\mathcal{U}u$ intersecting $(y, s_+(x))$, then $\mathcal{U}^{-1}\breve{\mathcal{H}}(y,\infty) \subset \mathcal{H}(x,\infty)$. But this would mean that $s_+(x) \leq y$. It follows that $\mathcal{U}\mathcal{H}(x,\infty) = \breve{\mathcal{H}}(s_+(x),\infty)$ in all cases.

The function s_+ has range $\operatorname{supp}\breve{w} \setminus \{\sup\operatorname{supp}\breve{w}\}$, since if not let y be in this set but not in the range of s_+. An argument analogous to that defining s_+ determines $x \in \operatorname{supp} w$ such that $\breve{\mathcal{H}}(y,\infty) = \mathcal{U}\mathcal{H}(x,\infty)$. Since x can not be in the domain of s_+ we must have $x = \sup\operatorname{supp} w$, so that $\mathcal{H}(x,\infty) = \{0\}$ and thus also $\breve{\mathcal{H}}(y,\infty) = \{0\}$. This contradicts the choice of y.

Analogous reasoning proves the existence of the function s_-. $\square$

We can now show that $\mathcal{U}$ is given by a so called *Liouville transform.*

Lemma 6.6. *There exist real-valued maps r, s defined in $\operatorname{supp} w$ such that r does not vanish and $s : \operatorname{supp} w \to \operatorname{supp}\breve{w}$ is increasing and bijective and such that $u = r\mathcal{U}u \circ s$ on $\operatorname{supp} w$ for any $u \in \mathcal{H}$.*

Proof. If $\operatorname{supp} w = \{x\}$, then $\dim\mathcal{H} = 1$ so also $\dim\breve{\mathcal{H}} = 1$. It follows that also $\operatorname{supp}\breve{w}$ is a singleton, say $\{s\}$. It is clear that $\mathcal{H}$ consists of all multiples of $g_0(x,\cdot)$ and $\breve{\mathcal{H}}$ of all multiples of $\breve{g}_0(s,\cdot)$. It follows that for all $u \in \mathcal{H}$ we have $u(x) = r\breve{u}(s)$ where $r = g_0(x,x)/\breve{g}_0(s,s)$ which proves the lemma in this case, so now assume $\operatorname{supp} w$ has at least two points.

If $x \in \operatorname{supp} w$ and $v \in \mathcal{H}$ with $v(x) = 1$ we may, given any $u \in \mathcal{H}$, write $u = u_- + u_+ + u(x)v$ where $u_- \in \mathcal{H}(-\infty,x)$ and $u_+ \in \mathcal{H}(x,\infty)$. Applying $\mathcal{U}$ we obtain from Lemma 6.5 that $\breve{u} = \breve{u}_- + \breve{u}_+ + u(x)\breve{v}$ where $\breve{u}_- \in \breve{\mathcal{H}}(-\infty, s_-(x))$ unless $x = \inf\operatorname{supp} w$ in which case $u_- = 0$ and thus $\breve{u}_- = 0$. Similarly $\breve{u}_+ \in \breve{\mathcal{H}}(s_+(x),\infty)$ unless $x = \sup\operatorname{supp} w$ in which case $u_+ = 0$ and thus $\breve{u}_+ = 0$.

If $s_\pm$ are both defined at x we can not have $s_-(x) < s_+(x)$ since then the restrictions of elements of $\breve{\mathcal{H}}$ to $(s_-(x), s_+(x))$ would be a one-dimensional set, which implies that $(s_-(x), s_+(x))$ is an unbounded gap in $\operatorname{supp}\breve{w}$, contradicting the fact that $s_\pm(x)$ are in $\operatorname{supp}\breve{w}$.

A similar reasoning but starting from $\breve{u} \in \breve{\mathcal{H}}$ and using the inverses of $s_\pm$ shows that we can not have $s_-(x) > s_+(x)$ either, so that we define $s = s_+ = s_-$ whenever one of $s_\pm$ is defined. It now follows that $\breve{u}(s(x)) = \breve{v}(s(x))u(x)$, and $\breve{v}(s(x)) \neq 0$ since not all elements of $\breve{\mathcal{H}}$ vanish at $s(x)$. We may now set $r(x) = 1/\breve{v}(s(x))$ and the proof is finished. $\square$

Since $s : \operatorname{supp} w \to \operatorname{supp}\breve{w}$ is bijective and increasing it follows that (a,b) is a gap in $\operatorname{supp} w$ if and only if $(s(a), s(b))$ is a gap in $\operatorname{supp}\breve{w}$, and similarly if $a = -\infty$ or $b = \infty$. Thus gaps in $\operatorname{supp} w$ and $\operatorname{supp}\breve{w}$ are in a one-to-one correspondence. We now need to define the functions r, s also in gaps of $\operatorname{supp} w$ and prove the other claimed properties of these functions. The key to this is the following proposition.

Proposition 6.7. *If x and y are in* $\operatorname{supp} w$*, then*

$$g_0(x,y) = r(x)r(y)\breve{g}_0(s(x),s(y)).$$

Proof. Suppose $\breve{u} \in \breve{\mathcal{H}}$ and $u = \mathcal{U}^{-1}\breve{u}$. Since $s(x) \in \operatorname{supp} \breve{w}$ it follows that $\breve{g}_0(s(x),\cdot) \in \breve{\mathcal{H}}$ and, by Lemma 6.6, $u(x) = r(x)\breve{u}(s(x))$ so that

$$\langle \breve{u}, \mathcal{U} g_0(x,\cdot)\rangle = \langle u, g_0(x,\cdot)\rangle = u(x) = r(x)\breve{u}(s(x)) = r(x)\langle \breve{u}, \breve{g}_0(s(x),\cdot)\rangle.$$

Thus $\mathcal{U} g_0(x,\cdot) = r(x)\breve{g}_0(s(x),\cdot)$. Since $y \in \operatorname{supp} w$ Lemma 6.6 also shows that $g_0(x,y) = r(y)\mathcal{U} g_0(x,\cdot)(s(y))$, and combining these formulas completes the proof. □

The proposition has the following corollary.

Corollary 6.8. *If $x \in \operatorname{supp} w$, then*

$$F_\pm(x) = r(x)\breve{F}_\pm(s(x)). \tag{6.1}$$

Proof. Suppose $x, y \in \operatorname{supp} w$ and $y \le x$. Then, by Proposition 6.7,

$$\frac{F_+(x)}{r(x)\breve{F}_+(s(x))} = \frac{r(y)\breve{F}_-(s(y))}{F_-(y)}.$$

This implies that both sides are independent of x and y and thus equal a constant C. The corollary is proved if we can prove that $C = 1$.

Now let λ be an eigenvalue of $\breve{T}$ so that $\breve{f}_+(\cdot,\lambda)$ is an eigenfunction and according to Proposition 6.3 $f_+(\cdot,\lambda)$, given by $f_+(x,\lambda) = r(x)\breve{f}_+(s(x),\lambda)$ for $x \in \operatorname{supp} w$, the corresponding eigenfunction for T. We then have

$$C\frac{f_+(x,\lambda)}{F_+(x)} = \frac{\breve{f}_+(s(x),\lambda)}{\breve{F}_+(s(x))}$$

for all $x \in \operatorname{supp} w$. If $\operatorname{supp} w$ is bounded above, choose $x = \sup\operatorname{supp} w$. Then we have $f_+(x,\lambda) = F_+(x)$ and $\breve{f}_+(s(x),\lambda) = \breve{F}_+(s(x))$ so that $C = 1$. If $\operatorname{supp} w$ is not bounded above we take a limit as $x \to \infty$ in $\operatorname{supp} w$ and arrive at the same conclusion. □

If we can extend the definitions of r and s to continuous functions such that (6.1) continues to hold for all x it follows that $u = r\mathcal{U}u \circ s$ for all $u \in \mathcal{H}$ even in gaps of $\operatorname{supp} w$. This is a consequence of two facts. Firstly, the formula $u = r\breve{u} \circ s$ then gives a bijective map of the solutions of $-u'' + qu = 0$ to the solutions of $-\breve{u}'' + \breve{q}\breve{u} = 0$ and, secondly, elements of $\mathcal{H}$ and $\breve{\mathcal{H}}$ are determined in gaps of $\operatorname{supp} w$ respectively $\operatorname{supp} \breve{w}$ as described in Corollary 3.5.

With K as in Definition 2.4 and $\breve{K}$ defined similarly we must define s so that $K = \breve{K} \circ s$, so Proposition 2.5 and the normalization of $F_\pm$ and $\breve{F}_\pm$ show that $s(0) = 0$ and we have $s = \breve{K}^{-1} \circ K$. Thus s is strictly increasing of class C^1 with range $\mathbb{R}$ and a strictly positive derivative $s' = (\breve{F}_+ \circ s/F_+)^2$, which is locally absolutely continuous. Furthermore we must define $r = F_+/\breve{F}_+ \circ s$. This gives $r > 0$ and shows that r is locally absolutely continuous with a derivative of locally bounded variation as well as $r^2 s' = 1$ so that $s(x) = \int_0^x r^{-2}$. With these definitions (6.1) holds for all x.

Differentiating $F_+ = r\breve{F}_+ \circ s$ we obtain $F_+' = rs'\breve{F}_+' \circ s + r'\breve{F}_+ \circ s = \breve{F}_+' \circ s/r + r'\breve{F}_+ \circ s$. Differentiating once more we obtain

$$qF_+ = F_+'' = s'\breve{F}_+'' \circ s/r - r'\breve{F}_+' \circ s/r^2 + r's'\breve{F}_+' \circ s + r''\breve{F}_+ \circ s$$
$$= r^{-3}\breve{q} \circ s\breve{F}_+ \circ s + r''\breve{F}_+ \circ s = r^{-4}\breve{q} \circ sF_+ + r''F_+/r.$$

It follows that

$$\breve{q} \circ s = r^3(-r'' + qr).$$

A similar calculation, using that according to Proposition 6.3 $Tu = v$ precisely if $\breve{T}\breve{u} = \breve{v}$, shows that we also have

$$\breve{w} \circ s = r^4 w.$$

This uses that the range of T is $\mathcal{H}$, so that there always are choices of v different from 0 in a neighborhood of any given point.

This completes the proof of Theorem 5.3. To prove Corollary 5.4 we need only note that if $q = \breve{q}$, then $K = \breve{K}$ so that s is the identity and $r \equiv 1$. Thus $\breve{w} = w$.

Appendix A. Some technical lemmas

We begin by quoting a standard fact.

Lemma A.1. *Suppose f, g are entire functions of zero exponential type such that f/g is entire. Then f/g is also of zero exponential type.*

The lemma is a special case of the corollary to Theorem 12 in Chapter I of Levin [14]. We shall also need the following lemma.

Lemma A.2. *Suppose f is entire and for every $\varepsilon > 0$ satisfies*

$$\operatorname{Im}(z)f(z) = \mathcal{O}(e^{\varepsilon|z|})$$

for large $|z|$. Then f is of zero exponential type.

Proof. Put $u = \log^+|f|$. Then, with $z = re^{i\theta}$,

$$0 \le u(r,\theta) \le \varepsilon r + \mathcal{O}(1) + \log(|\sin\theta|^{-1})$$

for large r. The last term is locally integrable, so we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,d\theta \le \varepsilon r + \mathcal{O}(1).$$

Now, since u is subharmonic and non-negative we have, by the Poisson integral formula,

$$0 \le u(z) \le \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\,u(re^{i\theta})\,d\theta \le \frac{3}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta$$

if $|z| \le r/2$, since then

$$0 \le \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \le \frac{r^2 - |z|^2}{(r - |z|)^2} = \frac{r + |z|}{r - |z|} \le 3.$$

It follows that $0 \le u(z) \le 6\varepsilon|z| + \mathcal{O}(1)$ if $|z| = r/2$, so $|f(z)| = \mathcal{O}(e^{6\varepsilon|z|})$ for large $|z|$. Thus f is of zero exponential type. □

Our next lemma estimates the Stieltjes transform of certain measures.

Lemma A.3. *Suppose $d\mu$ is a (signed) Lebesgue-Stieltjes measure and that $h(\lambda) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-\lambda}$ is absolutely convergent for $\operatorname{Im}\lambda \neq 0$. As $\lambda \to \infty$ we then have $h(\lambda) = \mathcal{O}(|\lambda|/|\operatorname{Im}\lambda|)$ and for any $\delta > 0$ we have $h(\lambda) = o(1)$ as $\lambda \to \infty$ in the double sector $|\operatorname{Im}\lambda| \ge \delta|\operatorname{Re}\lambda|$.*

Proof. We have

$$|h(\lambda)| \le \int_{\mathbb{R}} \left|\frac{t-i}{t-\lambda}\right| \frac{|d\mu|(t)}{|t-i|}.$$

Here the first factor may be easily estimated by $(2|\lambda|+1)/|\operatorname{Im}\lambda|$ so[6] the first statement follows. Furthermore, the first factor tends boundedly to 0 as $\lambda \to \infty$ in the sector $|\operatorname{Im}\lambda| \ge \delta|\operatorname{Re}\lambda|$, so the second statement follows. □

We now turn to the auxiliary functions of the previous section.

Theorem A.4. *If $\hat{u}_+(\lambda)/\lambda$ is entire so is $A_+(u,x,\cdot)$, and if $\hat{u}_+$ is also of zero exponential type so is $A_+(u,x,\cdot)$. Similarly for $A_-(u,x,\cdot)$, depending on properties of $\hat{u}_-$.*

Proof. Let A denote the function $A_+(u,x,\cdot)$, *i.e.*,

$$A(\lambda) = (R_\lambda u)(x) + \frac{\hat{u}_+(\lambda) f_-(x,\lambda)}{\lambda W(\lambda)}$$

where $W(\lambda) = [f_-(\cdot,\lambda), f_+(\cdot,\lambda)]$. Thus A is meromorphic with poles possible at the eigenvalues of T, which are also the zeros of W. There is no pole at 0, since this is no eigenvalue and $\hat{u}_+$ vanishes there. We have

$$R_\lambda u(x) = \sum \frac{\hat{u}_+(\lambda_n) f_+(x,\lambda_n)}{(\lambda_n - \lambda)\|f_+(\cdot,\lambda_n)\|^2},$$

so the residue at $\lambda = \lambda_n$ is

$$-\hat{u}_+(\lambda_n)\frac{f_+(x,\lambda_n)}{\|f_+(\cdot,\lambda_n)\|^2} = -\hat{u}_+(\lambda_n)\frac{f_-(x,\lambda_n)}{\langle f_-(\cdot,\lambda_n), f_+(\cdot,\lambda_n)\rangle}.$$

Since $\lambda_n W'(\lambda_n) = \langle f_-(\cdot,\lambda_n), f_+(\cdot,\lambda_n)\rangle$ by Lemma 5.1 the residues of the two terms in A cancel and A is entire.

It is also clear that $f(\lambda) = R_\lambda u(x) W(\lambda)$ is entire, and since $\operatorname{Im}(\lambda) R_\lambda$ is bounded we obtain the same growth estimates for $\operatorname{Im}(\lambda) f$ as for W. Since W is of zero exponential type, so is f by Lemma A.2. It follows that A is the quotient of two functions of zero exponential type if $\hat{u}_+$ is of zero exponential type. Thus A is itself of zero exponential type by Lemma A.1.

Similarly one proves the statements about $A_-(u,x,\cdot)$. □

We shall also need the following result.

[6] The best possible t-independent estimate is $(|\lambda+i| + |\lambda-i|)/(2|\operatorname{Im}\lambda|)$.

Lemma A.5. *Suppose* $\lambda \mapsto A_+(u, z, \lambda)$ *is an entire function of zero exponential type for every* $z \leq x$ *and that it vanishes identically for* $z = x$. *Then it vanishes identically for all* $z \leq x$.

Similarly, if $\lambda \mapsto A_-(u, z, \lambda)$ *is an entire function of zero exponential type for every* $z \geq x$ *and vanishes identically for* $z = x$, *then it vanishes identically for all* $z \geq x$.

Proof. Suppose $A_+(u, x, \cdot) = 0$. Then

$$A_+(u, z, \lambda) = R_\lambda u(z) - \psi_{(-\infty,x]}(z, \lambda)\lambda R_\lambda u(x),$$

where $\psi_{(-\infty,x]}(z, \lambda) = f_-(z, \lambda)/(\lambda f_-(x, \lambda))$ is the Weyl solution for (2.1) on $(-\infty, x]$ with a Dirichlet condition at x. This function tends to 0 as $\lambda \to \infty$ along the imaginary axis (see [2, Corollary 3.12]), while the operator λR_λ stays bounded, so it is clear that $A_+(v, z, \lambda) \to 0$ as $\lambda \to \infty$ on the imaginary axis. Since $A_+(v, z, \cdot)$ is entire of zero exponential type it follows by the theorems of Phragmén-Lindelöf and Liouville that $A_+(v, z, \cdot) = 0$.

Similar arguments apply in the case of A_-. □

The next lemma is crucial but a very slight extension of a lemma by de Branges. We shall give a full proof, however, since there is an oversight in the proof by de Branges which will be corrected below. We are not aware of the oversight being noted in the literature, but a correct proof may also be found in the Diplomarbeit of Koliander [13].

Lemma A.6. *Suppose* F_j *are entire functions of zero exponential type, and assume that for some* $\alpha \geq 0$ *we have*

$$\min(|F_1(\lambda)|, |F_2(\lambda)|) = o(|\lambda|^\alpha)$$

uniformly in $\operatorname{Re}\lambda$ *as* $|\operatorname{Im}\lambda| \to \infty$, *as well as* $\min(|F_1(i\nu)|, |F_2(i\nu)|) = o(1)$ *as* $\nu \to \pm\infty$. *Then* F_1 *or* F_2 *vanishes identically.*

This is a simple consequence of the following lemma, which is essentially de Branges' [4, Lemma 8, p. 108].

Lemma A.7. *Let* F_j *be entire functions of zero exponential type, and assume that* $\min(|F_1(z)|, |F_2(z)|) = o(1)$ *uniformly in* $\operatorname{Re} z$ *as* $|\operatorname{Im} z| \to \infty$. *Then* F_1 *or* F_2 *is identically zero.*

Proof of Lemma A.6. Suppose first that F_1 is a polynomial not identically zero. Then, by assumption, $F_2(i\nu) = o(1)$ as $\nu \to \pm\infty$, so by the theorems of Phragmén-Lindelöf and Liouville it follows that F_2 vanishes identically. Similarly if F_2 is a polynomial.

In all other cases F_1, F_2 both have infinitely many zeros, so if $n \geq \alpha$ and $z_1, \dots, z_n$ are zeros of F_1 we put $G_1(\lambda) = F_1(\lambda)/\prod_1^n(\lambda - z_j)$. Defining G_2 similarly we now have $\min(|G_1(\lambda)|, |G_2(\lambda)|) = o(1)$ uniformly in $\operatorname{Re}\lambda$ as $\operatorname{Im}\lambda \to \pm\infty$, while G_1, G_2 are still entire of zero exponential type. By Lemma A.7 it follows that G_1 or G_2 is identically zero, and the lemma follows. □

To prove Lemma A.7 we need some additional lemmas.

Lemma A.8. *Suppose F is entire of exponential type. If there is a constant C and a sequence $r_j \to \infty$ such that $|F(z)| = \mathcal{O}(1)$ as $j \to \infty$ for $|\operatorname{Im} z| \geq C$ and $|z| = r_j$, then F is constant.*

Proof. Setting $u = \log^+ |F|$ we have $u(z) = \mathcal{O}(1)$ if $|\operatorname{Im} z| \geq C$ and $|z| = r_j$. If $z = r_j e^{i\theta}$ the condition $|\operatorname{Im} z| \leq C$ means $|\sin\theta| \leq C/r_j$, and the measure of the set of $\theta \in [0, 2\pi]$ satisfying this is $\mathcal{O}(1/r_j)$ as $j \to \infty$, whereas $|F(z)| \leq e^{\mathcal{O}(|z|)}$ so that $u(r_j e^{i\theta}) = \mathcal{O}(r_j)$. Thus $\int_0^{2\pi} u(r_j e^{i\theta})\, d\theta = \mathcal{O}(1)$ as $j \to \infty$.

It follows that F is bounded, using the Poisson integral formula in much the same way as in the proof of Lemma A.2, so that F is constant. □

Next we prove a version of de Branges' Lemma 7 on p. 108 of [4], with the added assumption that $0 < p < 1$, with p as below. Without the extra assumption the lemma is not true[7]. If F is an entire function we define u as before and

$$V(r) = \int_0^{2\pi} (u(re^{i\theta}))^2\, d\theta.$$

Furthermore, let $x = \log r$ so that $u(re^{i\theta}) = u(e^{x+i\theta})$ is a continuous, subharmonic and non-negative function of (x, θ), with period 2π in θ, and put $v(x) = V(e^x)$. Let $M = \{(x, \theta) : u(e^{x+i\theta}) > 0\}$. The set M has period 2π in θ, and we define $p(x)$ so that $2\pi p(x)$ is the measure of the trace

$$M(x) = \{\theta \in [0, 2\pi) : (x, \theta) \in M\}.$$

The function p is lower semi-continuous, and we have $p(x) \leq 1$. Now assume one may choose a so that $p(x) > 0$ for $x \geq a$. Thus p is locally in $[a, \infty)$ bounded away from 0, so that $1/p$ is upper semi-continuous, positive and locally bounded. We may therefore define the strictly increasing function

$$s(x) = \int_a^x \exp\Big(\int_a^t 1/p\Big)\, dt.$$

Lemma A.9. *Suppose $0 < p(x) < 1$ for all $x \geq a$. Then the quantity v is a convex function of $s > 0$.*

Proof. We may think of u as defined on a cylindrical manifold $\mathcal{C}$ with coordinates $(x, \theta) \in \mathbb{R} \times [0, 2\pi)$ of which M is an open subset. In M the function u is harmonic, and the boundary ∂M is a level set of $|F|$. The boundary is therefore of class C^1 except where the gradient of $|F|$ vanishes. However, the length of the gradient equals $|F'|$, as is easily seen, and the exceptional points are therefore locally finite in number. We may therefore use integration by parts (the divergence theorem or the general Stokes theorem) for the set M.

Assuming $\varphi \in C_0^\infty(\mathcal{C})$ and integrating by parts we obtain

$$\int_M \Delta\varphi u^2 = \int_{\partial M} \Big(u^2 \frac{\partial\varphi}{\partial n} - 2\varphi u \frac{\partial u}{\partial n}\Big) + \int_M \varphi \Delta u^2 = 2\int_M \varphi |\operatorname{grad} u|^2,$$

[7] The original statement of de Branges is correct if one defines $p(x) = \infty$ whenever $p(x) = 1$ according to de Branges. This is not an unnatural definition, but will not help in proving his Theorem 35 nor our Lemma A.7.

since u vanishes on ∂M and is harmonic in M. Now suppose φ is independent of θ. Then we may write the above as

$$\int_{\mathbb{R}} \varphi'' v = \int_{\mathbb{R}} \varphi(x)\Big(2\int_{M(x)} (u_x^2 + u_\theta^2)\Big)\, dx,$$

so that (in the sense of distributions) $v''(x) = 2\int_{M(x)}(u_x^2 + u_\theta^2)$. A similar calculation shows that $v'(x) = \int_{M(x)} 2uu_x$.

The function s has a C^1 inverse, so we may think of x, and thus v, as a function of s. We obtain $v' = s'\frac{dv}{ds}$ and $v'' = (s')^2\frac{d^2v}{ds^2} + s''\frac{dv}{ds}$. Thus $(s')^2\frac{d^2v}{ds^2} = v'' - v's''/s' = v'' - v'/p$. We need to prove the positivity of this. Now

$$\begin{aligned} v''(x) - v'(x)/p(x) &= 2\int_{M(x)} (u_x^2 + u_\theta^2 - uu_x/p) \\ &= 2\int_{M(x)} ((u_x - u/2p)^2 + u_\theta^2 - u^2/4p^2)\, d\theta \\ &\geq 2\Big(\int_{M(x)} u_\theta^2 - \frac{1}{4p^2}\int_{M(x)} u^2\Big). \end{aligned}$$

Positivity therefore follows if we have the inequality

$$\int_{M(x)} u_\theta^2 \geq \frac{1}{4p^2(x)}\int_{M(x)} u^2. \tag{A.1}$$

Since $p(x) < 1$ the function $\theta \mapsto u(e^{x+i\theta})$ has a zero, so that u vanishes at the endpoints of all components of the open set $M(x)$. If I is such a component we therefore have $\int_I (u_\theta)^2 \geq (\pi/|I|)^2 \int_I u^2$ where $|I|$ is the length of I.

This just expresses the fact that the smallest eigenvalue of $-u'' = \lambda u$ with Dirichlet boundary conditions on I is $(\pi/|I|)^2$. We have $(\pi/|I|)^2 \geq (2p)^{-2}$ since $|I| \leq 2\pi p$, so adding up the inequalities for the various components of $M(x)$ we obtain (A.1), and the proof is finished. □

Proof of Lemma A.7. Suppose first that F_1 is bounded and therefore constant. If this constant is not zero the assumption implies that $F_2(i\nu) \to 0$ as $\nu \to \pm\infty$. Since F_2 is of zero exponential type the Phragmén-Lindelöf principle shows that F_2 is bounded and has limit zero along the imaginary axis and therefore is the constant 0. Similarly if F_2 is bounded. We may thus assume that F_1 and F_2 are both unbounded.

If there is a sequence $r_j \to \infty$ such that $F_1(z)$ satisfies the assumptions of Lemma A.8, then F_1 is constant according to Lemma A.8. Similarly for F_2.

We may thus also assume that for $k = 1, 2$ and every large r the inequality $|F_k(z)| \leq 1$ is violated for some z with $|z| = r$ and $|\operatorname{Im} z| > C$. Since F_k is analytic and thus continuous, the opposite inequalities must hold on some open θ-sets for $z = re^{i\theta}$ and every large r.

But if $|F_1(z)| > 1$ we must have $|F_2(z)| \leq 1$ for large $|z|$ and $|\operatorname{Im} z| > C$ and *vice versa*. It follows that for some a we have $0 < p_k(x) < 1$, $k = 1, 2$, for $x \geq a$.

By Cauchy-Schwarz $\frac{1}{2\pi}\int_0^{2\pi} u_1(re^{i\theta})\,d\theta \leq \left(\frac{1}{2\pi}\int_0^{2\pi} u_1^2(re^{i\theta})\,d\theta\right)^{1/2}$, so it follows that if v_1 is bounded, then so is F_1, using the Poisson integral formula in much the same way as in the proof of Lemma A.2. Thus v_1 must be unbounded, and since it is non-negative and convex as a function of s_1 there is a constant $c > 0$ such that $v_1(x) \geq cs_1(x)$ for large x. Similarly we may assume $v_2(x) \geq cs_2(x)$ for large x. We shall show that this contradicts the assumption of order for F_1, F_2.

Using the convexity of the exponential function we obtain for large $x > a$

$$\big(V_1(r(x)) + V_2(r(x))\big)/2 \geq c\int_a^x \exp\Big(\int_a^t (1/p_1 + 1/p_2)/2\Big)\,dt. \tag{A.2}$$

Now, by assumption $\min(u_1(re^{i\theta}), u_2(re^{i\theta})) = 0$ for large r and $C \leq r|\sin\theta|$ so that then u_1 or u_2 equal zero. The measure of the θ-set not satisfying $r|\sin\theta| \geq C$ for a given r is less than $2\pi C/r$. It follows that $p_1 + p_2 \leq 1 + C/r$. Since

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{p_1 + p_2}{p_1 p_2} \geq \frac{4}{p_1 + p_2} \geq \frac{4r}{r + C} = \frac{4e^x}{e^x + C}$$

the integral in (A.2) is at least $\frac{1}{2}(e^{2x} - e^{2a})/(e^a + C)^2$. Thus $V_1(r) + V_2(r) \geq c'r^2$ for some constant $c' > 0$ and large r. The assumption of order for F_k means, however, that $V_k(r) = o(r^2)$. This contradiction proves the lemma. $\square$

References

[1] Christer Bennewitz. Spectral asymptotics for Sturm-Liouville equations. *Proc. London Math. Soc. (3)*, 59(2):294–338, 1989.

[2] C. Bennewitz, B. M. Brown, and R. Weikard. Inverse spectral and scattering theory for the half-line left-definite Sturm-Liouville problem. *SIAM J. Math. Anal.*, 40(5):2105–2131, 2009.

[3] Bennewitz, C., Brown, B. M. and Weikard, R. Scattering and inverse scattering for a left-definite Sturm-Liouville problem. *J. Differential Equations 253 (2012), no. 8, 2380–2419.*

[4] Louis de Branges Hilbert spaces of entire functions Prentice-Hall, Englewood Cliffs, 1968.

[5] Camassa, Roberto and Holm, Darryl D. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett. 71 (1993), no. 11, 1661–1664.*

[6] A. Constantin and H. P. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math., 52 (1999), pp. 949–982.

[7] Constantin, A. On the scattering problem for the Camassa-Holm equation. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457 (2001), pp. 953–970.*

[8] Adrian Constantin. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. *Ann. Inst. Fourier (Grenoble)*, 50(2):321–362, 2000.

[9] Adrian Constantin and Joachim Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.*, 181(2):229–243, 1998.

[10] Eckhardt, Jonathan and Teschl, Gerald. On the isospectral problem of the dispersionless Camassa-Holm equation. *Adv. Math. 235 (2013), 469–495.*

[11] Eckhardt, Jonathan. Direct and inverse spectral theory of singular left-definite Sturm-Liouville operators. *J. Differential Equations 253 (2012), no. 2, 604634.*

[12] B. Fuchssteiner and A. S. Fokas. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D*, 4(1):47–66, 1981/82.

[13] Koliander, G. Hilbert Spaces of Entire Functions in the Hardy Space Setting. *Diplomarbeit, Technische Universität Wien*, 2011.

[14] B. Ja. Levin. *Distribution of zeros of entire functions*, volume 5 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., revised edition, 1980. Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman.

[15] Z. Jiang, L. Ni, and Y. Zhou. *Wave breaking of the Camassa-Holm equation*, J. Nonlinear Sci., 22(2): 235–245, 2012.

C. Bennewitz
Centre for Mathematical Sciences
Box 118
221 00 Lund
Sweden
e-mail: `christer.bennewitz@telia.com`

B. M. Brown
School of Computer Science
Cardiff University, Cardiff
CF24 3XF, U.K.
e-mail: `malcolm@cs.cf.ac.uk`

R. Weikard
Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170, USA.
e-mail: `rudi@math.uab.edu`