SCATTERING AND INVERSE SCATTERING FOR A LEFT-DEFINITE STURM-LIOUVILLE PROBLEM

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ABSTRACT. This work develops a scattering and an inverse scattering theory for the Sturm-Liouville equation $-u'' + qu = \lambda wu$ where w may change sign but $q \geq 0$. Thus the left-hand-side of the equation gives rise to a positive quadratic form and one is led to a left-definite spectral problem. The crucial ingredient of the approach is a generalized transform built on the Jost solutions of the problem and hence termed the Jost transform and the associated Paley-Wiener theorem linking growth properties of transforms with support properties of functions.

One motivation for this investigation comes from the Camassa-Holm equation for which the solution of the Cauchy problem can be achieved by the inverse scattering transform for $-u'' + \frac{1}{4}u = \lambda wu$.

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1. Introduction

In this paper we will consider the direct and inverse scattering problem for the differential equation

$$-u'' + qu = \lambda wu, \tag{1.1}$$

in the case when w is not required to have a fixed sign. Instead we assume $q \geq 0$. This situation is known as the left definite case.

An important motivation for considering this is the study of the spectral problem associated with the Camassa-Holm equation. This is the equation

$$-u'' + \tfrac{1}{4}u = \lambda wu, \tag{1.2}$$

where $w = \psi_{xx} - \psi + \kappa$, κ is a constant and ψ satisfies the Camassa-Holm equation (7.1). The Camassa–Holm equation is an integrable system in a similar sense as the Korteweg–de Vries (KdV) equation. It was first derived as an abstract bi-Hamiltonian system by Fuchssteiner and Fokas [23]. Subsequently, it was shown by Camassa and Holm [11] that it may serve as an integrable model for shallow water waves.

In contrast to the KdV equation the Camassa–Holm equation can model breaking waves, *i.e.*, smooth initial data may develop singularities in finite time; cf. Constantin and Escher [17] and Constantin [15]. This, however, happens only when the initial w is not of fixed sign, and it is this fact which motivates us to consider (1.1) without the assumption that w is positive.

The well developed theory of scattering and inverse scattering for the Schrödinger equation is of crucial importance to the theory of the KdV equation. In the same way scattering/inverse scattering theory for (1.2) is important for dealing with the Camassa–Holm equation.

The problem of inverse scattering for (1.2) is considerably more difficult than for the Schrödinger equation, which may be viewed as a rather mild perturbation of the equation $-u'' = \lambda u$. In case of (1.2) one deals with a perturbation of the equation $-u'' + \frac{1}{4}u = \lambda u$, which changes the coefficient containing the eigenvalue parameter λ. It appears that the methods used so far for dealing with the Schrödinger equation are no longer applicable.

One may treat the scattering/inverse scattering problem for (1.2) by transforming the equation to the Schrödinger equation and using the known theory for this, see Constantin [16] and Constantin and Lenells [19]. However, such a transformation requires considerable smoothness of w and, which is more serious, requires $w > 0$.

From the physical point of view the full-line case where w decays at infinity and the periodic case are most interesting. The former was treated by Fokas [22] and Constantin and various co-authors, for example in [16], [18], and [19]. The latter was addressed by Constantin and McKean [13], Constantin [14], and Vaninsky [30].

We will be interested in the full line problem where the solution ψ decays at infinity. If $\kappa \neq 0$ one may, after appropriate scaling, assume $\kappa = 1$ which we will from now on. The so called zero dispersion limit where $\kappa = 0$ may be treated by similar methods, but we refer to the recent paper by Eckhardt [21] which deals with this case by methods involving de Branges spaces, also crucially using the fact that in this situation the spectrum is discrete. In (1.1) we will thus assume that $w - 1$ decays at infinity, which is reflected in Assumption 1.2.

The key to a spectral theory with w of indefinite sign is to note that even if w changes sign, the quadratic form $\int_{\mathbb{R}}(|u'|^2 + \frac{1}{4}|u|^2)$ associated with the left hand side of (1.2) is positive and well adapted to serve as a norm-square of a Hilbert space in which to treat (1.2). Problems of this nature have a long history and their study seems to have been initiated by Weyl [31], who called such problems *polar*.

Later many authors have dealt with more or less general left-definite problems. In particular we mention a series of papers by Niessen, Schneider, and their collaborators on singular left-definite so-called S-hermitian systems; see, *e.g.*, [28], and Bennewitz [4] and the references cited there. For a more recent contribution, see Kong, Wu, and Zettl [25]. However, papers in inverse spectral theory for left-definite problems are much more scarce; one example is Binding, Browne, and Watson [10].

Closer to our present purpose, but dealing with a half-line problem, is our paper [3], where a theory modeled on standard Titchmarsh-Weyl theory is given. This may be extended to a full line theory in standard fashion, but this will not serve our purpose since such a theory does not interact smoothly with scattering theory. We will therefore in this paper construct a spectral theory closely associated with scattering theory. Our main results are uniqueness theorems for inverse scattering for the equation (1.1) and some closely related results given in Section 6.1.

The plan of the paper is as follows. After introducing the basic assumptions for the paper and stating some auxiliary results needed later, we define in Section 2 a Hilbert space and a selfadjoint operator in this which is a realization of (1.1). We discuss properties of the operator and the underlying Hilbert space. In Section 3 we discuss the direct scattering process for (1.1) and introduce appropriate Jost solutions and a scattering matrix. Of particular importance is the high energy asymptotics for the Jost solutions which we give in Theorem 3.3. We end the section by preparing the introduction of a generalized Fourier transform which we call the *Jost transform* associated with (1.1).

The Jost transform is defined in Section 4 where essential properties are discussed and a full spectral theory based on the Jost transform is given. Our approach to inverse spectral and scattering theory is based on a generalized form of the classical Paley-Wiener theorem (see also [3] and Bennewitz [7]) which is valid for the Jost transform, and we deal with this in Section 5.

Our uniqueness theorem for inverse scattering, which is at the same time an inverse spectral theorem, is given in Section 6 where we also discuss similarities and differences to the standard uniqueness theorem for inverse scattering for the Schrödinger equation. Finally, in Section 7 we give the time-evolution of the scattering data and thereby exhibit how the inverse scattering is used to solve the Cauchy problem of the Camassa-Holm equation. An appendix deals with several technicalities from analytic function theory, needed primarily for the proof of the Paley-Wiener theorem. Some of the results and techniques in the appendix we have not been able to find in the literature and may be new.

1.1. **Basic assumptions.**

Minimal requirements on the coefficients q and w in (1.1) are given in the following assumption which will be in force throughout the entire paper even when it is not explicitly mentioned.

Assumption 1.1. The coefficients q and w are locally integrable, real-valued functions on $\mathbb{R}$. The function q is nonnegative and both q and w are supported on sets of positive measure.

This assumption will be enough for the basic spectral theory given in Section 2, but from Section 3 onwards, and in order to be able to discuss scattering, we will need the following more restrictive assumption.

Assumption 1.2. The coefficients q and w satisfy Assumption 1.1. In addition it is assumed that there is a constant $q_0 \geq 0$ such that $q - q_0$ and $w - 1$ are in $L^1(\mathbb{R})$, For inverse scattering we shall need a further restriction, as given in the following assumption.

Assumption 1.3. The coefficients q and w satisfy Assumption 1.2. In addition the first moment $\int_{\mathbb{R}} |x(q(x) - q_0 w(x))|dx$ of $q - q_0 w$ is finite.

This is assumed from Section 5 onwards. It is used earlier occasionally, but then always explicitly mentioned. The assumption is only essential at one point, when proving, in Theorem A.2, analyticity properties at q_0 of the function (5.1).

Note that, if $1/p > 0$ is locally integrable, the change of variables $t = \int_0^x 1/p$ turns the equation $-(pu')'+qu = \lambda wu$ into $-v''+Qv = \lambda W v$ where $v(t) = u(x(t))$, $Q(t) = p(x(t))q(x(t))$ and $W(t) = p(x(t))w(x(t))$. Therefore, our results concerning equation (1.1) pertain actually to a more general class of equations of the form $-(pu')' + qu = \lambda wu$.

We use the following notation in the paper: We denote the open upper half-plane by $\mathbb{C}_+$ and if $\omega \subset \mathbb{R}$ we denote the characteristic function of ω by χ_ω. Furthermore, $[f, g] = fg' - f'g$ denotes the Wronskian of f and g. Recall that if f and g solve the same homogeneous equation $u'' = hu$, then $[f, g]$ is constant.

Sometimes in the sequel we will encounter so called Nevanlinna functions. These are functions N analytic in $\mathbb{C} \setminus \mathbb{R}$ with $\operatorname{Im}\lambda \operatorname{Im} N(\lambda) \geq 0$ and $N(\overline{\lambda}) = \overline{N(\lambda)}$. The typical example is the Titchmarsh-Weyl m-function, crucial in the spectral theory of half-line Sturm-Liouville equations. The fundamental fact about Nevanlinna functions is as follows (see, *e.g.*, Akhiezer and Glazman [2]).

Lemma 1.4. *For any Nevanlinna function N there are uniquely determined constants $A \in \mathbb{R}$ and $B \geq 0$ and a uniquely determined positive measure $d\rho$ with $\int_{\mathbb{R}}(1 + t^2)^{-1}\, d\rho(t) < \infty$ such that*

$$N(\lambda) = A + B\lambda + \int_{-\infty}^{\infty} \Big(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\Big)\, d\rho(t) = A + B\lambda + \int_{-\infty}^{\infty} \Big(\lambda + \frac{1+\lambda^2}{t-\lambda}\Big)\frac{d\rho(t)}{1+t^2}.$$

We will primarily need the following fact about Nevanlinna functions.

Proposition 1.5. *If N is Nevanlinna the function $\lambda \mapsto (\operatorname{Im}\lambda)\, N(\lambda)$ is bounded by a second order polynomial in $|\lambda|$ for all non-real λ.*

In particular, $N(\lambda)/\lambda$ is bounded as $\lambda \to \infty$ in sets where $|\lambda|/\operatorname{Im}\lambda$ is bounded and $(\operatorname{Im}\lambda)N(\lambda)$ is bounded as λ approaches a real value.

Proof. This follows from the (second) representation formula of Lemma 1.4 since $|\operatorname{Im}\lambda/(t - \lambda)| \leq 1$ if $t \in \mathbb{R}$. □

We shall also use the following well known consequence of the reflection principle (see Ahlfors [1]).

Proposition 1.6. *Suppose f is analytic in a connected domain $\Omega \subset \mathbb{C}_+$, with a non-empty open real set $\omega \subset \partial\Omega \cap \mathbb{R}$.*

(1) *If $\operatorname{Re} f$ extends continuously to ω and vanishes there, then f extends analytically to $\Omega \cup \Omega^* \cup \omega$, where $\Omega^* = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$.*
(2) *If f extends continuously to ω, then f is uniquely determined by its values on ω.*

2. Definition of operator and resolvent

2.1. A Hilbert space.

We introduce the set $\mathcal{H}_1$ of locally absolutely continuous functions on $\mathbb{R}$ for which both $|u'|^2$ and $q|u|^2$ are integrable. The form

$$\langle u, v\rangle = \int_{\mathbb{R}} (u'\overline{v'} + qu\overline{v})$$

is an inner product on $\mathcal{H}_1$ and $\|\cdot\|$ is the associated norm. In fact $\mathcal{H}_1$ is complete in the metric induced by $\langle\cdot,\cdot\rangle$ and hence a Hilbert space. The proof of completeness is analogous to the corresponding one presented in our previous paper [3] and depends on the following lemma which will also be used on other occasions in the sequel.

Lemma 2.1. *For any $u \in \mathcal{H}_1$ we have $u(x) = o(\sqrt{|x|})$ as $x \to \pm\infty$. Moreover, if I is a bounded interval there is a constant C_I such that $|u(x)| \le C_I\|u\|$ for $x \in I$.*

Finally, if there is a constant $q_0 > 0$ such that the negative part of $q - q_0$ is in $L^1(\mathbb{R})$, then there is even a constant $C_{\mathbb{R}}$ such that $|u(x)| \le C_{\mathbb{R}}\|u\|$ for all $x \in \mathbb{R}$.

Proof. The proof uses the identity $u(x) = u(y) + \int_y^x u'$. Cauchy's inequality shows that

$$|u(x)| \le |u(y)| + \left|(x-y)\int_y^x |u'|^2\right|^{1/2}. \tag{2.1}$$

This gives $\overline{\lim}_{x\to\infty} |u(x)|/\sqrt{x} \le \left(\int_y^\infty |u'|^2\right)^{1/2}$, which is arbitrarily small for large y. Treating the case $x \to -\infty$ similarly we obtain the first statement.

For the second claim assume $x \in I$ and choose the bounded interval $J \supset I$ so that $\int_J q > 0$. Multiplying (2.1) by $q(y)$ and integrating with respect to y over J we obtain, again using Cauchy's inequality, that

$$|u(x)|\int_J q \le \|\sqrt{q}\,u\|_2\sqrt{\int_J q}+\sqrt{|J|}\|u'\|_2\int_J q,$$

where $\|\cdot\|_2$ is the norm of $L^2(\mathbb{R})$ and $|J|$ is the length of J. Dividing by $\int_J q$ we now obtain the second claim with $C_I=\sqrt{|J|+1/\int_J q}$.

For the final claim, note that the additional hypothesis implies the existence of a constant B such that $Bq_0\ge 1+\int_{\mathbb{R}}(q-q_0)_-$, where the minus sign denotes negative part. Thus $\int_J q\ge 1$ as soon as $|J|\ge B$ so that the final claim follows with $C_{\mathbb{R}}=\sqrt{1+B}$. □

Remark. If $q-q_0\in L^1(\mathbb{R})$ for some constant $q_0>0$ we obtain from the lemma that

$$\Big|\int_{\mathbb{R}}(q-q_0)|u|^2\Big|\le C_{\mathbb{R}}^2\int_{\mathbb{R}}|q-q_0|\,\|u\|^2.$$

Since the lemma in particular shows that point evaluations are uniformly bounded linear forms in $H^1(\mathbb{R})$ we have a similar inequality for $u\in H^1(\mathbb{R})$ with $\|u\|$ replaced by the norm of $H^1(\mathbb{R})$. Because

$$\int_{\mathbb{R}}(|u'|^2+q|u|^2)=\int_{\mathbb{R}}(q-q_0)|u|^2+\int_{\mathbb{R}}(|u'|^2+q_0|u|^2)$$

it follows that replacing q by q_0 gives a norm which is equivalent to the norm of $H^1(\mathbb{R})$, and also equivalent to $\|\cdot\|$. Thus in this case $\mathcal{H}_1=H^1(\mathbb{R})$ as equivalently normed spaces. In particular, we even have $u(x)=o(1)$ as $x\to\pm\infty$ in this case.

A crucial property of our Hilbert space is that compactly supported elements are dense. It will be useful later on to have an explicit construction of compactly supported approximations of elements in $\mathcal{H}_1$. To this end we define functions φ_n, continuous with support $[-2n,2n]$, identically equal to 1 in $[-n,n]$, and linear in $[-2n,-n]$ and $[n,2n]$.

Lemma 2.2. *If $u\in\mathcal{H}_1$ then so is $\varphi_j u$ and $\|u-\varphi_j u\|\to 0$ as $j\to\infty$.*

Proof. We must show that $(u-\varphi_j u)'\to 0$ in $L^2(\mathbb{R})$ and $(u-\varphi_j u)^2 q\to 0$ in $L^1(\mathbb{R})$. The second statement is an immediate consequence of the dominated convergence theorem. To prove the first statement we have

$$((1-\varphi_j)u)'=(1-\varphi_j)u'-\varphi_j' u,$$

where the first term tends to 0 in L^2, again by the dominated convergence theorem, and the second term is zero except when $j<|x|<2j$. According to Lemma 2.1 $\varphi_j' u=\pm u/j=o(1/\sqrt{j})$ in these intervals. Thus its L^2-norm is $o(1)$. □

The following simple proposition will occasionally be useful.

Proposition 2.3. *Suppose q is locally integrable and non-negative. Then a non-trivial solution of $-f'' + qf = 0$ which has a zero can have no other zeros, nor can its derivative vanish anywhere.*

Proof. Suppose $f(a) = 0$ and that either f or f' vanishes in $b \neq a$. An integration by parts then shows that

$$\int_a^b (|f'|^2 + q|f|^2) = 0.$$

Since $q \geq 0$ this shows that f is constant between a and b, and thus vanishes there, and hence everywhere. $\square$

2.2. **A selfadjoint relation.**

We are interested in investigating equations of the form

$$-u'' + qu = wg \tag{2.2}$$

where (u, g) are pairs in a certain subspace T_1 of $\mathcal{H}_1 \oplus \mathcal{H}_1$. Suppose now that $u, g, v \in \mathcal{H}_1$. If v is compactly supported, u' is locally absolutely continuous, and (u, g) satisfies (2.2) an integration by parts yields

$$\langle u, v \rangle = \int_{\mathbb{R}} wg\overline{v}.$$

Therefore we investigate the functional $u \mapsto \int_{\mathbb{R}} u\overline{v}$ on $\mathcal{H}_1$ defined for any fixed function v in the set of compactly supported functions in $L^1(\mathbb{R})$, which we denote by L_0. Using Lemma 2.1 one shows that this functional is, in fact, continuous[1]. Thus, by Riesz' representation theorem, there exists, for any such v, a $v^* \in \mathcal{H}_1$ so that $\int_{\mathbb{R}} u\overline{v} = \langle u, v^* \rangle$. The relationship between v and v^* is linear, *i.e.*, there exists an operator $\mathcal{G} : L_0 \to \mathcal{H}_1$ such that $\mathcal{G}v = v^*$ and $\langle u, \mathcal{G}v \rangle = \int_{\mathbb{R}} u\overline{v}$.

Since, for any $x \in \mathbb{R}$, the map $u \mapsto u(x)$ is a bounded linear functional on $\mathcal{H}_1$ there is a function $g_0(x, \cdot) \in \mathcal{H}_1$ such that

$$u(x) = \langle u, \overline{g_0(x, \cdot)} \rangle.$$

Hence

$$(\mathcal{G}v)(x) = \langle \mathcal{G}v, \overline{g_0(x, \cdot)} \rangle = \int_{\mathbb{R}} vg_0(x, \cdot).$$

We can now define the set T_1 mentioned above.

$$T_1 = \{(u, g) \in \mathcal{H}_1 \oplus \mathcal{H}_1 : \langle u, v \rangle = \langle g, \mathcal{G}(wv) \rangle \text{ for all } v \in L_0 \cap \mathcal{H}_1\}.$$

[1]It is clear that the functional is bounded also if v is a compactly supported measure, or even a compactly supported element of $H^{-1}(\mathbb{R})$ since $\mathcal{H}_1 \subset H^1_{\mathrm{loc}}(\mathbb{R})$.

We remind the reader about the following facts on linear relations (see, *e.g.*, [4]). A (closed) linear subset E of $\mathcal{H}_1 \oplus \mathcal{H}_1$ is called a (closed) linear relation on $\mathcal{H}_1$. The adjoint E^* of E is defined as

$$E^* = \{(u^*, v^*) \in \mathcal{H}_1 \oplus \mathcal{H}_1 : \langle u^*, v\rangle = \langle v^*, u\rangle \text{ for all } (u, v) \in E\}.$$

E is called symmetric if $E \subset E^*$ and self-adjoint if $E = E^*$. The following facts hold: E^* is closed, E^{**} is the closure of E, and $E \subset F$ implies $F^* \subset E^*$.

Thus we see that T_1 is the adjoint of the relation

$$T_c = \{(\mathcal{G}(wv), v) : v \in L_0 \cap \mathcal{H}_1\},$$

which clearly is symmetric if w is realvalued.

One can now show that T_1 is a differential relation. More precisely, the following statement holds.

Proposition 2.4. *We have $(u, g) \in T_1$ if and only if u and $g \in \mathcal{H}_1$, u' is locally absolutely continuous, and $-u'' + qu = wg$.*

Proof. If $(u, g) \in T_1$, then for any $\varphi \in C_0^1(\mathbb{R})$ we have

$$\int_{\mathbb{R}} (u'\overline{\varphi'} + qu\overline{\varphi}) = \int_{\mathbb{R}} wg\overline{\varphi}$$

which after an integration by parts gives

$$\int_{\mathbb{R}} \Big(u'(x) - \int_a^x (qu - wg)\Big)\overline{\varphi'} = 0.$$

It follows from du Bois Reymond's lemma that $u'(x) - \int_a^x (qu - wg)$ is constant. Thus u' is locally absolutely continuous and differentiation gives $-u'' + qu = wg$.

The converse is an immediate consequence of integration by parts. □

Proposition 2.5. *The dimension of $\mathcal{D}_\lambda = \{(u, \lambda u) \in T_1\}$ is zero whenever $\operatorname{Im}\lambda \neq 0$ or $\lambda = 0$.*

Proof. The definition of T_1 shows that $(u, 0) \in T_1$ if and only if $\langle u, v\rangle = 0$ for all compactly supported elements $v \in \mathcal{H}_1$. Since these elements are dense in $\mathcal{H}_1$ by Lemma 2.2 it follows that we must have $u = 0$.

Now suppose $(u, \lambda u) \in T_1$ and let φ_j be the functions introduced in Lemma 2.2. We then obtain

$$\langle \varphi_j u, u\rangle = \langle \mathcal{G}(w\varphi_j u), \lambda u\rangle = \overline{\lambda} \int_{\mathbb{R}} w\varphi_j |u|^2.$$

Multiplying by λ and taking imaginary part we obtain $\operatorname{Im}\big(\lambda\langle \varphi_j u, u\rangle\big) = 0$. Now $\varphi_j u \to u$ as $j \to \infty$ so we obtain $\operatorname{Im}\lambda \|u\|^2 = 0$. Thus $u = 0$. □

Corollary 2.6. *T_1 is a self-adjoint relation.*

Proof. Let $T_0 = T_1^*$ be the closure of T_c. Theorem 1.4 in [4] established a simple generalization of the von Neumann formula for symmetric operators, *i.e.*, $T_1 = T_0 \dotplus \mathcal{D}_\lambda \dotplus \mathcal{D}_{\overline{\lambda}}$ as a direct sum, for any non-real λ. Since $\mathcal{D}_\lambda = \{0\}$ we have $T_1 = T_1^*$. □

Let $\mathcal{H}_+$ be the set of locally absolutely continuous functions u for which $|u'|^2 + q|u|^2$ is integrable over any interval bounded from below, and $\mathcal{H}_-$ the set for which this expression is integrable on any interval bounded from above. Of course $\mathcal{H}_+ \cap \mathcal{H}_- = \mathcal{H}_1$. If u solves $-u'' + qu = \lambda wu$ for some $\lambda \in \mathbb{C}$, then $|u'|^2 + q|u|^2$ is always locally integrable, so in this case belonging to $\mathcal{H}_\pm$ is just a restriction at ∞ respectively $-\infty$.

Proposition 2.7. *Suppose* $\operatorname{Im} \lambda \neq 0$ *or* $\lambda = 0$*. Then the set of solutions of equation* (1.1) *in* $\mathcal{H}_+$ *as well as the set of solutions in* $\mathcal{H}_-$ *is one-dimensional.*

This is a special case of [5, Theorem 2.3], see also [8].

2.3. **Operator and resolvent.**

We now associate with the self-adjoint relation T_1 a self-adjoint operator T defined in a subspace $\mathcal{H}$ of $\mathcal{H}_1$ and study its spectral theory following Bennewitz [4]. Thus we define $\mathcal{H}_\infty = \{g \in \mathcal{H}_1 : (0, g) \in T_1\}$, which is clearly a closed subspace of $\mathcal{H}_1$, and $\mathcal{H} = \mathcal{H}_1 \ominus \mathcal{H}_\infty$. We denote the orthogonal projections of $\mathcal{H}_1$ onto $\mathcal{H}$ and $\mathcal{H}_\infty$ by $E_\mathbb{R}$ and E_∞ respectively. Now, if $(u, g) \in T_1$ then $u \in \mathcal{H}$ and $(u, E_\mathbb{R} g) = (u, g) - (0, E_\infty g) \in T_1$. Therefore we define the domain of T as $D_T = \{u \in \mathcal{H}_1 : \exists g \in \mathcal{H}_1 : (u, g) \in T_1\}$, and $Tu = E_\mathbb{R} g$ if $(u, g) \in T_1$. By Lemma 1.14 and Theorem 1.15 of [4] the domain D_T is a dense linear subset of $\mathcal{H}$ and

$$T_1 \cap (\mathcal{H} \oplus \mathcal{H}) = \{(u, E_\mathbb{R} g) : (u, g) \in T_1\}$$

is the graph of the self-adjoint operator T in $\mathcal{H}$.

We may now apply the spectral theorem to T. Denote the elements of the resolution of the identity for the operator T by E_ω. We extend the domain of the projection E_ω to all of $\mathcal{H}_1$ by setting $E_\omega \mathcal{H}_\infty = 0$. In the present case it is an immediate consequence of Proposition 2.4 that the space $\mathcal{H}_\infty$ consists of those elements $g \in \mathcal{H}_1$ for which $wg = 0$ almost everywhere. We have assumed that w is not identically equal to zero in order to avoid the trivial case in which $\mathcal{H}_\infty = \mathcal{H}_1$ and $\mathcal{H} = \{0\}$. On the other hand, if w is supported everywhere, then $\mathcal{H} = \mathcal{H}_1$.

Let R_λ be the resolvent of $T : \mathcal{H} \to \mathcal{H}$ and extend the domain of R_λ to all of $\mathcal{H}_1$ by setting $R_\lambda \mathcal{H}_\infty = 0$. It is easily verified that if $g \in \mathcal{H}_1$, then $R_\lambda g = u$ precisely if $(u, \lambda u + g) \in T_1$ and that the extended resolvent, although no longer injective, still has the fundamental properties $(R_\lambda)^* = R_{\overline{\lambda}}$, $\|R_\lambda\| \le 1/|\operatorname{Im}\lambda|$ and $R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu$.

Assume $\operatorname{Im}\lambda \neq 0$ or $\lambda = 0$. Let $\psi_\pm(\cdot,\lambda)$ be nontrivial solutions of equation (1.1) so that $\psi_+(\cdot,\lambda)$ is in $\mathcal{H}_+$ and $\psi_-(\cdot,\lambda)$ is in $\mathcal{H}_-$ (cf. Proposition 2.7). Recall that the Wronskian $[u,v] = uv' - u'v$ of solutions u and v to (1.1) is independent of x. Then define

$$g(x,y,\lambda) = \frac{\psi_-(\min(x,y),\lambda)\psi_+(\max(x,y),\lambda)}{[\psi_+(\cdot,\lambda),\psi_-(\cdot,\lambda)]}. \tag{2.3}$$

Note that the Wronskian only vanishes on eigenvalues and that g does not depend on the particular choice made for $\psi_\pm$.

Proposition 2.8. *The kernel $g_0(x,y)$ of the evaluation operator equals $g(x,y,0)$ and is real for all x and y.*

Proof. Both real and imaginary parts of $\psi_+(\cdot,0)$ are solutions in $\mathcal{H}_+$ of (1.1) for $\lambda = 0$, and both can not be trivial. We may therefore assume that $\psi_+(\cdot,0)$, and similarly $\psi_-(\cdot,0)$, are realvalued. Thus also $g(x,y,0)$ is realvalued.

A straightforward computation now shows that $g(\cdot,\cdot,0)$ has the property that $u(x) = \langle u, \overline{g(x,\cdot,0)}\rangle = \langle u, g(x,\cdot,0)\rangle$ if $u \in \mathcal{H}_1$ is compactly supported. The density of such functions gives the desired conclusion. $\square$

Theorem 2.9. *When $\operatorname{Im}\lambda \neq 0$ the resolvent R_λ of T is given by*

$$(R_\lambda u)(x) = \langle u, \overline{G(x,\cdot,\lambda)}\rangle = \langle u, \overline{g(x,\cdot,\lambda)}\rangle/\lambda - u(x)/\lambda$$

where $G(x,\cdot,\lambda) = (g(x,\cdot,\lambda) - g(x,\cdot,0))/\lambda$ equals $R_\lambda g(x,\cdot,0)$, so that $g(x,\cdot,\lambda) = \lambda R_\lambda g(x,\cdot,0) + g(x,\cdot,0)$.

Proof. Fix $x \in \mathbb{R}$ and $\lambda \notin \mathbb{R}$ and let $F = G(x,\cdot,\lambda)$. Then F is continuous at x. It is clearly in $\mathcal{H}_+ \cap \mathcal{H}_- = \mathcal{H}_1$. Next, using that G is constructed as a difference, one checks that F' is locally absolutely continuous and that it satisfies $-F'' + qF = w(\lambda F + g(x,\cdot,0))$. Hence $(F, \lambda F + g(x,\cdot,0)) \in T_1$, or

$$G(x,\cdot,\lambda) = R_\lambda g(x,\cdot,0).$$

Since $g(x,y,0)$, q and w are realvalued, also $(\overline{F}, \overline{\lambda F} + g(x,\cdot,0)) \in T_1$, *i.e.*, $\overline{G(x,\cdot,\lambda)} = R_{\overline{\lambda}} g(x,\cdot,0)$. The proof is finished upon noticing that $(R_\lambda u)(x) = \langle R_\lambda u, g(x,\cdot,0)\rangle = \langle u, R_{\overline{\lambda}} g(x,\cdot,0)\rangle$ since the adjoint of R_λ is $R_{\overline{\lambda}}$. $\square$

3. Direct scattering

In this section we discuss the direct scattering process associated with the operator T. The results presented in Sections 3.1, 3.3 are completely analogous (with similar proofs) to corresponding results for the one-dimensional Schrödinger equation. We lay them out in order to provide easy reference and set notation. In Section 3.2 we discuss the growth of the Jost solutions as functions of k. This growth is considerably more complicated when w is allowed to deviate from 1 as compared to the case where $w = 1$. We believe the results in Section 3.2 to be new. In Section 3.4 we discuss the eigenvalues of T. Finally, in Section 3.5 we treat the connection between spectral measures and transmission coefficients. Deift and Trubowitz allude to this connection for the Schrödinger equation in [20] but, to the best of our knowledge, it has not been expounded in the literature yet.

3.1. **Jost solutions.**

In one-dimensional scattering theory the Jost solutions $f_\pm$ are fundamental. Their definition and existence is the subject of the following lemma.

Lemma 3.1. *Suppose $q - q_0$ and $w - 1$ are integrable. If k is in $\overline{\mathbb{C}_+} \setminus \{0\}$, there exist solutions $f_\pm(\cdot, k)$ of (1.1) with $\lambda = k^2 + q_0$ having the following properties:*

1. *$f_\pm(x, k) \sim e^{\pm ikx}$ and $f_\pm'(x, k) \sim \pm ike^{\pm ikx}$ as x tends to $\pm\infty$,*
2. *$f_\pm(x, \cdot)$, $f_\pm'(x, \cdot)$ are analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+} \setminus \{0\}$.*

If in addition the first moment of $q - q_0 w$ is finite, then (1) and the continuity in (2) hold for $k \in \overline{\mathbb{C}_+}$.

Proof. This is standard (see, for instance, Deift and Trubowitz [20]). One introduces $g_+(x, k) = f_+(x, k)e^{-ikx}$ which satisfies the differential equation $g_+'' + 2ikg_+' = (q - \lambda w + k^2)g_+$ and the integral equation

$$g_+(x, k) = 1 + \int_x^\infty \frac{e^{2ik(t-x)} - 1}{2ik} Q(t, k)g_+(t, k)dt \tag{3.1}$$

where

$$Q(\cdot, k) = q - \lambda w + k^2 = q - q_0 w + k^2(1 - w) \tag{3.2}$$

is integrable. One then solves the integral equation by successive approximations from its desired initial values $g_+(\infty) = 1$, $g_+'(\infty) = 0$ using the estimate $|e^{2ik(t-x)} - 1| \le 2$ for $t \ge x$.

If the first moment of $q - q_0 w$ is finite, we may make use of the estimate $|e^{2ik(t-x)} - 1| \le 2|k|(t - x)$ which holds when $t \ge x$ and $\mathrm{Im}(k) \ge 0$. This allows once more to use successive approximation to show the existence and continuity of $f_+(x,\cdot)$ and $f'_+(x,\cdot)$ in $\overline{\mathbb{C}_+}$. For $k=0$ (3.1) is then replaced by

$$g_+(x,0) = 1 + \int_x^\infty (t-x)(q(t) - q_0 w(t)) g_+(t,0)dt. \qquad (3.3)$$

The proofs for f_- are completely analogous. □

In the sequel we shall always assume that λ and $k \in \overline{\mathbb{C}_+}$ are connected via $\lambda = q_0 + k^2$. Later the following estimates will be useful.

Lemma 3.2. *For* $\operatorname{Im} k \geq 0$, $k \neq 0$ *the following estimates hold:*

$$|g_\pm(x,k)| \leq \exp\big(\|q - q_0 w\|_1/|k| + |k|\|w-1\|_1\big), \qquad (3.4)$$

$$|[f_+(\cdot,k), f_-(\cdot,k)]| \leq 2|k| \exp\big(\|q - q_0 w\|_1/|k| + |k|\|w-1\|_1\big), \qquad (3.5)$$

where $\|\cdot\|_1$ *is the norm of* $L^1(\mathbb{R})$.

Proof. Applying Gronwall's inequality to (3.1) we obtain

$$|g_+(x,k)| \leq \exp\Big(|k|^{-1} \int_x^\infty |Q|\Big).$$

Differentiating (3.1) and inserting the estimate of g_+ this gives

$$|g'_+(x,k) + ikg_+(x,k)| \leq |k| \exp\Big(|k|^{-1} \int_x^\infty |Q|\Big).$$

Similarly estimates for g_- give (3.4) and (3.5) since

$$[f_+, f_-] = g_+(g'_- - ikg_-) - (g'_+ + ikg_+)g_-. \qquad (3.6)$$

□

3.2. High energy asymptotics of Jost functions.

For small $\delta > 0$ define the set

$$S(\delta) = \{k \in \mathbb{C} : \operatorname{Im} k \geq 1, \delta \leq |\operatorname{Re} k|/|k| \leq 1 - \delta\}.$$

We shall examine the asymptotic behavior of $f_\pm(x,k)$ and $f'_\pm(x,k)$ as k tends to infinity in $S(\delta)$. We note that, if $w-1$ is integrable, the set where w is negative has finite Lebesgue measure. Also $|1 - \sqrt{|w|}| \leq |1-w|$ pointwise. In the following theorem we interpret $\sqrt{w}$ as $\pm i\sqrt{|w|}$ when $\pm\operatorname{Re}(k) > 0$ and $w < 0$, and use the ordinary square root when $w \geq 0$. It follows that $1 - \sqrt{w}$ is integrable.

Theorem 3.3. *Suppose $q - q_0$ and $w - 1$ are integrable. For $\delta > 0$ we have*

$$f_+(x,k) = \exp(ik(x + \int_x^\infty (1-\sqrt{w})) + o(|k|)),$$
$$f_+'(x,k) = \exp(ik(x + \int_x^\infty (1-\sqrt{w})) + o(|k|)),$$
$$f_-(x,k) = \exp(-ik(x - \int_{-\infty}^x (1-\sqrt{w})) + o(|k|)),$$
$$f_-'(x,k) = \exp(-ik(x - \int_{-\infty}^x (1-\sqrt{w})) + o(|k|)),$$
$$[f_+(\cdot,k), f_-(\cdot,k)] = \exp(ik \int_{-\infty}^\infty (1-\sqrt{w}) + o(|k|))$$

as $k \to \infty$ in S_δ if $\sqrt{w}$ is interpreted as mentioned above.

Proof. Recall that $\lambda = k^2 + q_0$ and let

$$m_+(x,\lambda) = f_+'(x,k)/(\lambda f_+(x,k))$$

be the (left definite) Dirichlet m-function for the interval $[x,\infty)$, see our paper [3]. According to Theorem 4.2 of [6] $m_+(x,\lambda) \sim w(x)/\sqrt{-\lambda w(x)}$ (using the principal branch of the square root) for every Lebesgue point x of w for which $w(x) \neq 0$. Also, if x is a Lebesgue point of w with $w(x) = 0$, then $k\, m_+(x,\lambda)$ tends to zero. Thus $\lambda\, m_+(\cdot,\lambda)/(ik)$ converges pointwise almost everywhere to $\sqrt{w}$ (using the square root as defined above) as k tends to infinity in $S(\delta)$. Recalling the definition of Q from (3.2), we see that $k^{-2}Q(\cdot,k)/(1+\lambda\, m(\cdot,\lambda)/(ik))$ tends to $(1-w)/(1+\sqrt{w}) = 1-\sqrt{w}$ almost everywhere.

Now, for $k \in S_\delta$ we have firstly that $|k^{-2}Q(\cdot,k)| \le |1-w| + |q - q_0 w|$ which is integrable. Secondly, since both $\lambda \mapsto ik/\lambda$ and $m_+(x,\cdot)$ are Nevanlinna functions their imaginary parts are of the same sign. This implies

$$\begin{aligned}\frac{1}{|1+\lambda\, m_+(\cdot,\lambda)/(ik)|} &\le \frac{|k/\lambda|}{|\operatorname{Im}(ik/\lambda)|}\\ &= \frac{|k\lambda|}{|\operatorname{Re} k|(|k|^2+q_0)} \le \frac{|k|}{|\operatorname{Re} k|} \le \frac{1}{\delta}.\end{aligned} \tag{3.7}$$

Thus $k^{-2}Q(\cdot,k)/(1+\lambda\, m_+(\cdot,\lambda)/(ik))$ has a bound in $L^1(\mathbb{R})$ independent of $k \in S_\delta$. For any interval I the dominated convergence theorem therefore implies that

$$\lim_{S_\delta \ni k \to \infty} \int_I \frac{k^{-2}Q}{1+\lambda\, m_+(\cdot,\lambda)/(ik)} = \int_I (1-\sqrt{w}). \tag{3.8}$$

Since $g_+' + 2ikg_+ = (\lambda\, m_+ + ik)g_+$ we have

$$g_+'(x,k) + 2ikg_+(x,k) = 2ik \exp\Big(ik \int_x^\infty \frac{k^{-2}Q}{1+\lambda\, m_+(\cdot,\lambda)/(ik)}\Big)$$

as is easily verified. Now

$$f_+(x,k) = e^{ikx} g_+(x,k) = \frac{e^{ikx}(g_+'(x,k) + 2ikg_+(x,k))}{\lambda\, m_+ + ik}.$$

Here $(\lambda\, m_+ + ik)^{-1}$ equals $(-\lambda)^{-1}$ times a Nevanlinna function so by Proposition 1.5 this factor is certainly $e^{o(|k|)}$ as $k \to \infty$ in S_δ proving our first assertion. The second follows in the same way from the fact that $f_+'(x,k)/f_+(x,k) = \lambda\, m_+(x,k)$. Of course, we may deal analogously with f_- and f_-' so that only our last assertion remains to be proven.

By (3.6) and using that as $x \to -\infty$ we have $g_-(x,k) \to 1$ and $g_-'(x,k) \to 0$, while (3.4) shows that $g_+(x,k)$ is bounded, we find that

$$\begin{aligned}[f_+(\cdot,k), f_-(\cdot,k)] &= -\lim_{x\to-\infty}(g_+'(x,k) + 2ikg_+(x,k)) \\ &= -2ik \exp\Big(ik \int_{-\infty}^\infty \frac{k^{-2}Q}{1+\lambda\, m(\cdot,\lambda)/(ik)}\Big). \end{aligned} \tag{3.9}$$

Our last claim now follows from (3.8). □

Corollary 3.4. *Suppose $q - q_0$ and $w - 1$ are integrable. For $\operatorname{Re} k \neq 0$ and $\operatorname{Im} k \geq 0$ we have*

$$\begin{aligned}|[f_+(\cdot,k), f_-(\cdot,k)]| &\geq 2|k| \exp\Big(-\frac{1}{|\operatorname{Re} k|}\int_{-\infty}^\infty |Q|\Big) \\ &\geq 2|k| \exp\big(-\tfrac{|k|}{|\operatorname{Re} k|}(\|q - q_0 w\|_1/|k| + |k|\|w-1\|_1)\big).\end{aligned}$$

Proof. This follows immediately from (3.9) and (3.7) noting that these relationships hold as long as $\operatorname{Im} k \geq 0$ and $\operatorname{Re} k \neq 0$. □

3.3. **Transmission and reflection coefficients.**

For real $k \neq 0$ the functions $f_-(\cdot,k)$ and $f_-(\cdot,-k) = \overline{f_-(\cdot,k)}$ form a basis of solutions as do the functions $f_+(\cdot,k)$ and $f_+(\cdot,-k) = \overline{f_+(\cdot,k)}$. Indeed, the asymptotic behavior of $f_\pm$ shows that $[\overline{f_+}, f_+] = [f_-, \overline{f_-}] = 2ik$. From the identity $[f,g][r,s] = [f,r][g,s] - [f,s][g,r]$ we obtain

$$|[f_-, f_+]|^2 = 4k^2 + |[f_-, \overline{f_+}]|^2 > 0.$$

Hence f_+ and f_- also form a basis of solutions so that we may find coefficients $\mathfrak{T}_\pm(k)$, $\mathfrak{R}_\pm(k)$ satisfying

$$\begin{cases} \mathfrak{T}_- f_- = \mathfrak{R}_+ f_+ + \overline{f_+} \\ \mathfrak{T}_+ f_+ = \mathfrak{R}_- f_- + \overline{f_-}. \end{cases} \tag{3.10}$$

Taking the Wronskian with f_+ in the first and f_- in the second equation shows that $\mathfrak{T}_+ = \mathfrak{T}_-$, which we will denote by $\mathfrak{T}$ from now on. It also shows that

$$[f_+, f_-] = -2ik/\mathfrak{T}(k),$$
$$\frac{1}{2ik}\,[\overline{f_+}, f_-] = \mathfrak{R}_+(k)/\mathfrak{T}(k) = -\overline{(\mathfrak{R}_-(k)/\mathfrak{T}(k))},$$

where the second line is obtained by taking the Wronskian with $\overline{f_+}$ in the first equation and $\overline{f_-}$ in the second. Equation (3.9) becomes $|\mathfrak{T}|^2 + |\mathfrak{R}_+|^2 = |\mathfrak{T}|^2 + |\mathfrak{R}_-|^2 = 1$. Also $\mathfrak{T}(-k) = \overline{\mathfrak{T}(k)}$ and $\mathfrak{R}_\pm(-k) = \overline{\mathfrak{R}_\pm(k)}$.

The coefficient $\mathfrak{T}$ is called the transmission coefficient while $\mathfrak{R}_\pm$ are called reflection coefficients. They are primary data observed in a scattering experiment. It is customary to collect them in the unitary matrix

$$\begin{pmatrix} \mathfrak{T} & \mathfrak{R}_+ \\ \mathfrak{R}_- & \mathfrak{T} \end{pmatrix}$$

which is called the *scattering matrix.*

We shall later need the following theorem. The proof follows the one by Klaus [24] for the case of the Schrödinger operator closely.

Theorem 3.5. *Under Assumption 1.3 and if $[f_+, f_-](0) = 0$ we have $[f_+, f_-](k) \sim -ik(\alpha+1/\alpha)$ as $k \to 0$ in $\operatorname{Im} k \geq 0$, where α is determined by $f_-(x, 0) = \alpha f_+(x, 0)$.*

To prove the theorem we consider the solution $u(x, k)$ of (1.1) with the same initial data as $f_+(\cdot, 0)$. Thus $u(0, k) = a$, $u'(0, k) = b$, where $a = f_+(0, 0)$, $b = f'_+(0, 0)$. Since $u(x, 0) = f_+(x, 0) = g_+(x, 0) = f_-(x, 0)/\alpha$ we obtain from (3.1) and the corresponding formula for f_- that

$$\begin{aligned} a = 1 + \int_0^\infty t(q(t) - q_0 w(t))u(t, 0)\,dt \\ = 1/\alpha - \int_{-\infty}^0 t(q(t) - q_0 w(t))u(t, 0)\,dt \end{aligned} \tag{3.11}$$

and

$$b = -\int_0^\infty (q - q_0 w)u(\cdot, 0) = \int_{-\infty}^0 (q - q_0 w)u(\cdot, 0). \tag{3.12}$$

The function u satisfies the integral equation

$$u(x, k) = a\cos(kx) + b\,\frac{\sin(kx)}{k} + \int_0^x \frac{\sin k(x-t)}{k}\,Q(t, k)u(t, k)\,dt, \tag{3.13}$$

which for $k = 0$ takes the form

$$u(x,0) = a + bx + \int_0^x (x-t)(q(t) - q_0 w(t))u(t,0)\,dt. \tag{3.14}$$

We shall need to estimate the difference $\Delta u(x,k) = u(x,k) - u(x,0)$.

Lemma 3.6. *There is a constant C such that for $k \in \mathbb{R}$ we have $|\Delta u(x,k)| \le C \min(1,|kx|)(|k| + \min(1,|kx|))$.*

Proof. We consider only the case $x \ge 0$; the case $x \le 0$ is similar. Splitting the first integral in (3.12) into integrals over $(0,x)$ and (x,∞), the second term is less than $x^{-1}\int_x^\infty |t(q-q_0w)u(t,0)|$, so that $bx = -x\int_0^x (q-q_0w)u(\cdot,0) + o(1)$ as $x \to \infty$.

Using this when expressing Δu by the help of (3.13) and (3.14) we find a constant A such that

$$\begin{aligned}|\Delta u(x,k)| \le{}& A\min(1,|kx|)(|k| + \min(1,|kx|)) \\ &+ \min(1,|kx|)\int_0^x (|k||1-w| + |q-q_0w|/|k|)|\Delta u|.\end{aligned}$$

Here we use that $1 - \cos kx$ and $1 - \sin kx/kx$ may be estimated by $\min(2,(kx)^2)$ and $|\sin kx|$ by $\min(1,|kx|)$. With $s(x) = x - (\sin kx)/k$ we also use $|s(x) - s(x-t)| = t|s'(\xi)| = t|1-\cos k\xi| \le t\min(2,(kx)^2)$. Setting $v(x,k) = |\Delta u(x,k)/\min(1,|kx|)|$ we now obtain

$$v(x,k) \le A(|k| + \min(1,|kx|)) + \int_0^x (|k||1-w| + |t(q-q_0w)|)v(t,k)\,dt,$$

so that the claim finally follows from Gronwall's inequality. $\square$

Proof of Theorem 3.5. Choosing $f = f_+(\cdot,k)$, $g = f_-(\cdot,k)$, $r = u(\cdot,k)$, and $s = ax - b$ in the identity

$$[f,g][r,s] = [f,r][g,s] - [g,r][f,s]$$

and evaluating it at $x = 0$ gives

$$[f_+(\cdot,k), f_-(\cdot,k)] = \big(\alpha[f_+(\cdot,k),u(\cdot,k)] - [f_-(\cdot,k),u(\cdot,k)]\big)(1 + o(1))$$

after dividing by $[r,s] = a^2 + b^2 \ne 0$ and using the continuity of $f_\pm(0,\cdot)$ and $f'_\pm(0,\cdot)$ at $k = 0$.

The expressions $[f_\pm(\cdot,k),u(\cdot,k)]$ do not depend on x and may therefore be obtained by taking the limit at plus or minus infinity. Since $f'_+(x,k)/f_+(x,k) \to ik$ as x tends to ∞ and since $u(\cdot,k)$ and $f_+(\cdot,k)$

are bounded we get

$$\begin{aligned}[f_+(\cdot,k),u(\cdot,k)] &= \lim_{x\to\infty} f_+(x,k)(u'(x,k)-iku(x,k))\\ &= \lim_{x\to\infty} e^{ikx}(u'(x,k)-iku(x,k))\end{aligned}$$

and similarly

$$[f_-(\cdot,k),u(\cdot,k)] = \lim_{x\to-\infty} e^{-ikx}(u'(x,k)+iku(x,k)).$$

We will show below that

$$\lim_{x\to\infty} e^{ikx}(u'(x,k)-iku(x,k)) = -ik(1+o(1)) \tag{3.15}$$

as $k\to 0$ in $\mathbb{R}$. Similarly

$$\lim_{x\to-\infty} e^{-ikx}(u'(x,k)+iku(x,k)) = ik/\alpha\,(1+o(1)) \tag{3.16}$$

as $k\to 0$ in $\mathbb{R}$. Thus

$$[f_+(\cdot,k),f_-(\cdot,k)] = -(\alpha+1/\alpha)ik(1+o(1))$$

as $k\to 0$ in $\mathbb{R}$. However, if $\operatorname{Im} k>0$ we still have $[f_+,f_-]$ bounded as $k\to 0$ so the Phragmén-Lindelöf principle (see, *e.g.*, 5.63 in Titchmarsh [29]) shows that the same asymptotic formula is valid even for $k\to 0$ in $\operatorname{Im} k\geq 0$, which was to be shown.

It remains to calculate the limits (3.15) and (3.16). We will deal only with the first; the second is treated similarly.

From (3.13), using (3.11) and (3.12), we obtain

$$\begin{aligned}e^{ikx}(u'(x,k)-iku(x,k)) &= -ika+b+\int_0^x e^{ikt}Q(t,k)u(t,k)\,dt\\ \to -ik+\int_0^\infty &\big(e^{ikt}Q(t,k)u(t,k)-(ikt+1)(q-q_0w)(t)u(t,0)\,dt\big)\end{aligned}$$

as $x\to\infty$. We shall estimate the last integral. It equals

$$\begin{aligned}\int_0^\infty e^{ikt}Q(t,k)\Delta u(t,k)\,dt+k^2\int_0^\infty e^{ikt}(1-w(t))u(t,0)\,dt\\ +k\int_0^\infty \frac{e^{ikt}-ikt-1}{kt}\,t(q-q_0w)(t)u(t,0)\,dt.\end{aligned}$$

The second term is clearly $\mathcal{O}(|k|^2)$ as $k\to 0$. In the last integral the first factor is bounded by 2, so that dominated convergence shows the corresponding term to be $o(|k|)$ as $k\to 0$.

By Lemma 3.6 the integrand of the first term may be estimated by

$$C|k|(|k||1-w(t)|+|t(q-q_0w)(t)|)(|k|+\min(1,|kt|)),$$

and the integral of this is $o(|k|)$ as $k \to 0$ by dominated convergence. $\square$

Remark. The theorem shows that the transmission coefficient $\mathfrak{T}(k)$ is continuous at 0, tending to some real value in $[-1, 1]$ at $k = 0$. One may similarly show continuity at 0 for the reflection coefficients, analogously to what is done in Klaus [24], but we will not need this.

3.4. **Eigenvalues.**

If $k \neq 0$ is real then the asymptotics of $f_+(\cdot, k)$ show that the real and imaginary parts are linearly independent solutions of (1.1), and that no linear combination of them can be in $\mathcal{H}_1$. Hence there are no eigenvalues in (q_0, ∞).

We showed in Proposition 2.5 that q_0 is not an eigenvalue if it is equal to 0. The example $w(x) = 1$ and $q(x) = 2x^2(x^2 + 5)/(x^2 + 1)^2$ shows that q_0 may well be an eigenvalue if $q_0 > 0$ since $u(x) = 1/(x^2 + 1)$ defines an eigenfunction for this case.

If $q_0 > 0$ and $f \in \mathcal{H}_1$ is a solution of $f'' = (q - q_0 w)f$, then according to Lemma 2.1 f is bounded, so that f'' is integrable. Thus f' has limits at $\pm\infty$ which must be 0 since $f' \in L^2(\mathbb{R})$. Two solutions in $\mathcal{H}_1$ therefore have Wronskian zero, so they are linearly dependent. If q_0 is an eigenvalue it is therefore simple.

If in addition the first moment of $q - q_0 w$ is finite, then f must be a multiple of $f_+(\cdot, 0)$, which is asymptotically equal to 1 at ∞. Thus it is not in $\mathcal{H}_1$, so in this case q_0 is not an eigenvalue. Moreover, eigenvalues will not accumulate[2] at q_0, the proof of which follows the one for the half-line case, see part (3) of Theorem 7.1 in our paper [3].

Now suppose $\lambda < q_0$ so $\operatorname{Im} k > 0$. Then $f_+(x, k)$ is asymptotic to e^{ikx} so it is non-zero for all large x, say $x \geq a$. Thus for $x > a$ another solution is $f_+(x, k) \int_a^x (f_+(\cdot, k))^{-2}$ which is easily seen to be asymptotic to $-\frac{1}{2ik} e^{-ikx}$ at infinity. Thus the only linear combination of these solutions in $\mathcal{H}_+$ are multiples of $f_+(\cdot, k)$, so that all eigenvalues are simple. Moreover, an adaptation of the proof of Theorem 7.1 part (2) in [3] for the half-line case shows that there will be no negative eigenvalues if $w \geq 0$, but infinitely many negative eigenvalues which accumulate at negative infinity if w is negative on a set of positive measure.

Given $k \in \mathbb{C}_+$ the functions $f_\pm(\cdot, k)$ are eigenfunctions associated with the eigenvalue $k^2 + q_0$ precisely if the Wronskian $[f_+(\cdot, k), f_-(\cdot, k)]$ is zero, and $k^2 + q_0$ must then be real, *i.e.*, k is necessarily purely imaginary. Thus, all zeros of the Wronskian in $\mathbb{C}_+$ are purely imaginary.

---

[2]If $q_0 = 0$ this is true even if the first moment of q is not finite.

Moreover, since zeros of analytic functions are isolated the eigenvalues below q_0 may accumulate only at q_0 or negative infinity.

The function $k \mapsto [f_+(\cdot,k), f_-(\cdot,k)]$ is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+} \setminus \{0\}$. According to Proposition 1.6 it is uniquely determined by its values on the real axis. Thus $\mathfrak{T}$, initially defined on $\mathbb{R} \setminus \{0\}$, has a unique meromorphic extension to the upper half-plane, and all eigenvalues, except possibly q_0, are determined by the poles of $\mathfrak{T}$ in $\mathbb{C}_+$.

We collect these results in the following theorem.

Theorem 3.7. *Suppose $q - q_0$ and $w - 1$ are integrable. Then the eigenvalues of T all lie in the interval $(-\infty, q_0]$. They are all simple and can not accumulate at any point inside the interval $(-\infty, q_0)$.*

There are infinitely many negative eigenvalues if w is negative on a set of positive measure, otherwise none. The number $\lambda = k^2 + q_0 < q_0$ is an eigenvalue of T if and only if k is a pole of $\mathfrak{T}$.

Zero it is not an eigenvalue, nor will eigenvalues accumulate at 0 if $q_0 = 0$. The same conclusion holds if $q_0 > 0$ but the first moment of $q - q_0 w$ is finite.

3.5. **Eigenfunction expansion.**

Let γ be an axis-parallel rectangle cutting out an open interval I from $\mathbb{R}$. Assume that the endpoints of I are different from q_0 and any of the eigenvalues of T. When $u, v \in \mathcal{H}_1$ we get from the spectral theorem and the special case $v = g_0(x,\cdot)$ that

$$\langle E_I u, v\rangle = -\frac{1}{2\pi i}\oint_\gamma \langle R_\lambda u, v\rangle \, d\lambda,$$
$$E_I u(x) = -\frac{1}{2\pi i}\oint_\gamma R_\lambda u(x)\, d\lambda,$$

where $\omega \mapsto E_\omega$ denotes the spectral decomposition of T. Now suppose $u \in \mathcal{H}_1$ has compact support and define

$$\hat{u}_\pm(k) = \langle u, \overline{f_\pm(\cdot,k)}\rangle$$

when $\operatorname{Im} k \geq 0$, $k \neq 0$.

Integration by parts shows that $\hat{u}_\pm(k) = \lambda \int u w f_\pm(\cdot,k)$, and hence $\hat{u}_\pm$ vanishes at $k = i\sqrt{q_0}$, corresponding to $\lambda = 0$ (at least if $q_0 \neq 0$ or $q_0 = 0$ and q has a finite first moment), is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+} \setminus \{0\}$. We also define $\hat{u}_+(0)$ when q_0 is an eigenvalue by setting

$$\hat{u}_+(0) = \langle u, f_0\rangle$$

where f_0 is a realvalued and normalized eigenfunction.

Remark. As long as $\operatorname{Im} k > 0$ or $k \geq 0$ the equation $\lambda = q_0 + k^2$ gives a one-to-one correspondence between λ and k, so we can view either of these variables as a function of the other. These are clearly analytic for $\operatorname{Im} k > 0$ respectively $\lambda \notin [q_0, \infty)$.

In the rest of this section and the next we will always think of k as the function of λ defined in this way. We may therefore think of $\mathfrak{T}$ (really $\mathfrak{T} \circ k$) as a function of λ, meromorphic outside $[q_0, \infty)$ and with continuous limits as λ approaches (q_0, ∞) from above or below. Similarly for $f_\pm$ and $\hat{u}_\pm$ if $u \in \mathcal{H}_1$ has compact support.

If $\operatorname{Im} k > 0$ then $f_+(\cdot, k)$ is in $\mathcal{H}_+$ and $f_-(\cdot, k)$ is in $\mathcal{H}_-$. Therefore, if λ is not an eigenvalue and not in $[q_0, \infty)$, we will choose $\psi_+(x, \lambda) = f_+(x, k)$ and $\psi_-(x, \lambda) = f_-(x, k)$ in (2.3), the main ingredient of Green's function. Thus, if $u \in \mathcal{H}_1$, $\operatorname{Im} k > 0$, and $\operatorname{Re} k \neq 0$, Theorem 2.9 implies that

$$(R_\lambda u)(x) = -\frac{\mathfrak{T}(k)}{2ik\lambda}(f_+(x,k)\langle u, \overline{f_-(\cdot,k)}\rangle^x + f_-(x,k)\langle u, \overline{f_+(\cdot,k)}\rangle_x) - \frac{u(x)}{\lambda} \tag{3.17}$$

where $\langle u, v\rangle^x$ and $\langle u, v\rangle_x$ are the integrals of $u'\overline{v'} + qu\overline{v}$ over $(-\infty, x)$ and (x, ∞) respectively. If u has compact support we may write (3.17) as

$$(R_\lambda u)(x) = -\frac{\mathfrak{T}(k)}{2ik\lambda}\,\hat{u}_+(k) f_-(x,k) + (\langle u, \overline{\varphi(\cdot, x, \lambda)}\rangle^x - u(x))/\lambda. \tag{3.18}$$

where

$$\varphi(y, x, \lambda) = \frac{\mathfrak{T}(k)}{2ik}\,(f_-(x,k)f_+(y,k) - f_+(x,k)f_-(y,k)).$$

It is easily verified that the function $\varphi(\cdot, x, \lambda)$ solves $-\varphi'' + q\varphi = \lambda w \varphi$ and satisfies the initial conditions $\varphi(x, x, \lambda) = 0$ and $\varphi'(x, x, \lambda) = 1$ so that $\varphi(y, x, \cdot)$ and $\varphi'(y, x, \cdot)$ are entire functions, locally uniformly in (x, y). Integration by parts shows that the last two terms in (3.18) together equal $\int_{-\infty}^x uw\varphi(\cdot, x, \lambda)$. This expression, together with its x-derivative, is an entire function of λ locally uniformly in x. Consequently we arrive at the following expressions

$$E_I u(x) = -\frac{1}{2\pi}\oint_\gamma \hat{u}_+(k) f_-(x,k)\frac{\mathfrak{T}(k)}{2k\lambda}\,d\lambda, \tag{3.19}$$

$$\langle E_I u, v\rangle = -\frac{1}{2\pi}\oint_\gamma \hat{u}_+(k)\overline{\hat{v}_-(-\overline{k})}\frac{\mathfrak{T}(k)}{2k\lambda}\,d\lambda, \tag{3.20}$$

if also v has compact support.

Suppose now that I is to the left of q_0. Then all singularities inside γ occur where $\mathfrak{T}$ has a pole (recall that $\lambda = 0$ is not an eigenvalue and that $\hat{u}_+$ vanishes for $\lambda = 0$ if $q_0 > 0$). To find the corresponding residues let $\lambda = \lambda_n + i\nu$ where λ_n is an eigenvalue, so that

$$-i\nu\langle R_\lambda f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle = \langle f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle.$$

Setting $u = f_+(\cdot, k_n)$ in (3.17) and taking an inner product with $v = f_-(\cdot, k_n)$ gives then

$$i\nu\mathfrak{T}(k) = 2ik\lambda_n \frac{\langle f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle}{\langle h(\cdot, k), f_-(\cdot, k_n)\rangle}$$

where

$$h(x, k) = f_+(x, k)\langle f_+(\cdot, k_n), \overline{f_-(\cdot, k)}\rangle^x + f_-(x, k)\langle f_+(\cdot, k_n), \overline{f_+(\cdot, k)}\rangle_x.$$

We note that $h(x, k_n) = f_+(x, k_n)\langle f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle$ so that $\mathfrak{T} \circ k$ has a simple pole at $\lambda = \lambda_n$ with residue

$$\operatorname{Res}_{\lambda_n} \mathfrak{T} \circ k = \frac{2ik_n\lambda_n}{\langle f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle} \tag{3.21}$$

Equation (3.19) now gives

$$\begin{aligned} E_I u(x) &= \sum_{\lambda_n \in I} \frac{\hat{u}_+(k_n) f_-(x, k_n)}{\langle f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle} \\ &= \sum_{\lambda_n \in I} \frac{\hat{u}_-(k_n) f_-(x, k_n)}{\|f_-(\cdot, k_n)\|^2} = \sum_{\lambda_n \in I} \frac{\hat{u}_+(k_n) f_+(x, k_n)}{\|f_+(\cdot, k_n)\|^2}. \end{aligned}$$

and (3.20) implies

$$\begin{aligned} \langle E_I u, v\rangle &= \sum_{\lambda_n \in I} \frac{\hat{u}_+(k_n)\overline{\hat{v}_-(k_n)}}{\langle f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle} \\ &= \sum_{\lambda_n \in I} \frac{\hat{u}_-(k_n)\overline{\hat{v}_-(k_n)}}{\|f_-(\cdot, k_n)\|^2} = \sum_{\lambda_n \in I} \frac{\hat{u}_+(k_n)\overline{\hat{v}_+(k_n)}}{\|f_+(\cdot, k_n)\|^2}. \end{aligned}$$

This persists[3] if the right endpoint of I approaches q_0.

Now suppose that I lies to the right of q_0. Here we may move the top and bottom of γ towards $\mathbb{R}$ with no change of the integral because of analyticity, and since the integrand has continuous limits from above and below in I we may shrink the height to zero. In this way we obtain

[3]Since the left hand side has a finite limit. The right hand side has positive terms for $u = v$ so the series is absolutely convergent.

by dominated convergence

$$E_I u(x) = \int_I (\hat{u}_+(k) f_-(x,k)\mathfrak{T}(k) + \hat{u}_+(-k) f_-(x,-k)\mathfrak{T}(-k)) \frac{d\lambda}{4\pi k\lambda},$$
$$\langle E_I u, v\rangle = \int_I (\hat{u}_+(k)\overline{\hat{v}_-(-k)}\mathfrak{T}(k) + \hat{u}_+(-k)\overline{\hat{v}_-(k)}\mathfrak{T}(-k)) \frac{d\lambda}{4\pi k\lambda}.$$

The scattering relations give $\hat{u}_+(-k) = \mathfrak{T}(k)\hat{u}_-(k) - \mathfrak{R}_+(k)\hat{u}_+(k)$ and a similar formula for $\hat{v}_-(-k)$ so that

$$E_I u(x) = \frac{1}{2\pi}\int_I (\hat{u}_+(k)\overline{f_+(x,k)} + \hat{u}_-(k)\overline{f_-(x,k)})|\mathfrak{T}(k)|^2 \frac{d\lambda}{2k\lambda},$$
$$\langle E_I u, v\rangle = \frac{1}{2\pi}\int_I (\hat{u}_+(k)\overline{\hat{v}_+(k)} + \hat{u}_-(k)\overline{\hat{v}_-(k)})|\mathfrak{T}(k)|^2 \frac{d\lambda}{2k\lambda}.$$

Again, this formula holds even if the left endpoint of I is q_0.

Finally, if $q_0 \in I$ we split $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ by introducing two new vertical sides at $\operatorname{Re}\lambda = q_0 \pm \varepsilon$, γ_1 being the leftmost and γ_3 the rightmost rectangle. The corresponding intervals are denoted I_1, I_2, I_3 (where again the endpoints of each interval avoid q_0 and the eigenvalues of T). We have

$$-\frac{1}{2\pi i}\oint_{\gamma_2} R_\lambda u(x)\, d\lambda \to E_{\{q_0\}} u(x),$$
$$-\frac{1}{2\pi i}\oint_{\gamma_2} \langle R_\lambda u, v\rangle\, d\lambda \to \langle E_{\{q_0\}} u, v\rangle$$

as $\varepsilon \downarrow 0$, which is zero unless q_0 is an eigenvalue, in which case $E_{\{q_0\}}$ is the projection onto the one-dimensional eigenspace. The contributions from I_1 and I_3, determined above may be added to obtain $E_I u(x)$ respectively $\langle E_I u, v\rangle$.

We may now also expand I to all of $\mathbb{R}$ to get the following Parseval relation for compactly supported functions $u, v \in \mathcal{H}_1$:

$$\langle E_{\mathbb{R}} u, v\rangle = \sum_{\lambda_n < q_0} \frac{\hat{u}_+(k_n)\overline{\hat{v}_+(k_n)}}{\|f_+(\cdot,k_n)\|^2} + \langle E_{\{q_0\}} u, v\rangle$$
$$+ \int_{(q_0,\infty)} (\hat{u}_+(k)\overline{\hat{v}_+(k)} + \hat{u}_-(k)\overline{\hat{v}_-(k)}) \frac{|\mathfrak{T}(k)|^2}{4\pi k\lambda}\, d\lambda.$$

We also have the inversion formula for compactly supported $u \in \mathcal{H}_1$:

$$E_{\mathbb{R}} u(x) = \sum_{\lambda_n < q_0} \frac{\hat{u}_+(k_n) f_+(x, k_n)}{\|f_+(\cdot, k_n)\|^2} + E_{\{q_0\}} u(x) \\ + \int_{(q_0,\infty)} (\hat{u}_+(k)\overline{f_+(x,k)} + \hat{u}_-(k)\overline{f_-(x,k)}) \frac{|\mathfrak{T}(k)|^2}{4\pi k \lambda}\, d\lambda.$$

4. The Jost transform

The results of the previous section will be the key for the construction of a generalized Fourier transform which we shall call the Jost transform. Let $\mathbb{L}$ be the union of the positive imaginary k-axis and the nonnegative real axis. We recall that any $\lambda \in \mathbb{R}$ corresponds to a unique $k \in \mathbb{L}$ via $\lambda = q_0 + k^2$, so we may think of $k \in \mathbb{L}$ as a function of λ. In the following we will always tacitly assume that $\lambda = k^2 + q_0$. Similarly, for the real variable t we will, without further ado, use s to denote the root of $t - q_0$ in $\mathbb{L}$.

For the discussion below recall that the eigenvalues of T which are smaller than q_0 are given by $\lambda_n = k_n^2 + q_0$ with purely imaginary numbers $k_n \in \mathbb{L}$.

In the following we consider $\mathbb{C}^2$ as a space of rows and denote the first and second components of its elements with subscripts $+$ and $-$, respectively. Let $L^2_{\mathcal{J}}$ be the set of functions $\hat{u} : \mathbb{L} \to \mathbb{C}^2$ for which the quadratic form associated with

$$\langle \hat{u}, \hat{v} \rangle_{\mathcal{J}} = \sum_{\lambda_n < q_0} \frac{\hat{u}_+(k_n)\overline{\hat{v}_+(k_n)}}{\|f_+(\cdot, k_n)\|^2} + \hat{u}_+(0)\overline{\hat{v}_+(0)} \\ + \int_{(q_0,\infty)} (\hat{u}_+(s)\overline{\hat{v}_+(s)} + \hat{u}_-(s)\overline{\hat{v}_-(s)}) \frac{|\mathfrak{T}(s)|^2}{4\pi s t}\, dt \quad (4.1)$$

is finite (Recall that we always have $t = q_0 + s^2$ with $s \in \mathbb{L}$). The term containing $\hat{u}_+(0)$ should be dropped unless q_0 is an eigenvalue. More precisely, in $L^2_{\mathcal{J}}$ we identify, as usual, any two functions $\hat{u}$ and $\hat{v}$ for which $\langle \hat{u} - \hat{v}, \hat{u} - \hat{v} \rangle_{\mathcal{J}} = 0$. Thus an element $(\hat{u}_+, \hat{u}_-) \in L^2_{\mathcal{J}}$ is determined a.e. on $(0, \infty)$, and $\hat{u}_+$ is also determined on all $k_n \in \mathbb{L}$ where $\lambda_n = q_0 + k_n^2$ is an eigenvalue, including at 0 if q_0 is an eigenvalue. Apart from this $\hat{u}$ is undetermined.

The space $L^2_{\mathcal{J}}$ can be thought of as a direct sum of three weighted L^2-spaces, each associated with one of the three summands on the right hand side of (4.1) and is thus a Hilbert space with inner product given by $\langle \cdot, \cdot \rangle_{\mathcal{J}}$. We denote the associated norm by $\| \cdot \|_{\mathcal{J}}$.

Define
$$F(x,k) = (f_+(x,k), f_-(x,k)).$$
Then, for a compactly supported function $u \in \mathcal{H}_1$, introduce the map $\mathcal{J}u : \mathbb{L} \to \mathbb{C}^2$ by setting
$$(\mathcal{J}u)(k) = \langle u, \overline{F(\cdot,k)} \rangle.$$
If q_0 is an eigenvalue this is not defined for $k = 0$, and then we define $(\mathcal{J}u)_+(0) = \langle u, f_0 \rangle$ and $F_+(\cdot, 0) = f_0$.

The considerations in the previous section prove the following statement.

Lemma 4.1. *For compactly supported u and $v \in \mathcal{H}_1$ we have $\mathcal{J}u$ and $\mathcal{J}v \in L^2_{\mathcal{J}}$. If E_I is the spectral projection for T associated with the interval I then we have the pointwise expansion*
$$E_I u(x) = \langle \chi_I(t) \mathcal{J}u(s), F(x,s) \rangle_{\mathcal{J}},$$
and the Parseval-type formula
$$\langle E_I u, v \rangle = \langle \chi_I(t) \mathcal{J}u(s), \mathcal{J}v(s) \rangle_{\mathcal{J}}.$$
In particular, $\langle E_{\mathbb{R}} u, v \rangle = \langle \mathcal{J}u, \mathcal{J}v \rangle_{\mathcal{J}}$.

Since compactly supported functions are dense in $\mathcal{H}_1$ the Jost transform extends to a map $\mathcal{J} : \mathcal{H}_1 \to L^2_{\mathcal{J}}$. More precisely the following theorem holds.

Theorem 4.2. *Assume q and w satisfy Assumption 1.2. Then the following statements are true.*

1. *The map $u \mapsto \mathcal{J}u$, defined for compactly supported $u \in \mathcal{H}_1$, extends by continuity to a map $\mathcal{J} : \mathcal{H}_1 \to L^2_{\mathcal{J}}$ called the* Jost transform.
2. *The mapping $\mathcal{J} : \mathcal{H}_1 \to L^2_{\mathcal{J}}$ has kernel $\mathcal{H}_\infty$ and $\langle E_\omega u, v \rangle = \langle \chi_\omega(t) \mathcal{J}u(s), \mathcal{J}v(s) \rangle_{\mathcal{J}}$ for all Borel sets $\omega \subset \mathbb{R}$. In particular* Parseval's formula *$\langle u, v \rangle = \langle \mathcal{J}u, \mathcal{J}v \rangle_{\mathcal{J}}$ holds if at least one of u and v is in $\mathcal{H}$.*
3. *The mapping $\mathcal{J} : \mathcal{H} \to L^2_{\mathcal{J}}$ is unitary.*
4. *For fixed $x \in \mathbb{R}$ the function $F(x, \cdot)$ is in $L^2_{\mathcal{J}}$. Moreover, if $\hat{u} \in L^2_{\mathcal{J}}$ then $x \mapsto \langle \hat{u}, F(x, \cdot) \rangle_{\mathcal{J}}$ represents an element u of $\mathcal{H}$. This map $\hat{u} \mapsto u$ is the adjoint of $\mathcal{J} : \mathcal{H}_1 \to L^2_{\mathcal{J}}$ and thus the inverse of $\mathcal{J}$ restricted to $\mathcal{H}$.*
5. *If $u \in D_T$, then $\mathcal{J}(Tu)(k) = \lambda(\mathcal{J}u)(k)$. Conversely, if $\hat{u}$ and $k \mapsto \lambda \hat{u}(k)$ are in $L^2_{\mathcal{J}}$, then $\mathcal{J}^*(\hat{u}) \in D_T$.*

Remark. It is clear from the theorem that T has absolutely continuous spectrum covering $[q_0, \infty)$ and eigenvalues at λ_n and possibly q_0, but no other spectrum.

Proof of Theorem 4.2, parts (1) and (2). Let u_n be a sequence of compactly supported functions in $\mathcal{H}_1$ converging to a given element $u \in \mathcal{H}_1$, *e.g.*, $u_n = u\varphi_n$ with the φ_n introduced in Lemma 2.2. By Parseval's formula (Lemma 4.1) the sequence $\mathcal{J}u_n$ is Cauchy and hence convergent. The limit is clearly independent of the chosen sequence u_n and is, by definition, $\mathcal{J}u$. This proves the first statement.

The Parseval type formula of Lemma 4.1 now extends by continuity to all $u, v \in \mathcal{H}_1$, and then in standard fashion to the case when I is replaced by an arbitrary Borel set. This proves (2). ☐

Before we prove part (3) of Theorem 4.2 we need to establish Lemma 4.3 below for which we rely on the following notation. For any two elements $\hat{u}$ and $\hat{v}$ of $L^2_{\mathcal{J}}$, we define the left-continuous function $\Xi_{\hat{u},\hat{v}}$ by setting

$$\Xi_{\hat{u},\hat{v}}(\lambda) = \sum_{\lambda_n < \lambda} \frac{\hat{u}_+(k_n)\overline{\hat{v}_+(k_n)}}{\|f_+(\cdot, k_n)\|^2}$$

if $\lambda \leq q_0$ and

$$\Xi_{\hat{u},\hat{v}}(\lambda) = \Xi_{\hat{u},\hat{v}}(q_0) + \hat{u}_+(0)\overline{\hat{v}_+(0)} + \int_{q_0}^{\lambda} \hat{u}(s)\hat{v}(s)^* \frac{|\mathfrak{T}(s)|^2}{4\pi st} dt$$

if $\lambda > q_0$. The term containing $\hat{u}_+(0)$ should be dropped unless q_0 is an eigenvalue. $\Xi_{\hat{u},\hat{v}}$ is a function of bounded variation (with total variation at most $\|\hat{u}\|_{\mathcal{J}}\|\hat{v}\|_{\mathcal{J}}$) and thus gives rise to a complex measure on $\mathbb{R}$. Note that $\Xi_{\hat{u},\hat{v}}(t) = \langle E_{(-\infty,t)}u, v\rangle$ if $\hat{u} = \mathcal{J}u$ and $\hat{v} = \mathcal{J}v$.

Lemma 4.3. *The Jost transform of $R_\lambda u$ is $s \mapsto (\mathcal{J}u)(s)/(t-\lambda)$ provided*[4] *that* $\operatorname{Im}(\lambda) \neq 0$.

Proof. By the spectral theorem we have $\langle R_\lambda u, v\rangle = \int_{\mathbb{R}} \frac{d\langle E_{(-\infty,t)}u,v\rangle}{t-\lambda}$ and since $\langle E_{(-\infty,t)}u, v\rangle = \Xi_{\mathcal{J}u,\mathcal{J}v}(t)$ one gets

$$\langle R_\lambda u, v\rangle = \langle \mathcal{J}u/(t-\lambda), \mathcal{J}v\rangle_{\mathcal{J}}.$$

In particular,

$$\langle R_\lambda u, R_\lambda u\rangle = \langle \mathcal{J}u/(t-\lambda), \mathcal{J}R_\lambda u\rangle_{\mathcal{J}}. \tag{4.2}$$

We also have $R_\lambda - R_{\overline{\lambda}} = (\lambda - \overline{\lambda})R_{\overline{\lambda}}R_\lambda$ and $\langle R_\lambda u, R_\lambda u\rangle = \langle R_{\overline{\lambda}}R_\lambda u, u\rangle$ so that we may write

$$\langle R_\lambda u, R_\lambda u\rangle = \frac{1}{\lambda - \overline{\lambda}}(\langle R_\lambda u, u\rangle - \langle R_{\overline{\lambda}}u, u\rangle) = \|\mathcal{J}u/(t-\lambda)\|^2_{\mathcal{J}}. \tag{4.3}$$

[4]By continuity this extends to all λ outside the spectrum of T.

Equations (4.2) and (4.3) and Parseval's formula applied to the expansion of $\|\mathcal{J}u/(t-\lambda) - \mathcal{J}(R_\lambda u)\|_{\mathcal{J}}^2$ yields zero, thus proving the lemma. □

It is now easy to prove that $\mathcal{J}$ is surjective.

Lemma 4.4. *The Jost transform $\mathcal{H}_1 \to L^2_{\mathcal{J}}$ is surjective and its restriction to $\mathcal{H}$ is unitary.*

Proof. Suppose that $\hat{u} \in L^2_{\mathcal{J}}$ is orthogonal to all Jost transforms $\hat{v}$. Since $\hat{v}(s)/(t-\lambda)$ is also a transform for any non-real λ, we have

$$\langle \hat{u}, \hat{v}/(t-\lambda)\rangle_{\mathcal{J}} = 0$$

for all non-real λ. Thus the Stieltjes transform of the measure $d\,\Xi_{\hat{u},\hat{v}}$ is zero, so by the uniqueness of the Stieltjes transform it follows that this measure is the zero measure, *i.e.*, $\Xi_{\hat{u},\hat{v}}$ is a constant function. We need to prove that this implies that $\hat{u}$ is the zero element of $L^2_{\mathcal{J}}$.

We first consider the eigenvalues. Choosing v as an eigenfunction associated with λ_n, say $f_+(\cdot, k_n)$, we get $\hat{v}_+(k_n) = \|f_+(\cdot, k_n)\|^2 > 0$. Hence $\hat{u}_+(k_n) = 0$. It follows in the same way that $\hat{u}_+(0) = 0$ if q_0 is an eigenvalue.

Since $\mathfrak{T}(s) \neq 0$ for $s \in \mathbb{R} \setminus \{0\}$ we have $\hat{u}\hat{v}^* = 0$ a.e. in $(0, \infty)$ (with respect to Lebesgue measure) for any given fixed compactly supported function $v \in \mathcal{H}_1$ and $\hat{v} = \mathcal{J}v$. Thus there is a set N of zero measure such that $\hat{u}\hat{v}^* = 0$ outside N whenever

$$v = v_j = \min\{1, j(x-a)^+, j(b-x)^+\}$$

where $j \in \mathbb{N}$, $a, b \in \mathbb{Q}$, $a < b$, and the superscript $+$ denotes the positive part of a function. For any fixed $s \notin N$ we get, after an integration by parts, $\hat{v}_\pm(s) = t\int_{\mathbb{R}} wvf_\pm(\cdot, s)$ so that

$$\overline{\hat{u}(s)\hat{v}_j(s)^*} = t\int_{\mathbb{R}} wv_jy = 0$$

where $y(s) = \overline{\hat{u}_+(s)}f_+(\cdot, s) + \overline{\hat{u}_-(s)}f_-(\cdot, s)$. Letting j go to infinity the dominated convergence theorem shows that $\int_a^b wy = 0$ and hence that $y = 0$ on the support of w. Since y is a solution to a linear and homogeneous differential equation it follows that y is identically equal to zero. Now $f_\pm(\cdot, s)$ are linearly independent for real $s \neq 0$, so we obtain $\hat{u}(s) = 0$ for a.a. s. □

Proof of Theorem 4.2, part (4). If I is a compact interval not containing q_0 and $u \in \mathcal{H}_1$ we have

$$E_I u(x) = \langle E_I u, g_0(x, \cdot)\rangle = \langle \chi_I(t)\mathcal{J}u(s), \mathcal{J}(g_0(x, \cdot))(s)\rangle_{\mathcal{J}},$$

and if $u \in \mathcal{H}_1$ is compactly supported Lemma 4.1 gives

$$E_I u(x) = \langle \chi_I(t) \mathcal{J} u(s), F(x,s) \rangle_{\mathcal{J}}.$$

Clearly $\chi_I(t)F(x,s)$ is in $L^2_{\mathcal{J}}$, so the second formula holds in general for compact intervals not containing q_0 by the density in $\mathcal{H}_1$ of compactly supported elements. Comparing the two formulas and using that $\mathcal{J}$ is surjective we therefore obtain $\mathcal{J}(g_0(x,\cdot)) = F(x,\cdot)$ which completes the proof, except if q_0 is an eigenvalue when a similar calculation shows $\mathcal{J}(g_0(x,\cdot))_+(0) = f_0(x)$. □

Remark. Since we have $\mathcal{J}(g_0(x,\cdot)) = F(x,\cdot)$ we obtain from Theorem 2.9 and Lemma 4.3 that $\mathcal{J}(G(x,\cdot,\lambda))(s) = F(x,s)/(t-\lambda)$ and $\mathcal{J}(g(x,\cdot,\lambda)(s) = \lambda F(x,s)/(t-\lambda) + F(x,s) = tF(x,s)/(t-\lambda)$.

Finally, we turn to the remaining part (5) of Theorem 4.2.

Lemma 4.5. *If $u \in D_T$ then $\mathcal{J}(Tu)(k) = \lambda(\mathcal{J}u)(k)$. Conversely, if $\hat{u}$ and $k \mapsto \lambda\hat{u}(k)$ are in $L^2_{\mathcal{J}}$, then $\mathcal{J}^*(\hat{u})$ is in D_T.*

Proof. We have $u \in D_T$ if and only if for some non-real λ and some $v \in \mathcal{H}_1$ we have $u = R_\lambda(v - \lambda u)$. Taking transforms we get $u \in D_T$ if and only if $u \in \mathcal{H}$ and $\hat{u}(s) = (\hat{v}(s) - \lambda\hat{u}(s))/(t-\lambda)$ or $t\hat{u}(s) = \hat{v}(s)$ for some $\hat{v} \in L^2_{\mathcal{J}}$. □

The proof of Theorem 4.2 is now complete. We conclude this section by presenting the inversion formula in a different form. Suppose $(\hat{u}_+, \hat{u}_-) \in L^2_{\mathcal{J}}$. We may write Theorem 4.2(4) as

$$\begin{aligned} u(x) = \sum_n \frac{\hat{u}_+(k_n) f_-(x,k_n)}{\langle f_+(\cdot,k_n), f_-(\cdot,k_n)\rangle} + \hat{u}_+(0)\overline{f_0(x)} \\ + \int_0^\infty (\hat{u}_+(s)\overline{f_+(x,s)} + \hat{u}_-(s)\overline{f_-(x,s)}) \frac{|\mathfrak{T}(s)|^2}{2\pi t}\, ds \end{aligned}$$

where the sum is taken over all n for which $\lambda_n = k_n^2 + q_0$ is an eigenvalue below q_0 and the term containing $\hat{u}_+(0)$ should be dropped unless q_0 is an eigenvalue. The scattering relations (3.10) between $f_\pm(\cdot,k)$ and $\overline{f_\pm(\cdot,k)} = f_\pm(\cdot,-k)$ translate into

$$\mathfrak{T}(k)\hat{u}_\mp(k) = \mathfrak{R}_\pm(k)\hat{u}_\pm(k) + \hat{u}_\pm(-k) \tag{4.4}$$

for real $k \neq 0$. Thus, recalling from (3.21) that

$$i\lambda_n \langle f_+(\cdot,k_n), f_-(\cdot,k_n)\rangle^{-1} = \frac{1}{2k_n} \operatorname{Res}_{\lambda_n} \mathfrak{T} \circ k = \operatorname{Res}_{k_n} \mathfrak{T},$$

and that $\mathfrak{T}\overline{\mathfrak{R}_-} + \overline{\mathfrak{T}}\mathfrak{R}_+ = 0$, we may express the inverse transform by

$$u(x) = \sum_n \hat{u}_+(k_n) f_-(x, k_n) \frac{\operatorname{Res}_{k_n} \mathfrak{T}}{i\lambda_n} + \hat{u}_+(0) f_0(x) + \frac{1}{2\pi} ⨍_{-\infty}^{\infty} \hat{u}_+(s) f_-(x, s) \frac{\mathfrak{T}(s)}{t}\, ds \quad (4.5)$$

where $⨍_{-\infty}^{\infty} f(s)\, ds = \int_0^\infty (f(s) + f(-s))\, ds$ denotes a principal value integral and the term containing $\hat{u}_+(0)$ should be dropped unless q_0 is an eigenvalue.

5. A Paley-Wiener theorem for the Jost transform

In this section we shall prove a Paley-Wiener theorem for the Jost transform, *i.e.*, we relate support properties of $u \in \mathcal{H}$ to growth properties of $\hat{u} = \mathcal{J}u$. *In this section and in Section 6 we shall always assume Assumption 1.3.* In particular q_0 is not an eigenvalue and terms containing $\hat{u}_+(0)$ will be dropped from formulas where they previously occurred.

We shall require the following definition.

Definition 5.1. An element $(\hat{u}_+, \hat{u}_-) \in L^2_{\mathcal{J}}$ is said to be in the class $\mathcal{C}(a, b)$ if it has an extension to the closed upper half k-plane with the following properties:

(1) $\hat{u}_\pm$ are analytic in $\operatorname{Im} k > 0$, continuous in $\operatorname{Im} k \geq 0$ and $\hat{u}_\pm(k)/\lambda$ is continuous at $\lambda = 0$, *i.e.*, at $k = i\sqrt{q_0}$.
(2) If $\lambda_n = q_0 + k_n^2$ is an eigenvalue we have $\hat{u}_-(k_n) = \alpha_n \hat{u}_+(k_n)$ where $f_-(\cdot, k_n) = \alpha_n f_+(\cdot, k_n)$.
(3) For $k \in \mathbb{R} \setminus \{0\}$ we have

$$\hat{u}_\pm(-k) = \mathfrak{T}(k)\hat{u}_\mp(k) - \mathfrak{R}_\pm(k)\hat{u}_\pm(k)$$

(4) There is a constant c_1 such that $|\hat{u}_\pm(k)| \leq e^{c_1|k|}$ for large k with $\operatorname{Im} k \geq 0$.
(5) $\hat{u}_+(k) = o(|\lambda f_+(a, k)|)$ and $\hat{u}_-(k) = o(|\lambda f_-(b, k)|)$ as k^2 tends to infinity on the imaginary axis.

Remark. The requirement (3) involves some redundancy; if the formula is true for the upper sign it will automatically be true for the lower sign and *vice versa* by the standard scattering relations. Alternatively, if both formulas are true for $k > 0$, then they are true in general.

We shall also find the following definition useful.

Definition 5.2.

(1) A *gap* in the support[5] of a function w is a component of the complement of $\operatorname{supp} w$.
(2) For $a \in \mathbb{R}$ we define $a_- = a_-(w)$ and $a_+ = a_+(w)$ as $a_- = a_+ = a$ unless a is in the closure of a gap of $\operatorname{supp} w$, in which case (a_-, a_+) is that gap.

Since $w - 1$ is integrable it follows that the complement of $\operatorname{supp} w$ has finite measure, in particular every gap in the support of w is an open interval of finite length. We can now state our main theorem.

Theorem 5.3. *Suppose q and w satisfy Assumption 1.3 and that $a < b$. Then $\hat{u} \in \mathcal{C}(a, b)$ if the support of $u \in \mathcal{H}_1$ is contained in $[a, b]$.*

Conversely, if $\hat{u} \in \mathcal{C}(a, b)$ then the support of $u = \mathcal{J}^* \hat{u} \in \mathcal{H}$ is contained in $[a_-, b_+]$.

The replacement of a and b by a_- respectively b_+ in the last statement is partly explained by the following lemma.

Lemma 5.4. *Let (a, b) be a gap in the support of w. Then the restriction of any $u \in \mathcal{H}$ to $[a, b]$ is a solution of $-u'' + qu = 0$ uniquely determined by $u(a)$ and $u(b)$.*

Proof. Any $\varphi \in C_0^1(a, b)$ is in $\mathcal{H}_\infty$, so we have $0 = \int_a^b (u'\overline{\varphi'} + qu\overline{\varphi})$. As in the proof of Proposition 2.4 it follows that u' is locally absolutely continuous and $-u'' + qu = 0$ there.

That a solution of $-u'' + qu = 0$ is determined by its values in two different points is an immediate consequence of the fact that, according to Proposition 2.3, no non-trivial solution can have two different zeros. □

It follows that the support of $u \in \mathcal{H}$ can not begin or end inside a gap of $\operatorname{supp} w$.

Proof of Theorem 5.3. Assume first that $\operatorname{supp} u \subset [a, b]$. An integration by parts gives

$$\hat{u}_\pm(k) = \lambda \int_a^b u w f_\pm(\cdot, k),$$

which immediately gives an extension of the domain of $\hat{u}$ to $\overline{\mathbb{C}_+}$ with properties (1) and (2) of Definition 5.2. Property (3) follows from the scattering relations (3.10) and (4) follows from the estimate (3.4). Finally, we have $f_+(x, k) = \lambda f_+(a, k) \frac{f_+(x,k)}{\lambda f_+(a,k)}$ where we denote the last factor by $\psi_{[a,\infty)}(x, \lambda)$, since this is the *Weyl solution* for the left definite

[5]This refers to the essential support of w, *i.e.*, the support in the sense of distributions.

Dirichlet problem on $[a,\infty)$ for the equation (1.1) (see [3]). Thus we have

$$\hat{u}_+(k) = \lambda f_+(a,k)\langle u, \psi_{[a,\infty)}(\cdot,\overline{\lambda})\rangle.$$

It is shown in our paper [3] that if $q \not\equiv 0$ in (a,∞), then $\psi_{[a,\infty)}(\cdot,\lambda) \to 0$ in the appropriate Hilbert space as $k^2 \to \infty$ along the imaginary axis. This proves (5).

An extra argument is needed in the case $q_0 = 0$, in which case we may have $q \equiv 0$ in $[a,\infty)$, so that we do not have a genuine left definite problem on $[a,\infty)$. We note that in this case $\psi_{[a,\infty)}$ may also be considered the Weyl solution for a modified equation where q is the Dirac measure at a, which does give a genuine left definite problem. The scalar product with u is unchanged since u vanishes in $(-\infty,a]$. Although the case when q is a measure is not explicitly considered in [3], it is easy to see that the results remain the same.

Assume now that $\hat{u} \in \mathcal{C}(a,b)$ and define the auxiliary functions

$$A_\pm(x,\lambda) = (R_\lambda u)(x) + \frac{\mathfrak{T}(k)}{2ik\lambda}\hat{u}_\pm(k) f_\mp(x,k). \tag{5.1}$$

We show in the appendix that A_+ for fixed x is entire of order at most $1/2$ as a function of λ, and that it tends to zero as $\lambda - q_0 \to \infty$ along $i\mathbb{R}$ if $x < a$. The Phragmén-Lindelöf theorem then shows that $A_+(x,\cdot)$ is bounded if $x < a$ and thus, by Liouville's theorem, it is constant. The limit along $i\mathbb{R}$ being zero we obtain $A_+(x,\lambda) = 0$ for $x < a$.

Applying the differential equation to A_+ shows that $wu = 0$ in $(-\infty,a)$, so that $u(x) = 0$ except in gaps of $\operatorname{supp} w$ if $x \le a$. Since u vanishes at the endpoints of any gaps contained in $(-\infty,a)$ it follows by Lemma 5.4 that u vanishes in all such gaps. We conclude that $\operatorname{supp} u \subset [a_-,\infty)$.

Similar calculations involving A_- show that $\operatorname{supp} u \subset (-\infty,b_+]$, so that the final conclusion is $\operatorname{supp} u \subset [a_-,b_+]$. $\square$

6. Uniqueness of the inverse scattering problem

Let $\mathrm{Op}(q_0)$ be the class of self-adjoint operators, as defined in Section 2, associated with coefficients satisfying Assumption 1.3 with a given nonnegative number q_0. We will consider two operators in $\mathrm{Op}(q_0)$ which we denote by T and $\breve{T}$. If some entity is associated with T then there is a corresponding entity associated with $\breve{T}$ and we will use the accent $\breve{}$ on the latter one for distinction. In particular, while the coefficients q and w are associated with T the coefficients $\breve{q}$ and $\breve{w}$ are associated with $\breve{T}$ and $\breve{T}$ is an operator in $\breve{\mathcal{H}}$ rather than in $\mathcal{H}$. To avoid cumbersome notation we use $\|\cdot\|$ to denote the norm of either $\mathcal{H}_1$ or $\breve{\mathcal{H}}_1$. It will always be clear from the context which is meant. Similar conventions will be used when we later consider an operator $\tilde{T}$.

The definition of the norm of $L^2_{\mathcal{J}}$ requires knowledge of the constant q_0 which defines the relation between λ and k, the absolute value $|\mathfrak{T}(k)|$ for $k \in \mathbb{R}_+$ of the transmission coefficient, the eigenvalues $\lambda_n = q_0 + k_n^2$ and the numbers $\|f_+(\cdot, k_n)\|$, $n = 1, 2, \ldots$. The latter numbers we call the *norming constants* of T.

Remark. Recall that the eigenvalues of T are determined by the poles of the transmission coefficient $\mathfrak{T}$, which are all simple.

If $\mathfrak{T}$ is known the equation (3.21) shows that if one of $\|f_+(\cdot, k_n)\|$, $\|f_-(\cdot, k_n)\|$ or the proportionality constant α_n in $f_-(\cdot, k_n) = \alpha_n f_+(\cdot, k_n)$ is known, then the two others are determined. Knowing $\mathfrak{T}$ it is therefore immaterial whether we consider the numbers $\|f_+(\cdot, k_n)\|$, $\|f_-(\cdot, k_n)\|$ or α_n as norming constants, and we will use whichever is most convenient in each case.

6.1. **Statement of results.**

The main result of this section is the following theorem.

Theorem 6.1. *Suppose T and $\breve{T}$ in* $\mathrm{Op}(q_0)$ *have the same scattering matrices and norming constants. Then there is a strictly increasing and continuously differentiable function $s : \mathbb{R} \to \mathbb{R}$ such that $r = 1/\sqrt{s'}$ and r' are locally absolutely continuous. Moreover, $s(x) - x$ and $r(x) - 1$ tend to zero as $x \to \pm\infty$ and*

$$\breve{q} \circ s = r^3(-r'' + qr),$$
$$\breve{w} \circ s = r^4 w.$$

Conversely, if the operators T and $\breve{T}$ have coefficients related in this way, then they have the same scattering matrix and norming constants.

It is to be expected that one can not uniquely recover the two coefficients q and w from the scattering data. The following corollaries show how the conclusion may be improved with some *a priori* information on the coefficients.

Corollary 6.2. *Suppose T and $\breve{T}$ in* $\mathrm{Op}(q_0)$ *have the same scattering matrix and norming constants, and that $|\breve{w}| = |w|$. Then $\breve{w} = w$ and $\breve{q} = q$ on the support of w. In particular, if* $\mathrm{supp}(w) = \mathbb{R}$ *or if $q = \breve{q}$ in all gaps of* $\mathrm{supp}\, w$*, then $T = \breve{T}$.*

Corollary 6.3. *Suppose T and $\breve{T}$ in* $\mathrm{Op}(q_0)$ *have the same scattering matrix and norming constants, and that $\breve{q} = q$. Then $T = \breve{T}$,* i.e., *$\breve{w} = w$.*

Recall that for the one-dimensional Schrödinger equation it is customary to use only the reflection coefficient $\mathfrak{R}_+$, the location of the eigenvalues (only finitely many in this case) and the norming constants $\|f_+(\cdot,k_n)\|$ as data.

Since $|\mathfrak{R}_+|^2+|\mathfrak{T}|^2=1$ the absolute value of $\mathfrak{T}(k)$ for real $k\neq 0$ is determined if $\mathfrak{R}_+$ is known, and the poles of $\mathfrak{T}$ are determined by the eigenvalues. In the Schrödinger case it is also known that $\mathfrak{T}(k)\to 1$ as $k\to\infty$ in the closed upper half plane, and altogether this determines $\mathfrak{T}$, and therefore the full scattering matrix, uniquely.

According to Theorem 3.3 we do not have this simple behavior of $\mathfrak{T}$ at infinity in the present case. In the next theorem, where w^+ and w^- denote the positive and negative parts respectively of the function w, we show to what extent $\mathfrak{T}$ is determined by the location of its poles and its absolute value on the real axis.

Theorem 6.4. *Suppose the operators T and $\breve{T}$ have transmission coefficients with the same poles and satisfy $|\mathfrak{T}(k)|=|\breve{\mathfrak{T}}(k)|$ for real $k\neq 0$. Then $\int_{-\infty}^{\infty}(\sqrt{w^-}-\sqrt{\breve{w}^-})=0$ and $\mathfrak{T}(k)=e^{i\alpha k}\breve{\mathfrak{T}}(k)$ for all $k\in\overline{\mathbb{C}_+}$, where*

$$\alpha=\int_{-\infty}^{\infty}(\sqrt{w^+}-\sqrt{\breve{w}^+})=\int_{-\infty}^{\infty}(\sqrt{|w|}-\sqrt{|\breve{w}|}).$$

We also have a sort of converse of this theorem.

Theorem 6.5. *Suppose $T\in\operatorname{Op}(q_0)$ with transmission coefficient $\mathfrak{T}$. Then, given any $\alpha\in\mathbb{R}$, there is another operator $\tilde{T}\in\operatorname{Op}(q_0)$ with transmission coefficient $\tilde{\mathfrak{T}}(k)=e^{i\alpha k}\mathfrak{T}(k)$.*

We may even find continuously differentiable functions σ and ρ with σ strictly increasing and ρ' locally absolutely continuous, $\rho=1/\sqrt{\sigma'}$, $\sigma(x)-x\to 0$ and $\rho(\pm x)\to 1$ as $x\to\infty$ while $\sigma(x)-x\to-\alpha$ as $x\to-\infty$, and such that if the coefficients of $\tilde{T}$ are given by

$$\tilde{q}\circ\sigma=\rho^3(-\rho''+q\rho),$$
$$\tilde{w}\circ\sigma=\rho^4 w,$$

then $\tilde{T}\in\operatorname{Op}(q_0)$, $\tilde{\mathfrak{R}}_+=\mathfrak{R}_+$, $\tilde{\mathfrak{T}}(k)=e^{i\alpha k}\mathfrak{T}(k)$, $\tilde{\mathfrak{R}}_-(k)=e^{2i\alpha k}\mathfrak{R}_-(k)$ and the norming constants satisfy $\|f_+(\cdot,k_n)\|=\|\tilde{f}_+(\cdot,k_n)\|$.

Thus, in addition to $|\mathfrak{T}(k)|$ for real k and the eigenvalues we must know $\int_{\mathbb{R}}(1-\sqrt{w^+})$ for $\mathfrak{T}$ to be determined. Nevertheless, we have the following corollary of Theorem 6.1.

Corollary 6.6. *Suppose $\mathfrak{R}_+=\breve{\mathfrak{R}}_+$, that T and $\breve{T}$ in $\operatorname{Op}(q_0)$ have the same eigenvalues and that $\|f_+(\cdot,k_n)\|=\|\breve{f}_+(\cdot,k_n)\|$ for all eigenvalues $\lambda_n=q_0+k_n^2$. Then there is a strictly increasing and continuously* *differentiable function $s : \mathbb{R} \to \mathbb{R}$ such that $r = 1/\sqrt{s'}$ and r' are locally absolutely continuous. Moreover, $s(x) - x$ and $r(\pm x) - 1$ tend to zero as $x \to \infty$, $s(x) - x \to \int_{\mathbb{R}}(\sqrt{w^+} - \sqrt{\breve{w}^+})$ as $x \to -\infty$ and*

$$\breve{q} \circ s = r^3(-r'' + qr),$$
$$\breve{w} \circ s = r^4 w.$$

Conversely, if the coefficients of the operators T and $\breve{T}$ are related in this way, then $\mathfrak{R}_+ = \breve{\mathfrak{R}}_+$, T and $\breve{T}$ have the same eigenvalues and $\|f_+(\cdot, k_n)\| = \|\breve{f}_+(\cdot, k_n)\|$ for all eigenvalues $\lambda_n = q_0 + k_n^2$.

Note that if we additionally assume that $\int_{\mathbb{R}}(\sqrt{w^+} - \sqrt{\breve{w}^+}) = 0$ we are back to Theorem 6.1. A corresponding result is of course valid if we suppose $\breve{\mathfrak{R}}_- = \mathfrak{R}_-$, that T and $\breve{T}$ have the same eigenvalues and that the norming constants satisfy $\|f_-(\cdot, k_n)\| = \|\breve{f}_-(\cdot, k_n)\|$.

6.2. **Proofs.**

We begin with the easy direction of Theorem 6.1.

Proof of the converse part of Theorem 6.1. To prove this let $f(x, k) = r(x)\breve{f}_+(s(x), k)$. Thus $e^{-ikx} f(x, k)$ is asymptotic to $r(x)e^{ik(s(x)-x)}$ which is asymptotic to 1 as $x \to \infty$. Furthermore it is easily verified that f satisfies $-f'' + qf = \lambda f$ so that $f = f_+$.

Similarly one shows that $r(x)\breve{f}_-(s(x), k) = f_-(x, k)$. It follows that the two equations have the same scattering matrix and thus also the same eigenvalues. If $\lambda_n = q_0 + k_n^2$ is such an eigenvalue and $\breve{f}_-(\cdot, k_n) = \alpha_n \breve{f}_+(\cdot, k_n)$ it follows that $f_-(\cdot, k_n) = \alpha_n f_+(\cdot, k_n)$ so that the two equations also have the same norming constants. $\square$

We next prove Theorem 6.4.

Proof of Theorem 6.4. The function $\mathfrak{T}/\breve{\mathfrak{T}} = [\breve{f}_+, \breve{f}_-]/[f_+, f_-]$ is analytic without zeros in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$, continuity at 0 following from Theorem 3.5. In addition $|\mathfrak{T}/\breve{\mathfrak{T}}| = 1$ on $\mathbb{R}$, so that we may define

$$F(k) = \log(\mathfrak{T}(k)/\breve{\mathfrak{T}}(k)).$$

Then $\operatorname{Re} F$ is continuous in $\overline{\mathbb{C}_+}$, vanishes on $\mathbb{R}$ and F is analytic in $\mathbb{C}_+$. Thus Proposition 1.6 shows that F extends to an entire function. Now, the estimate (3.5) applied to $[\breve{f}_+(\cdot, k), \breve{f}_-(\cdot, k)]$ and Theorem A.1 together with Lemma A.4 show that the entire function e^F is of order ≤ 1. It follows that F is a polynomial of degree ≤ 1. Thus we have $F(k) = i\alpha k + i\beta$ for some constants α, β. For real k this is purely imaginary, so these constants are both real. For purely imaginary k the coefficients $\mathfrak{T}(k)$ and $\breve{\mathfrak{T}}(k)$ are real so we have $e^{i\beta} = \pm 1$.

The functions $f_\pm(\cdot, i\sqrt{q_0})$ (note that $k = i\sqrt{q_0}$ corresponds to $\lambda = 0$) are strictly positive near $\pm\infty$ and $f'_\pm(\cdot, i\sqrt{q_0}) \to 0$ there. From the differential equation $f''_\pm(\cdot, i\sqrt{q_0}) = qf_\pm(\cdot, i\sqrt{q_0})$ we find that $f_\pm(\cdot, i\sqrt{q_0})$ are convex wherever they are positive. This shows that $f_\pm(\cdot, i\sqrt{q_0}) > 0$ and $\mp f'_\pm(\cdot, i\sqrt{q_0}) \geq 0$ throughout $\mathbb{R}$. Since 0 is not an eigenvalue we obtain

$$[f_+(\cdot, i\sqrt{q_0}), f_-(\cdot, i\sqrt{q_0})] > 0,$$

so that $\mathfrak{T}(i\sqrt{q_0})/\breve{\mathfrak{T}}(i\sqrt{q_0})$ is positive and hence $e^{i\beta} = 1$.

Finally, according to Theorem 3.3 we have

$$\mathfrak{T}(k) = \exp(-ik \int_{-\infty}^{\infty} (1 - \sqrt{w}) + o(|k|))$$

as $k \to \infty$ at least along some rays in $\mathbb{C}_+$ from which the value of α given in the theorem and $\int_{-\infty}^{\infty}(\sqrt{\breve{w}^-} - \sqrt{w^-}) = 0$ follow. $\square$

To prove the hard direction of Theorem 6.1 we note that by assumption the spaces $L^2_{\mathcal{J}}$ and $L^2_{\breve{\mathcal{J}}}$ are the same, so that the operator $\mathcal{U} = \breve{\mathcal{J}}^* \circ \mathcal{J} : \mathcal{H} \to \breve{\mathcal{H}}$ is unitary. We shall prove that $\mathcal{U}$ is (the inverse of) a *Liouville transform*. By this we mean a transform $\breve{u} \mapsto u$ given by $u(x) = r(x)\breve{u}(s(x))$ for certain fixed functions r and s. If $r : \mathbb{R} \to \mathbb{C}$ is never zero and $s : \mathbb{R} \to \mathbb{R}$ is strictly increasing and surjective, then the inverse of the map is also a Liouville transform of the same kind, as is the composite of two such maps.

To carry out the proof we require some preparation, and begin by a definition.

Definition 6.7.

(1) Let $\tau(x) = x + \displaystyle\int_x^\infty (1 - \sqrt{|w|})$ and $\breve{\tau}(x) = x + \displaystyle\int_x^\infty (1 - \sqrt{|\breve{w}|})$.

(2) Let $\alpha = \displaystyle\int_{\mathbb{R}} (\sqrt{|w|} - \sqrt{|\breve{w}|})$.

It follows from Theorem 3.3 that as $k^2 \to \infty$ in $i\mathbb{R}$ we have the formulas

$$\begin{aligned}
\log|f_+(x,k)| &= -\tfrac{|k|}{\sqrt{2}}(\tau(x) + o(1)),\\
\log|\breve{f}_+(x,k)| &= -\tfrac{|k|}{\sqrt{2}}(\breve{\tau}(x) + o(1)),\\
\log|f_-(x,k)| &= \tfrac{|k|}{\sqrt{2}}\Big(\tau(x) - \int_{\mathbb{R}}(1 - \sqrt{|w|}) + o(1)\Big),\\
\log|\breve{f}_-(x,k)| &= \tfrac{|k|}{\sqrt{2}}\Big(\breve{\tau}(x) + \alpha - \int_{\mathbb{R}}(1 - \sqrt{|w|}) + o(1)\Big).
\end{aligned} \tag{6.1}$$

Note that if $\mathfrak{T} = \breve{\mathfrak{T}}$, then according to Theorem 6.4 we have $\alpha = 0$. We will need the following simple technical lemma.

Lemma 6.8. *Suppose $u \in \mathcal{H}_1$ has compact support in $[a, \infty)$. Given $\varepsilon > 0$ we may then find $u_\varepsilon \in \mathcal{H}_1$ with compact support in (a, ∞) such that $\|u - u_\varepsilon\| < \varepsilon$. A similar statement is true if* $\operatorname{supp} u \subset (-\infty, a]$.

Proof. For $\delta > 0$ we replace u in the interval $[a, a+\delta]$ by zero and in the interval $[a+\delta, a+2\delta]$ by a linear function with value 0 at $a+\delta$ and $u(a+2\delta)$ at $a+2\delta$.

The modified function $\tilde{u}$ vanishes in $[a, a+\delta]$, is $(x-a-\delta)u(a+2\delta)/\delta$ in $[a+\delta, a+2\delta]$ and then equals u. Furthermore $u(a) = 0$ so

$$|u(a+2\delta)|^2 = \Big| \int_a^{a+2\delta} u' \Big|^2 \leq 2\delta \int_a^{a+2\delta} (|u'|^2 + q|u|^2) = o(\delta)$$

as $\delta \to 0$. The modification $\tilde{u} - u$ therefore has norm-square

$$\int_a^{a+2\delta} (|u' - \tilde{u}'|^2 + q|u - \tilde{u}|^2) \\ \leq 2 \int_a^{a+2\delta} (|u'|^2 + q|u|^2) + 2 \int_{a+\delta}^{a+2\delta} (|\tilde{u}'|^2 + q|\tilde{u}|^2).$$

The first term is $o(1)$, and so is the second since it is equal to

$$2\Big|\frac{u(a+2\delta)}{\delta}\Big|^2 \Big(\delta + \int_{a+\delta}^{a+2\delta} q|x-a-\delta|^2\Big) \leq \Big(1 + \delta \int_{a+\delta}^{a+2\delta} q\Big) o(1).$$

Thus the norm of the modification is arbitrarily small if δ is sufficiently small, and the modified function has support in (a, ∞).

The second statement is proved similarly. $\square$

We now prove a lemma establishing a connection between the supports of u and $\mathcal{U}u$.

Lemma 6.9. *Suppose $u \in \mathcal{H}$ has compact support in $[a, \infty)$ and suppose $\tau(a) = \breve{\tau}(\breve{a})$. Then* $\operatorname{supp} \mathcal{U}u \subset [\breve{a}_-(\breve{w}), \infty)$. *If $a = a_+(w)$ we even have* $\operatorname{supp} \mathcal{U}u \subset [\breve{a}_+(\breve{w}), \infty)$.

Similarly, if $\operatorname{supp} u \subset (-\infty, a]$ *we have* $\operatorname{supp} \mathcal{U}u \subset (-\infty, \breve{a}_+(\breve{w})]$, *and if $a = a_-(w)$ we even have* $\operatorname{supp} \mathcal{U}u \subset (-\infty, \breve{a}_-(\breve{w})]$.

Proof. We have $f_+(a, k) = o(|\breve{f}_+(\breve{a}_- - \varepsilon, k)|)$ for every $\varepsilon > 0$ by (6.1) so that, by the Paley-Wiener theorem 5.3, the support of $\mathcal{U}u$ is contained in $[\breve{a}_- - \varepsilon, \infty)$ for every $\varepsilon > 0$ and thus in $[\breve{a}_-, \infty)$.

Now, if $a = a_+$ Lemma 6.8 shows that we may, given $\varepsilon > 0$, find a compactly supported $u_\varepsilon \in \mathcal{H}_1$ with $\operatorname{supp} u_\varepsilon \subset (a_+, \infty)$ and $\|u - u_\varepsilon\| < \varepsilon$. Again by (6.1) $f_+(a+\delta, k) = o(|\breve{f}_+(\breve{a}_+, k)|)$ for every $\delta > 0$, and since

$(a+\delta)_- \geq a_+$ we have $\operatorname{supp} \mathcal{U}u_\varepsilon \subset [\breve{a}_+, \infty)$ which implies $\mathcal{U}u_\varepsilon(\breve{a}_+) = 0$. Since $\mathcal{U} : \mathcal{H}_1 \to \breve{\mathcal{H}}$ and point evaluations are continuous it follows that $\breve{u} = \mathcal{U}u$ vanishes at $\breve{a}_+$ and therefore throughout $[\breve{a}_-, \breve{a}_+]$, so $\operatorname{supp} \breve{u} \subset [\breve{a}_+, \infty)$.

The proof of the second part of the lemma is similar. □

The next lemma establishes the existence of the Liouville transform outside gaps of $\operatorname{supp} w$.

Lemma 6.10. *Suppose $\tau(a) = \breve{\tau}(\breve{a})$. Then $a_-(w) = a_+(w)$ if and only if $\breve{a}_-(\breve{w}) = \breve{a}_+(\breve{w})$. Furthermore, the equation $\tau(a) = \breve{\tau}(\breve{a})$ defines a strictly increasing and surjective function $s : \operatorname{supp} w \ni a \mapsto \breve{a} \in \operatorname{supp} \breve{w}$ such that there exists a non-vanishing function r defined in $\operatorname{supp} w$ with the property $u = r\breve{u} \circ s$ in $\operatorname{supp} w$ for every $u \in \mathcal{H}$, where $\breve{u} = \mathcal{U}u$.*

Proof. Suppose $a_- = a_+$ and let $v \in \mathcal{H}$ have compact support with $v(a) = 1$. We may then for any compactly supported $u \in \mathcal{H}$ write $u = u(a)v + u_+ + u_-$ where $u_\pm \in \mathcal{H}$ have compact supports with $\operatorname{supp} u_+ \subset [a, \infty)$, $\operatorname{supp} u_- \subset (-\infty, a]$. According to Lemma 6.9 we then have

$$\operatorname{supp} \breve{u}_+ \subset [\breve{a}_+, \infty) \quad \text{and} \quad \operatorname{supp} \breve{u}_- \subset (-\infty, \breve{a}_-],$$

so that the restriction of $\breve{u}$ to $[\breve{a}_-, \breve{a}_+]$ is $u(a)\breve{v}$. By the density of compactly supported elements in $\mathcal{H}$ this remains true for all $u \in \mathcal{H}$, so these restrictions span a one-dimensional space. But if $\breve{a}_- < \breve{a}_+$ this contradicts the fact that $\mathcal{U}$ is surjective, since then the corresponding space must have dimension two. Together with similar considerations involving $\mathcal{U}^{-1}$ this proves the first claim and the statement about s except at endpoints of gaps. It is also clear, again since $\mathcal{U}$ is surjective, that $\breve{v}(s(a)) \neq 0$ because not all elements of $\breve{\mathcal{H}}$ vanish in $s(a)$. We therefore set $r(a) = \breve{v}(s(a))$.

It remains to consider the case when a is an endpoint of a gap, say $a = a_-$. We may as before write $u = u(a)v + u_+ + u_-$, and then obtain, using Lemma 6.9, that $\breve{u}(\breve{a}_-) = u(a)\breve{v}(\breve{a}_-)$, so that we define $s(a_-) = \breve{a}_-$ and $r(a_-) = \breve{v}(\breve{a}_-)$. Similarly, if $a = a_+$ we define $s(a_+) = \breve{a}_+$ and $r(a_+) = \breve{v}(\breve{a}_+)$ which completes the proof. □

Lemma 6.10 is the most important step in the proof of Theorem 6.1. Note that gaps in $\operatorname{supp} w$ correspond to gaps in the domain of s while gaps in $\operatorname{supp} \breve{w}$ correspond to gaps in the range of s, and that these gaps are in a one-to-one correspondence.

It remains to define the Liouville transform in each gap.

Lemma 6.11. *Suppose (a_-, a_+) is a gap in* $\operatorname{supp} w$*. Then there is a Liouville transform mapping restrictions to $[s(a_-), s(a_+)]$ of elements $\breve{u} \in \breve{\mathcal{H}}$ to the restrictions to $[a_-, a_+]$ of the pre-images $u = \mathcal{U}^{-1}\breve{u}$.*

Proof. Let φ and θ be solutions of $-f'' + qf = 0$ with $\varphi(a_-) = 1$, $\varphi(a_+) = 0$ and $\theta(a_-) = 0$, $\theta(a_+) = 1$. Such solutions exist, are a basis for the solutions and have no zeros or zeros of their derivatives in (a_-, a_+) as shown in Proposition 2.3. Thus $\varphi' < 0$ and $\theta' > 0$ throughout $[a_-, a_+]$ so that $[\varphi, \theta] > 0$.

Now let $\breve{\varphi}$ and $\breve{\theta}$ be the analogous solutions of $-f'' + \breve{q}f = 0$ in $[s(a_-), s(a_+)]$. It is clear that $\breve{\varphi}/r(a_-)$ and $\breve{\theta}/r(a_+)$ are images under $\mathcal{U}$ of φ and θ respectively, in the following sense: Any element of $\mathcal{H}$ whose restriction to $[a_-, a_+]$ is φ is mapped to an element of $\breve{\mathcal{H}}$ whose restriction to $[s(a_-), s(a_+)]$ is $\breve{\varphi}/r(a_-)$, and similar for θ.

If we extend φ by 0 in $[a_+, \infty)$ and θ by 0 in $(-\infty, a_-]$, the images will have analogous properties. The scalar product of these extensions of φ and θ is $\int_{a_-}^{a_+}(\varphi'\overline{\theta'} + q\varphi\overline{\theta}) = \varphi'(a_+) < 0$. Since $\mathcal{U}$ is unitary this is equal to $(r(a_-)\overline{r(a_+)})^{-1}\int_{s(a_-)}^{s(a_+)}(\breve{\varphi}'\overline{\breve{\theta}'} + \breve{q}\breve{\varphi}\overline{\breve{\theta}}) = (r(a_-)\overline{r(a_+)})^{-1}\breve{\varphi}'(s(a_+))$, which is therefore negative. Thus we have $r(a_-)\overline{r(a_+)} > 0$ or $r(a_-)/r(a_+) > 0$.

We now need to define s and r so that $r(x)\breve{\varphi}(s(x))/r(a_-) = \varphi(x)$ and $r(x)\breve{\theta}(s(x))/r(a_+) = \theta(x)$ for $x \in [a_-, a_+]$. The requirements are equivalent to the equations

$$r(a_-)\breve{\theta}(s(x))/(\breve{\varphi}(s(x))r(a_+)) = \theta(x)/\varphi(x)$$
$$r(x) = r(a_-)\varphi(x)/\breve{\varphi}(s(x)).$$

Now, we saw above that $[\varphi, \theta] > 0$, and similarly $[\breve{\varphi}, \breve{\theta}] > 0$. Differentiating we obtain $(\theta/\varphi)' = [\varphi, \theta]/\varphi^2 > 0$, so θ/φ is strictly increasing with range $[0, \infty]$, and so is $\breve{\theta}/\breve{\varphi}$. Since also $r(a_-)/r(a_+) > 0$ the first equation defines s uniquely as a strictly increasing function mapping $[a_-, a_+]$ onto $[s(a_-), s(a_+)]$, so that r is uniquely defined by the second equation. ☐

We can now finish the proof of Theorem 6.1.

Proof of hard direction of Theorem 6.1. We have already defined s and r everywhere, and it only remains to prove the regularity and asymptotic properties of r and s, and the formulas for the coefficients.

The function s is strictly increasing and maps $\mathbb{R}$ onto $\mathbb{R}$ and is therefore continuous. We have a Liouville transform such that $u = r\breve{u} \circ s$, where $\breve{u} = \mathcal{U}u$, for every $u \in \mathcal{H}$. Thus, for every $\hat{u} \in L^2_{\mathcal{J}}$,

$$\langle \hat{u}, F(x, \cdot)\rangle_{\mathcal{J}} = u(x) = r(x)\breve{u}(s(x)) = r(x)\langle \hat{u}, \breve{F}(s(x), \cdot)\rangle_{\mathcal{J}}.$$

It follows that $F(\cdot,k) = r\breve{F}(s(\cdot),k)$, first for $k>0$ and then by unique analytic continuation (Proposition 1.6) in general. Now $f_+(\cdot,i\sqrt{q_0})$ and $\breve{f}_+(\cdot,i\sqrt{q_0})$ are positive, so that r is realvalued and strictly positive.

The function s is continuous and strictly increasing so $\breve{f}_\pm(s(x),i\sqrt{q_0})$ are continuous, of locally bounded variation and never vanish. Thus also r is continuous, of locally bounded variation and never vanishes.

Since $\mathfrak{T} = \breve{\mathfrak{T}}$, *i.e.* $[f_+,f_-] = [\breve{f}_+,\breve{f}_-]$, a simple calculation shows that the measure $r^2 ds$ is Lebesgue measure. Thus s is locally absolutely continuous and $r^2 s' = 1$. Since $r = f_+(\cdot,i\sqrt{q_0})/\breve{f}_+(s(\cdot),i\sqrt{q_0})$ also r is locally absolutely continuous.

Differentiating $f_+(x,k) = r(x)\breve{f}(s(x),k)$ gives

$$\begin{aligned} f_+'(x,k) &= r(x)s'(x)\breve{f}_+'(s(x),k) + r'(x)\breve{f}_+(s(x),k)\\ &= (r(x))^{-1}\breve{f}_+'(s(x),k) + r'(x)\breve{f}_+(s(x),k). \end{aligned}$$

Here the left hand side and the first term to the right are locally absolutely continuous, as is $\breve{f}_+(s(x),k)$. It follows that also r' is locally absolutely continuous. Differentiating again we obtain

$$\begin{aligned} (q-\lambda w)f_+ = f_+'' &= (r^{-1}\breve{f}_+'\circ s + r'\breve{f}_+\circ s)'\\ &= r^{-1}s'\breve{f}_+''\circ s + (-r'r^{-2} + r's')\breve{f}_+'\circ s + r''\breve{f}_+\circ s\\ &= (r^{-3}(\breve{q}\circ s - \lambda\breve{w}\circ s) + r'')\breve{f}_+\circ s\\ &= (r^{-4}(\breve{q}\circ s - \lambda\breve{w}\circ s) + r''/r)f_+. \end{aligned}$$

Since this is true for many λ we obtain

$$\breve{q}\circ s = r^3(-r'' + qr), \quad \breve{w}\circ s = r^4 w.$$

We also have

$$e^{\pm ikx} \sim f_\pm(x,k) = r(x)\breve{f}_\pm(s(x),k) \sim r(x)e^{\pm iks(x)}$$

as $x\to\pm\infty$, so that $r(x)e^{\pm ik(s(x)-x)} \to 1$ as $x\to\pm\infty$. Since this is true for many k we find that $s(x)-x\to 0$ and $r(x)\to 1$ as $x\to\pm\infty$. This finishes the proof. $\square$

We now turn to the corollaries.

Proof of Corollary 6.2. By assumption $\tau = \breve{\tau}$ so that $s(x) = x$ and thus $r=1$ on $\operatorname{supp} w$ from which the claim immediately follows. $\square$

Proof of Corollary 6.3. For $\lambda = 0$, *i.e.*, for $k = i\sqrt{q_0}$ the functions $f_\pm(\cdot,i\sqrt{q_0})$ and $\breve{f}_\pm(\cdot,i\sqrt{q_0})$ are realvalued, satisfy the same equation and have the same asymptotic behavior at $\pm\infty$ and are therefore equal. But, by the proof of Theorem 6.1, we also have $f_\pm(x,i\sqrt{q_0}) =$

$r(x)\breve{f}_\pm(s(x), i\sqrt{q_0})$. Thus, setting $E = f_-(\cdot, i\sqrt{q_0})/f_+(\cdot, i\sqrt{q_0})$, we get $E(s(x)) = E(x)$. Now $E' = [f_+(\cdot, i\sqrt{q_0}), f_-(\cdot, i\sqrt{q_0})]/f_+(\cdot, i\sqrt{q_0})^2$ where the Wronskian is constant and, since $\lambda = 0$ is not an eigenvalue, non-zero. Thus E is strictly monotone. It follows that $s(x) = x$ and therefore $r = 1$ and $\breve{w} = w$. □

Proof of Theorem 6.5. We consider the operator $\tilde{T}$ defined as in the statement of the theorem, where we additionally require of ρ that $1-\rho$ and $(1+|x|)\rho''(x)$ are integrable, in order that $\tilde{T} \in \operatorname{Op}(q_0)$. We may for example choose

$$\sigma(x) = x - \alpha(1 - \tfrac{2}{\pi}\arctan(\beta x))/2,$$

where $\beta > 0$ and sufficiently small.

Now suppose $f = \rho\tilde{f}_+(\sigma(\cdot), k)$. It is then easily verified that f satisfies $-f'' + qf = \lambda w f$ and

$$e^{-ikx} f(x) \sim \rho(x) e^{ik(\sigma(x)-x)} \sim 1$$

as $x \to \infty$, so that $f_+(\cdot, k) = \rho\tilde{f}_+(\sigma(\cdot), k)$. Similarly we find that $f_-(\cdot, k) = e^{i\alpha k}\rho\tilde{f}_-(\sigma(\cdot), k)$. It follows that the scattering matrix of $\tilde{T}$ is given by $\tilde{\mathfrak{R}}_+ = \mathfrak{R}_+$, $\tilde{\mathfrak{T}}(k) = e^{i\alpha k}\mathfrak{T}(k)$ and $\tilde{\mathfrak{R}}_-(k) = e^{2i\alpha k}\mathfrak{R}_-(k)$.

If $\lambda_n = q_0 + k_n^2$ and $f_-(\cdot, k_n) = \alpha_n f_+(\cdot, k_n)$ we obtain $\tilde{f}_-(\cdot, k_n) = e^{-i\alpha k_n}\alpha_n\tilde{f}_+(\cdot, k_n)$, and since $\tilde{\mathfrak{T}}(k) = e^{i\alpha k}\mathfrak{T}(k)$ we have

$$\langle f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle = e^{i\alpha k_n}\langle \tilde{f}_+(\cdot, k_n), \tilde{f}_-(\cdot, k_n)\rangle$$

because of (3.21). Thus

$$\begin{aligned}\|\tilde{f}_+(\cdot, k_n)\|^2 &= \frac{e^{i\alpha k_n}}{\alpha_n}\langle \tilde{f}_+(\cdot, k_n), \tilde{f}_-(\cdot, k_n)\rangle \\ &= \frac{1}{\alpha_n}\langle f_+(\cdot, k_n), f_-(\cdot, k_n)\rangle = \|f_+(\cdot, k_n)\|^2.\end{aligned}$$

□

Proof of Corollary 6.6. Assuming $\breve{\mathfrak{R}}_+ = \mathfrak{R}_+$ we obtain $|\breve{\mathfrak{T}}(k)| = |\mathfrak{T}(k)|$ for real k, and if the eigenvalues of $\breve{T}$ and T are the same we see from Theorem 6.4 that $\mathfrak{T}(k) = e^{i\alpha k}\breve{\mathfrak{T}}(k)$ where $\alpha = \int_{\mathbb{R}}(\sqrt{w^+} - \sqrt{\breve{w}^+})$.

We now define the operator $\tilde{T}$ as in Theorem 6.5 so that $\tilde{\mathfrak{R}}_+ = \mathfrak{R}_+$, $\tilde{\mathfrak{T}}(k) = e^{i\alpha k}\mathfrak{T}(k)$ and $\|\tilde{f}_+(\cdot, k_n)\| = \|f_+(\cdot, k_n)\|$. It follows that $\tilde{T}$ and $\breve{T}$ have the same scattering matrix and norming constants.

Applying Theorem 6.1 we find a Liouville transform taking $\breve{T}$ into $\tilde{T}$, and composing this with the transform constructed above, which takes $\tilde{T}$ into T, we obtain a Liouville transform taking $\breve{T}$ into T which is easily seen to have the properties stated.

The converse is proved by similar calculations as those in the proof of Theorem 6.5. $\square$

7. Application to the Camassa-Holm equation

The Camassa-Holm equation is

$$\psi_t - \psi_{txx} - 2\kappa\psi_x + 3\psi\psi_x = 2\psi_x\psi_{xx} + \psi\psi_{xxx},$$

where κ is a constant which is either zero or can be normalized to one by an appropriate scaling (setting $\psi(x,t) = \kappa\tilde{\psi}(x,\kappa t)$)[6]. Henceforth we will assume $\kappa = 1$. If one introduces $w = \psi_{xx} - \psi + \kappa$ the Camassa-Holm equation may be written more concisely as

$$w_t + 2\psi_x w + \psi w_x = 0 \quad\text{or}\quad \psi w_t + (\psi^2 w)_x = 0. \tag{7.1}$$

Associated with the Camassa-Holm equation is the left definite problem

$$-u_{xx} + \tfrac{1}{4}\,u = \lambda w u. \tag{7.2}$$

We now assume that w solves the Camassa-Holm equation, which we will consider on the whole real line, for solutions ψ which decay at infinity. We will assume that the decay is such that $w - 1 \in L^1(\mathbb{R})$. We are then in a position to discuss scattering for (7.2).

The Jost solutions of (7.2) will now also depend on time; thus the transmission and reflection coefficients, as well as the eigenvalues and the corresponding normalization constants all must be expected to depend on t. The time evolution of all these quantities is given by the following theorem; see Constantin [16].

Theorem 7.1. *The evolution of scattering data for the equation* (7.2) *when the weight w satisfies the Camassa-Holm equation is the following.*

(i) $\mathfrak{T}(k;t) = \mathfrak{T}(k;0)$,

(ii) $\mathfrak{R}_+(k;t) = e^{ikt/\lambda}\mathfrak{R}_+(k;0)$,

(iii) $\mathfrak{R}_-(k;t) = e^{-ikt/\lambda}\mathfrak{R}_-(k;0)$,

(iv) *Eigenvalues are constants of the motion.*

Moreover, if $\lambda_n = k_n^2 + 1/4$ is an eigenvalue and $\alpha_n(t)$ the proportionality constant in $f_-(\cdot,k_n;t) = \alpha_n(t) f_+(\cdot,k_n;t)$, then we have the following relationships.

(v) $\alpha_n(t) = e^{ik_n t/\lambda_n}\alpha_n(0)$,

(vi) $\|f_+(\cdot,k_n;t)\|^2 = e^{-ik_n t/\lambda_n}\|f_+(\cdot,k_n;0)\|^2$,

[6]One may also scale κ to 0 by setting $\psi(x,t) = \tilde{\psi}(x-\kappa t,t) + \kappa$, but note that if ψ decays at infinity then $\tilde{\psi}$ does not, so this is not very useful.

(vii) $\|f_-(\cdot, k_n; t)\|^2 = e^{ik_n t/\lambda_n} \|f_-(\cdot, k_n; 0)\|^2$.

The simplicity of the time evolution of the scattering data displayed by this theorem is of great significance for the solution of the Cauchy problem for the Camassa-Holm equation: Given an initial condition $w_0 = w(\cdot; 0)$ one investigates the scattering problem for the equation (7.2) with $w = w_0$. Next one evolves the scattering data as prescribed by Theorem 7.1 and, given the new scattering data, one solves the inverse scattering problem for equation (7.2) to obtain a $w(\cdot; t)$ which is the solution of the Camassa-Holm equation (7.1) evaluated[7] at time t. The process is summarized in the following commutative diagram.

$$\begin{array}{ccc} w(\cdot;0) & \xrightarrow{\textit{scattering}} & \text{scattering data} \\ \downarrow & & \downarrow \\ w(\cdot;t) & \xleftarrow{\textit{inverse scattering}} & \text{evolved scattering data} \end{array}$$

Appendix A. Technical details

We will here show that the auxiliary function $A_+(x, \lambda)$ of (5.1) for fixed x and under Assumption 1.3 is an entire function. We will also estimate the growth of $A_+(x, \cdot)$ at infinity. In the process we will also obtain estimates of $[f_+, f_-]$ which are needed to prove Theorem 6.4. Entirely similar considerations show analogous properties for $A_-(x, \lambda)$ so we give no details here.

Thinking of x as fixed we shall throughout abbreviate $A_+(x, \cdot)$ by A. We will need the following theorem.

Theorem A.1. *Suppose f is analytic for $|z| \geq 1$ and R_j is an increasing sequence of numbers (larger than 1) tending to infinity such that R_{j+1}/R_j remains bounded. Furthermore, suppose there are numbers n, $\alpha \geq 0$, and $C > 0$ such that*

$$|\operatorname{Im}(z) f(z)| \leq |z|^n \exp(C|z|^\alpha)$$

for all z lying on any of the circles $|z| = R_j$.

Then the following two statements are true:

1. *The entire part of f has growth order at most α.*
2. *If $\alpha = 0$, then the entire part of f is a polynomial of degree $< n$.*

Proof. We have $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ with

$$a_k = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz$$

[7] Note that given w there is only one decaying solution ψ of $-\psi'' + \psi = w - 1$.

and hence, using that $2i\,\mathrm{Im}\, z = z - \overline{z} = z - R^2/z$ if $|z| = R$,

$$\frac{1}{2\pi i}\int_{|z|=R} \frac{2i\,\mathrm{Im}(z) f(z)}{z^k}\, dz = a_{k-2} - R^2 a_k.$$

This expression may be estimated with the aid of our assumption to yield

$$|a_k| \le R_j^{-2}|a_{k-2}| + 2R_j^{n-1-k} \exp(CR_j^\alpha).$$

If $\alpha = 0$ and $k > n-1$ this shows that $a_k = 0$ proving our second claim.

Now suppose $\alpha > 0$. After enlarging C we may assume that $n = 1$ and that $R_j \le \exp(CR_j^\alpha)$. It follows then by induction that, for any fixed $k \in \mathbb{N}$, the inequality

$$|a_k| \le (k+1)BR^{-k}\exp(CR^\alpha) \tag{A.1}$$

holds whenever R is any of the numbers R_j and where B is the largest of the numbers 1, $|a_0|$, and $|a_1|/2$. If we choose j so that

$$R_{j-1}^\alpha \le \frac{k}{C\alpha} \le R_j^\alpha,$$

we have

$$|a_k| \le (k+1)BR_j^{-k}\exp(CR_j^\alpha) \le (k+1)B\left(\frac{k}{C\alpha}\right)^{-k/\alpha}\exp(\beta^\alpha k/\alpha)$$

if $\beta \ge R_j/R_{j-1}$. Since the growth order of an entire function is determined through its Taylor coefficients by (see, *e.g.*, Bieberbach [9], Levin [26], or Markushevich [27])

$$\limsup_{k\to\infty} \frac{k\log k}{-\log|a_k|}$$

it follows that the growth order of the entire part of f is at most α. □

We now turn to the auxiliary function

$$A(\lambda) = A_+(x,\lambda) = (R_\lambda u)(x) + \frac{\mathfrak{T}(k)}{2ik\lambda}\hat{u}_+(k) f_-(x,k).$$

Lemma A.2. *Under Assumption 1.3 the auxiliary function A is entire.*

Proof. Clearly A is analytic away from the real axis. We investigate the intervals $(-\infty, q_0)$ and $[q_0,\infty)$ separately. Starting with the former we note that A extends meromorphically to this interval and that possible poles may occur only at the eigenvalues of T (which correspond to the poles of $\mathfrak{T}$). Now consider any such eigenvalue $\lambda_n = k_n^2 + q_0$ and recall that λ_n cannot be zero. According to Theorem 2.9 we have $(R_\lambda u)(x) = -\mathfrak{T}(k)h(x,k)/(2ik\lambda) - u(x)/\lambda$ where $\lambda = k^2 + q_0$ and

$$h(x,k) = f_+(x,k)\langle u, \overline{f_-(\cdot,k)}\rangle^x + f_-(x,k)\langle u, \overline{f_+(\cdot,k)}\rangle_x.$$

Since $f_-(\cdot,k_n) = \alpha_n f_+(\cdot,k_n)$ we see that $h(x,k_n) = f_-(x,k_n)\hat{u}_+(k_n)$. Thus $A(\lambda) + u(x)/\lambda$ is a product of two factors one of which has a simple pole at λ_n while the other has a zero there making the singularity removable. Consequently A is analytic in $\mathbb{C}\setminus[q_0,\infty)$.

In the interval $[q_0,\infty)$ the function A is *a priori* undefined. We shall first show that A may be continued analytically across (q_0,∞). In order to see this we use Theorem 4.3 and the inverse transform (cf. equation (4.5)) to obtain

$$R_\lambda u(x) = \sum_n \frac{-i\hat{u}_+(k_n)f_-(x,k_n)\operatorname{Res}_{k_n}\mathfrak{T}}{\lambda_n(\lambda_n-\lambda)} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\hat{u}_+(s)f_-(x,s)\mathfrak{T}(s)}{t(t-\lambda)}\,ds,$$

if $\operatorname{Im}(\lambda)\neq 0$.

Now let J be a compact subinterval of (q_0,∞), corresponding to two intervals $\pm I$ in the k-plane; we have $0\notin I$. We modify the path of integration by replacing $\pm I$ by half-circles in the upper half plane. We denote the resulting contour by γ. By the residue theorem the change in the integral equals the negative of the second term in A if $\lambda\in\Lambda$, the neighborhood of the interior of J corresponding to k below γ. Thus for $\lambda\in\Lambda$

$$A(\lambda) = \sum_n \frac{-i\hat{u}_+(k_n)f_-(x,k_n)\operatorname{Res}_{k_n}\mathfrak{T}}{\lambda_n(\lambda_n-\lambda)} + \int_\gamma \frac{\hat{u}_+(s)f_-(x,s)\mathfrak{T}(s)}{2\pi t(t-\lambda)}\,ds,$$

which is clearly analytic in Λ. Thus A extends analytically to $\mathbb{C}\setminus\{q_0\}$.

Finally we show that q_0 is a removable singularity of A. To this end consider the function $f(z) = A(q_0+1/z)$. We will show first that f has, at worst, a simple pole at infinity by showing that $(\operatorname{Im} z)f(z)$ may be bound by a multiple of $|z|^2$ and then calling on Theorem A.1.

Calling on Lemma 2.1, we find that

$$|(R_\lambda u)(x)| \le C_{\{x\}}\|R_\lambda\|\|u\| \le C_{\{x\}}\|u\|/|\operatorname{Im}\lambda|.$$

Setting $\lambda = q_0 + 1/z$ this shows that $|\operatorname{Im}(z)(R_\lambda u)(x)| \le C_{\{x\}}\|u\||z|^2$ which gives an appropriate estimate for the first term of $\operatorname{Im}(z)f(z)$.

Looking now at the second term we find that $\hat{u}_+(k)/\lambda$, $f_-(x,k)$ and $[f_+,f_-](k)$ all have limits as $\lambda\to q_0$. If $[f_+,f_-](0)\neq 0$, the second term of A is bounded near q_0. If, however, $[f_+,f_-](0)=0$, we know from Theorem 3.5 that $[f_+, f_-](k)/k$ stays away from zero so that the second term of $\mathrm{Im}(z)f(z)$ behaves like $\mathcal{O}(|z|^{3/2}) = o(|z|^2)$.

Now that we know that f has at worst a simple pole at infinity we approach infinity along the imaginary axis where $|\,\mathrm{Im}(z)| = |z|$. But since q_0 is not an eigenvalue we have in fact $|(R_\lambda u)(x)| = o(|z|)$ and hence $f(z) = o(|z|)$ as $z \to \infty$ along $i\mathbb{R}$ so that the pole is actually removable. □

To estimate the growth of A at infinity we need the following lemma, which is Theorem 11 in Levin [26]. It relies on a result of Cartan [12] pertaining to the analogous question for polynomials.

Lemma A.3. *Let h be a holomorphic function in the disk $|z| \le 2\mathrm{e}R$ with $h(0) = 1$ and let η be an arbitrary positive number not exceeding $3\mathrm{e}/2$. Then, inside the disk $|z| \le R$, but outside of a family of excluded disks the sum of whose radii is not greater than $4\eta R$, we have*

$$\log|h(z)| \ge -H(\eta)\log M_h(2\mathrm{e}R)$$

where

$$H(\eta) = 2 + \log(\frac{3\mathrm{e}}{2\eta})$$

and

$$M_h(2\mathrm{e}R) = \max\{|h(z)| : |z| \le 2\mathrm{e}R\}.$$

We shall use the lemma to prove the following crucial fact.

Lemma A.4. *There is a strictly increasing sequence of positive numbers r_j and a positive number c_0 such that r_{j+1}/r_j remains bounded and*

$$|[f_+(\cdot,k), f_-(\cdot,k)]| \ge 2|k|e^{-2c_0|k|}$$

whenever k is on any of the semicircles of radius r_j.

Proof. We shall use the abbreviation $W(k) = [f_+(\cdot,k), f_-(\cdot,k)]$. Corollary 3.4 provides such an estimate as long as $|\,\mathrm{Re}\,k| \ge \delta|k|$ when δ is a fixed positive number (smaller than 1) — even on *all* semicircles. It remains to establish such a bound for k in the complementary sector about the imaginary axis. From the possible presence of zeros of W on the imaginary axis arises the necessity to restrict ourselves to certain circles avoiding these points.

For $\delta = 1/25$ (or smaller, but positive, if you like) let S'_δ be the sector $\{k \in \mathbb{C} : \mathrm{Im}\,k > 0, |\,\mathrm{Re}\,k| \le \delta|k|\}$. Let k_0 be any point on the right boundary of this sector and $k_j = 2^j k_0$. Let $B_j(r)$ be the disk $|k - k_j| \le r$. We note that the disk $B_j(4\mathrm{e}\delta|k_j|)$ lies entirely above the line $\mathrm{Im}\,k = |k_j|/2$ while the disk $B_j(2\delta|k_j|)$ intersects the line $\mathrm{Re}\,k =$

$-\delta|k|$ (the left boundary of S'_δ) in the points $-\overline{k_j}$ and $-\overline{k_j}(1-4\delta^2)$. The situation is sketched in Figure 1 except that there δ is chosen as 1/3 in order to be able to distinguish the intersection points on the left.

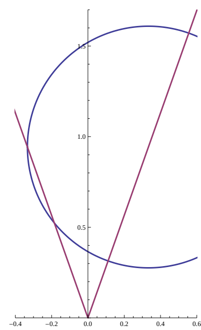

FIGURE 1. The disk $B_0(2\delta|k_0|)$ and the lines $\pm\operatorname{Re} k = \delta|k|$ for $\delta = 1/3$ and $|k_0| = 1$

Define $h(k) = W(k)/W(k_j)$. The function h is analytic in $B_j(4\mathrm{e}\delta|k_j|)$ and satisfies $h(k_j) = 1$. We also know from Lemma 3.2 and Corollary 3.4 that there is a number c', independent of k_j, such that

$$|h(k)| \leq e^{c'|k_j|}$$

as long as $k \in B_j(4\mathrm{e}\delta|k_j|)$. By Lemma A.3 we have for any $\eta \in (0,1)$ that

$$\log|h(k)| \geq -(2-\log\eta)c'|k_j|$$

provided $k \in B_j(2\delta|k_j|)$ but outside a family of excluded disks the sum of whose diameters is not greater then $24\mathrm{e}\eta\delta|k_j|$. Thus, if we choose η smaller than $\delta/(6e)$ and set $c = (2-\log\eta)c'$ we will be able to find a number $r_j \in [(1-4\delta^2)|k_j|, |k_j|]$ such that

$$|h(k)| \geq e^{-c|k_j|} \geq e^{-2c|k|}$$

for $|k| = r_j$ and $|\operatorname{Re} k| \leq \delta|k|$. Appealing again to Corollary 3.4 to obtain a lower bound on $|W(k_j)|$ we obtain our claim by setting $c_0 = c + (\|w-1\|_1 + \|q-q_0 w\|_1)/\delta$. $\square$

Lemma A.5. *The entire part of the auxiliary function* (5.1) *has growth order* $\leq 1/2$.

Proof. Instead of λ we shall use $z = \lambda - q_0 = k^2$ as our variable. In view of Theorem A.1 it is sufficient to show that $|\operatorname{Im}(z)A(z+q_0)| \leq |k|^2 e^{C|k|}$ for all k lying on a sequence of semi-circles $|k| = r_j$, $\operatorname{Im} k \geq 0$, where r_j is a strictly increasing sequence for which r_{j+1}/r_j remains bounded.

First note that, since $\|R_\lambda\| \leq 1/|\operatorname{Im}(\lambda)|$ and since, by Lemma 2.1, the evaluation operator on $\mathcal{H}_1$ is a bounded linear form on the space, we find that $|\operatorname{Im}(z)(R_{z+q_0}u)(x)|$ remains bounded on any such sequence. Hence we need to investigate the term $\mathfrak{T}(k)\hat{u}_+(k)f_-(x,k)/(2ik\lambda) = -\hat{u}_+(k)f_-(x,k)/(\lambda W(k))$, where $W(k) = [f_+(\cdot,k), f_-(\cdot,k)]$.

$|\hat{u}_+(k)|$ and $|f_-(x,k)|$ are each bounded by $e^{c_1|k|}$ for some constant c_1 and $|k| \geq 1$ either by assumption or else with the aid of the estimate (3.4). For $|W(k)|^{-1}$ use Lemma A.4 on the semicircles $|k| = r_j$. Thus the second term of the auxiliary function can not grow faster than exponentially in k and applying now Theorem A.1 shows that the entire part of A is of growth order at most $1/2$. □

Lemma A.6. *For any $x < a$ the auxiliary function* (5.1) *tends to zero as $k^2 \to \infty$ in $i\mathbb{R}$.*

Proof. If $x < a$ we have

$$A(\lambda) = (R_\lambda u)(x) - g(x,a,\lambda)\frac{\hat{u}_+(k)}{\lambda f_+(a,k)}$$

where g is the Green's function introduced in (2.3) (with $f_\pm$ used as $\psi_\pm$). $(R_\lambda u)(x)$ tends to zero as $\lambda - q_0$ tends to infinity in $i\mathbb{R}$. Since, by the spectral theorem, $-\lambda R_\lambda$ similarly tends strongly to the identity operator and since, from Theorem 2.9,

$$g(x,\cdot,\lambda) = \lambda R_\lambda g(x,\cdot,0) + g(x,\cdot,0),$$

$g(x,a,\lambda)$ tends to zero as does the term $\hat{u}_+(k)/(\lambda f_+(a,k))$ by assumption. Thus $A(\lambda) \to 0$. □

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