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Weak stability for an inverse Sturm–Liouville problem with finite spectral data and complex potential*

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Abstract

It is well known that knowing the Dirichlet–Dirichlet eigenvalues and the Dirichlet–Neumann eigenvalues determines uniquely the potential of a one-dimensional Schrödinger equation on a finite interval. We investigate here how well a potential may be approximated if only N of each type of eigenvalues are known to within an error ε.

1. Introduction

In this paper, we consider a stability result for the inverse problem associated with the Sturm–Liouville equation

$$-y'' + q_0(x)y = \lambda y, \qquad x \in (0, 1),$$

in which the potential $q_0 \in L^2(0, 1)$ is allowed to be complex valued and the spectral data consists of the first N Dirichlet–Dirichlet eigenvalues and the first N Dirichlet–Neumann eigenvalues, determined to within an accuracy ε.

With only finite given spectral data, the inverse problem will have infinitely many solutions, and a stability result may therefore seem either meaningless or impossible.

The usual philosophy in the numerical analysis literature is to construct recovery algorithms which select one of the infinitely many possible solutions. Numerical experiments are then carried out in which finite spectral data are generated from some known potential and the quality of the recovery procedure is assessed according to how closely the recovered potential approximates the original one in some norm.

This process cannot be meaningful unless one can prove that all of the infinitely many solutions to the finite data inverse problems are 'close', in some suitable sense. In this paper, we show that such a result does indeed hold, albeit in a rather weak norm. This norm can be

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strengthened using interpolation space estimates if one has further *a priori* information about the potential to be recovered; however, without such information our weak result appears to be reasonably tight, as we demonstrate with a numerical example.

There is a vast literature on numerical methods for inverse Sturm–Liouville problems. One approach is to discretize the problem on a finite difference grid and then solve an inverse eigenvalue problem for a matrix: see the review article of Chu [3] for methods for inverse matrix eigenproblems. The discretization approach does not work unless one is prepared either to 'correct' the spectral data by adding approximations to the finite difference errors before doing the recovery (see [12]) or use a special finite difference grid which minimizes these errors to start with (see [1]). Another approach is based on the transformation operators of Levitan and is due to Rundell and Sacks [15]. A further approach may be found in [2] and is based on a variational idea of Knowles. For reviews of reconstruction methods for inverse Sturm–Liouville problems see Rundell [14] and McLaughlin [8].

Stability results are rather less common, even with full spectral data. Ryabushko [16] estimates the difference in $L^2([0, 1])$ of two potentials q_1 and q_2 whose average value is zero and for which the Dirichlet–Dirichlet eigenvalues $\lambda_n(q_j)$ and the Dirichlet–Neumann eigenvalues $\mu_n(q_j)$ are known:

$$\|q_1 - q_2\|_2 \leqslant C(\|\lambda(q_1) - \lambda(q_2)\|_2 + \|\mu(q_1) - \mu(q_2)\|_2).$$

Another result in that direction is due to McLaughlin [7]: when the average values of the potentials are zero, then there is a local diffeomorphism between potentials in $L^2([0, 1])$ and sequences $\{\lambda_n - n^2\pi^2, \rho_n\}$ in $\ell^2 \times \ell^2$, where $\{\rho_n\}$ are the 'norming constants'. One may also find in Pöschel and Trubowitz [13] a formula for the derivative of the potential with respect to one eigenvalue, which yields stability in $L^2([0, 1])$ subject to perturbation of finitely many eigenvalues. The results which come closest in spirit to our main result here are those of Hochstadt [5] and Hitrik [4]. The former concerns the extent to which the potential is determined when one spectrum is completely known and only finitely many members of the other spectrum are known while the second deals with an inverse scattering problem on the full line where finitely many values of the reflection coefficient are known.

Notation. Throughout this paper we shall find it convenient to use the notation $f^{(n,m)}(x_0, y_0)$ to denote the value at (x_0, y_0) of the partial derivative $\frac{\partial^{n+m} f}{\partial x^n \partial y^m}$. The reason for this uncommon choice of notation is that it will be particularly important to indicate the points at which partial derivatives are evaluated.

By $\|\cdot\|_p$ we denote the standard norm in $L^p([0, 1])$ (with Lebesgue measure) or in $L^p(\mathbb{N})$ (with counting measure). Naturally, there can now be confusion about which case is present in a given instance. In section 8, we also use certain Sobolev norms. These we indicate by subscripts $\|\cdot\|_{H^r}$ for various values of r.

2. Statement of the main result

Assume q_0 and q are complex-valued functions in $L^2([0, 1])$. Let $\lambda_j(q)$, $j \in \mathbb{N}$ denote the eigenvalues of the boundary value problem

$$-y'' + qy = \lambda y, \qquad y(0) = 0, \qquad y(1) = 0$$

and assume that they are repeated according to their algebraic multiplicities. Similarly, let $\mu_j(q)$, $j \in \mathbb{N}$ be the eigenvalues of the boundary value problem

$$-y'' + qy = \lambda y, \qquad y(0) = 0, \qquad y'(1) = 0$$

also repeated according to their algebraic multiplicities. The quantities $\lambda_j(q_0)$ and $\mu_j(q_0)$ denote the eigenvalues of those problems where q is replaced by q_0. We will always assume that these eigenvalues are labelled in such a way that identical values are adjacent and that their moduli form nondecreasing sequences.

The solution of the initial value problem

$$-y'' + qy = \lambda y, \qquad y(0) = 0, \qquad y'(0) = 1$$

is denoted by $s(\lambda, \cdot)$. Similarly $s_0(\lambda, \cdot)$ denotes the corresponding object for the potential q_0. Note that the $\lambda_j(q)$ are the zeros of $s(\cdot, 1)$ while the $\mu_j(q)$ are the zeros of $s'(\cdot, 1)$. Moreover, the algebraic multiplicities of these eigenvalues are equal to their multiplicities as zeros of these functions.

We will prove the following theorem.

Theorem 2.1. *Assume q_0 and q are complex-valued functions in $L^2([0, 1])$ with the same mean value. Define $a_j = |\lambda_j(q) - \lambda_j(q_0)|$ and $b_j = |\mu_j(q) - \mu_j(q_0)|$ and let $\varepsilon_0 \geqslant 0$ and $N_0 \in \mathbb{N}$ be fixed. Then there exists a constant C, depending only on q_0, ε_0 and N_0 such that the following is true:*

If $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $N \geqslant N_0$ and $\max\{a_1, \ldots, a_N, b_1, \ldots, b_N\} \leqslant \varepsilon$ then

$$\left| \int_0^x (q(t) - q_0(t))\,\mathrm{d}t \right| \leqslant C \exp(\|q\|_2) \left(\varepsilon \log N + \frac{\|a\|_2 + \|b\|_2}{N^{1/2}} \right)$$

for all $x \in [0, 1]$.

3. The transformation operator

It is well known (see, e.g., [6]) that solutions of the differential equation $-y'' + q_0 y = \lambda y$ can be transformed to solutions of the corresponding equation with q_0 replaced by q by means of an integral operator, the so-called transformation operator, when $q, q_0 \in L^1([0, 1])$. To give a more precise definition, we introduce the sets

$$\mathcal{D}_0 = \{y \in AC([0, 1]) : y' \in AC([0, 1]), -y'' + q_0 y \in L^2([0, 1]), y(0) = 0\}$$

and

$$\mathcal{D} = \{Y \in AC([0, 1]) : Y' \in AC([0, 1]), -Y'' + qY \in L^2([0, 1]), Y(0) = 0\}.$$

Then the transformation operator $\mathcal{K} : L^2([0, 1]) \to L^2([0, 1])$ is of the form

$$Y(x) = (\mathcal{K}y)(x) = y(x) + \int_0^x K(x, t) y(t)\,\mathrm{d}t, \tag{1}$$

it maps $\mathcal{D}_0$ to $\mathcal{D}$, and the kernel K is determined by the requirement $-(\mathcal{K}y)'' + q\mathcal{K}y = \mathcal{K}(-y'' + q_0 y)$ for all $y \in \mathcal{D}_0$.

Define a function K_0 by

$$K_0(x, t) = \frac{1}{2} \int_{(x-t)/2}^{(x+t)/2} (q(s) - q_0(s))\,\mathrm{d}s \tag{2}$$

and suppose that a function H exists with the following properties:

(1) $H(x, \cdot) \in AC([0, x])$ for all $x \in (0, 1]$ and $H(\cdot, t) \in AC([t, 1])$ for all $t \in [0, 1)$. The same is true for $H^{(1,0)}$ and $H^{(0,1)}$.
(2) $H^{(2,0)}(x, t) - H^{(0,2)}(x, t) - (q(x) - q_0(t))H(x, t) = (q(x) - q_0(t))K_0(x, t)$ almost everywhere in $\{(x, t) : 0 \leqslant t \leqslant x \leqslant 1\}$.
(3) $H(x, 0) = 0$.
(4) $(H^{(1,0)} + H^{(0,1)})(x, x) = 0$ almost everywhere in $[0, 1]$.

Then, proceeding as Levitan in [6] with appropriate modifications, one shows that the operator $\mathcal{K}$ defined by setting $K = K_0 + H$ is the desired transformation operator. From these considerations, it is also clear that the given conditions on H are necessary.

We remark also that K satisfies the Volterra equation

$$K(x,t) = \frac{1}{2}\int_{(x-t)/2}^{(x+t)/2} (q(s) - q_0(s))\,\mathrm{d}s + \int_{(x-t)/2}^{(x+t)/2}\int_0^{(x-t)/2} (q(\alpha+\beta) - q_0(\alpha-\beta))K(\alpha+\beta,\alpha-\beta)\,\mathrm{d}\beta\,\mathrm{d}\alpha$$

which is solved by the series

$$K(x,t) = \sum_{n=0}^{\infty} K_n(x,t)$$

where the K_n are defined inductively by

$$K_n(x,t) = \int_{(x-t)/2}^{(x+t)/2}\int_0^{(x-t)/2} (q(\alpha+\beta) - q_0(\alpha-\beta))K_{n-1}(\alpha+\beta,\alpha-\beta)\,\mathrm{d}\beta\,\mathrm{d}\alpha$$

and K_0 is given by (2).

Since $K(x,0) = 0$, it is possible (and later useful) to extend K to the set

$$S := \{(x,t) : 0 \leqslant |t| \leqslant x \leqslant 1\}$$

by setting $K(x,-t) = -K(x,t)$. We will henceforth assume that S is the domain of K.

We prove next that K may be obtained by solving an initial value problem with initial conditions given on the line $x = 1$. Let $f = K(1,\cdot)$ and $g = K_x(1,\cdot)$ and note that f and g are odd functions on $[-1,1]$. Also, f is absolutely continuous and g is integrable. Let

$$\tilde{K}_0(x,t) = \frac{1}{2}\int_{x-t-1}^{x+t-1} (f'(s) + g(s))\,\mathrm{d}s \tag{3}$$

and

$$\tilde{K}_n(x,t) = \frac{1}{2}\int_x^1\int_{t+x-u}^{t-x+u} (q(u) - q_0(v))\tilde{K}_{n-1}(u,v)\,\mathrm{d}v\,\mathrm{d}u \tag{4}$$

for $(x,t) \in S$. We will study the series

$$\tilde{K}(x,t) = \sum_{n=0}^{\infty} \tilde{K}_n(x,t). \tag{5}$$

Lemma 3.1. *If $\tilde{K}_0$ is bounded in S, then the series (5) is uniformly convergent, and hence its sum solves the integral equation*

$$k(x,t) = \tilde{K}_0(x,t) + \frac{1}{2}\int_x^1\int_{t+x-u}^{t-x+u} (q(u) - q_0(v))k(u,v)\,\mathrm{d}v\,\mathrm{d}u \tag{6}$$

in the set S.

Proof. Define $Q = \left(\|q\|_2^2 + \|q_0\|_2^2\right)^{1/2}$ and note that

$$\int_x^1\int_{t+x-u}^{t-x+u} |q(u) - q_0(v)|^2\,\mathrm{d}v\,\mathrm{d}u \leqslant 4(1-x)\int_x^1 |q(u)|^2\,\mathrm{d}u + 4\int_x^1 \|q_0\|_2^2\,\mathrm{d}u \leqslant 4(1-x)Q^2 \tag{7}$$

and

$$\int_x^1 \int_{t+x-u}^{t-x+u} (1-u)^n \,\mathrm{d}v\,\mathrm{d}u = 2\int_x^1 (1-u)^n (u-x)\,\mathrm{d}u = \frac{2(1-x)^{n+2}}{(n+1)(n+2)} \leqslant \frac{2(1-x)^{n+2}}{(n+1)^2}. \quad (8)$$

We need estimates on the functions $\tilde{K}_j$ in (5). Since $\tilde{K}_0$ is bounded we obtain, using the Cauchy–Schwarz inequality and inequalities (7) and (8),

$$|\tilde{K}_1(x,t)| \leqslant \frac{1}{2}\|\tilde{K}_0\|_\infty \int_x^1 \int_{t+x-u}^{t-x+u} |q(u)-q_0(v)|\,\mathrm{d}v\,\mathrm{d}u \leqslant Q\|\tilde{K}_0\|_\infty (1-x)^{3/2}.$$

We shall prove by induction that

$$|\tilde{K}_j(x,t)| \leqslant \|\tilde{K}_0\|_\infty \frac{Q^j(1-x)^{3j/2}}{j!} \quad (9)$$

for $j \in \mathbb{N}$. We just established this estimate for $j=1$, so it remains only to show that if we assume (9) then we can derive the corresponding estimate with j replaced by $j+1$. Proceeding as before, i.e., using Cauchy–Schwarz and inequalities (7) and (8), we have

$$\begin{aligned} |\tilde{K}_{j+1}(x,t)| &\leqslant \|\tilde{K}_0\|_\infty \frac{Q^j}{2j!} \int_x^1 \int_{t+x-u}^{t+u-x} |q(u)-q_0(v)|(1-u)^{3j/2}\,\mathrm{d}v\,\mathrm{d}u \\ &\leqslant \|\tilde{K}_0\|_\infty \frac{Q^{j+1}(1-x)^{(3j+3)/2}}{(j+1)!} \end{aligned}$$

which is the required result. Thus, (9) holds for all $j \geqslant 1$.

Thus, the series (5) is uniformly and absolutely convergent, and hence determines the solution of the integral equation (6). □

Lemma 3.2. $\tilde{K}(x,t) = K(x,t)$.

Proof. Define $\tilde{H}(x,t) = \tilde{K}(x,t) - K_0(x,t)$ and recall that $K(x,t) = K_0(x,t) + H(x,t)$. We will use the following abbreviations:

$$\begin{aligned} I_1(x,t) &= H^{(0,1)}(x,t) + H^{(1,0)}(x,t) \\ &= \int_0^{(x-t)/2} \left(q\left(\frac{x+t}{2}+\beta\right) - q_0\left(\frac{x+t}{2}-\beta\right)\right) K\left(\frac{x+t}{2}+\beta, \frac{x+t}{2}-\beta\right) \mathrm{d}\beta, \\ \tilde{I}_1(x,t) &= \int_x^1 q(s)\tilde{K}(s, t-x+s)\,\mathrm{d}s - \int_t^{t-x+1} q_0(s)\tilde{K}(x-t+s, s)\,\mathrm{d}s, \\ \tilde{I}_2(x,t) &= \int_x^1 q(s)\tilde{K}(s, x+t-s)\,\mathrm{d}s - \int_{t+x-1}^{t} q_0(s)\tilde{K}(x+t-s, s)\,\mathrm{d}s. \end{aligned}$$

Since

$$f'(s) + g(s) = K^{(0,1)}(1,s) + K^{(1,0)}(1,s) = \frac{1}{2}(q-q_0)\left(\frac{1+s}{2}\right) + I_1(1,s)$$

we obtain

$$\tilde{H}(x,t) = \frac{1}{2}\int_{x-t-1}^{x+t-1} I_1(1,s)\,\mathrm{d}s + \frac{1}{2}\int_x^1 \int_{t+x-u}^{t+u-x} (q(u)-q_0(v))\tilde{K}(u,v)\,\mathrm{d}v\,\mathrm{d}u$$

which is absolutely continuous in either variable. Thus,

$$\tilde{H}^{(1,0)}(x,t) = \tfrac{1}{2}(I_1(1,x+t-1) - I_1(1,x-t-1) - \tilde{I}_1(x,t) - \tilde{I}_2(x,t))$$

and

$$\tilde{H}^{(0,1)}(x,t) = \tfrac{1}{2}(I_1(1,x+t-1) + I_1(1,x-t-1) + \tilde{I}_1(x,t) - \tilde{I}_2(x,t))$$

are also absolutely continuous. Differentiating further gives

$$\tilde H^{(2,0)}(x,t) - \tilde H^{(0,2)}(x,t) - (q(x) - q_0(t))\tilde H(x,t) = (q(x) - q_0(t))K_0(x,t).$$

By construction $H(1,t) = \tilde H(1,t)$ and $H^{(1,0)}(1,t) = \tilde H^{(1,0)}(1,t)$. Hence, the function $H - \tilde H$ is a solution of the Cauchy problem

$$k^{(2,0)}(x,t) - k^{(0,2)}(x,t) - (q(x) - q_0(t))k(x,t) = 0, \qquad k(1,t) = 0, \qquad k^{(1,0)}(1,t) = 0.$$

For the given Cauchy data k satisfies the integral equation

$$k(x,t) = \frac{1}{2}\int_x^1 \int_{t+x-u}^{t-x+u} (q(u) - q_0(v))k(u,v)\,\mathrm{d}v\,\mathrm{d}u$$

since, after several changes of variables,

$$\int_x^1 \int_{t+x-u}^{t-x+u} (k^{(2,0)}(u,v) - k^{(0,2)}(u,v))\,\mathrm{d}v\,\mathrm{d}u = 2k(x,t).$$

But lemma 3.1 shows now that $k = 0$ identically. □

Theorem 3.3. *Suppose that $q, q_0 \in L^2([0,1])$. Then*

$$\left|\int_0^x (q - q_0)(s)\mathrm{d}s\right| = 2|K(x,x)| \leqslant 4\exp(\|q\|_2 + \|q_0\|_2)\|\tilde K_0\|_\infty.$$

Proof. Since $K(x,x) = \sum_{n=0}^\infty \tilde K_n(x,x)$, the claim follows immediately from the estimate $|\tilde K_n(x,x)| \leqslant \|\tilde K_0\|_\infty Q^j/j!$ obtained in the proof of lemma 3.1. □

4. Asymptotic properties of the function s_0

Define $s_{0,0}(\lambda,x) = \sin(zx)/z$ where $z^2 = \lambda$. While this definition does not depend on the choice of a branch for the root, we will henceforth assume that $\mathrm{Im}(z) \geqslant 0$. One proves by induction that the kth derivative of $s_{0,0}$ with respect to its first argument is given by

$$s_{0,0}^{(k,0)}(\lambda,x) = \frac{f_k(zx)\,\mathrm{e}^{-\mathrm{i}zx} + g_k(zx)\,\mathrm{e}^{\mathrm{i}zx}}{z^{2k+1}} \tag{10}$$

where f_k and g_k are polynomials of degree k. This implies that

$$\left|s_{0,0}^{(k,0)}(\lambda,x)\right| \leqslant c_k\,\mathrm{e}^{\mathrm{Im}(z)x}|z|^{-k-1} \tag{11}$$

for appropriate constants c_k, $k \in \mathbb{N}_0$, $c_0 = 1$. Similarly, one obtains

$$\left|s_{0,0}^{(k,1)}(\lambda,x)\right| \leqslant c_k\,\mathrm{e}^{\mathrm{Im}(z)x}|z|^{-k}. \tag{12}$$

The following lemma shows that similar estimates hold for the function s_0, i.e., the solution of the initial value problem $-y'' + q_0 y = \lambda y$, $y(0) = 0$, $y'(0) = 1$, if λ is sufficiently large.

Lemma 4.1. *Given $q_0 \in L^1([0,1])$ there are positive constants $\eta_k, \tilde\eta_k$, $k \in \mathbb{N}_0$, depending only on $\|q_0\|_1$, such that*

$$\left|s_0^{(k,0)}(\lambda,x) - s_{0,0}^{(k,0)}(\lambda,x)\right| \leqslant \eta_k|z|^{-k-1}\,\mathrm{e}^{\mathrm{Im}(z)x}\left(\mathrm{e}^{\int_0^x |q_0(t)/z|\mathrm{d}t} - 1\right) \tag{13}$$

and

$$\left|s_0^{(k,1)}(\lambda,x) - s_{0,0}^{(k,1)}(\lambda,x)\right| \leqslant \tilde\eta_k|z|^{-k}\,\mathrm{e}^{\mathrm{Im}(z)x}\left(\mathrm{e}^{\int_0^x |q_0(t)/z|\mathrm{d}t} - 1\right) \tag{14}$$

for all $x \in [0,1]$, all $k \in \mathbb{N}_0$, and all λ satisfying $|\lambda| \geqslant 1$.

Proof. Define

$$g_0(z,t) = \mathrm{e}^{-t\,\mathrm{Im}(z)}|s_0(\lambda,t) - s_{0,0}(\lambda,t)|.$$

Replacing s_0 by $s_{0,0} + (s_0 - s_{0,0})$ under the integral in the variation of constants formula

$$s_0(\lambda,t) = s_{0,0}(\lambda,t) + \int_0^t s_{0,0}(\lambda,t-u)q_0(u)s_0(\lambda,u)\,\mathrm{d}u$$

one finds

$$g_0(z,t) \leqslant \frac{1}{|z|}\int_0^t |q_0(u)|g_0(z,u)\,\mathrm{d}u + \frac{1}{|z|^2}\int_0^t |q_0(u)|\,\mathrm{d}u. \tag{15}$$

Let $\phi(t) = \int_0^t |q_0(u)/z|\,\mathrm{d}u$, move the first term on the right of (15) to the left, and multiply by $|q_0(t)|\exp(-\phi(t))$. This will produce total derivatives on either side so that integration from 0 to x yields

$$\mathrm{e}^{-\phi(x)}\int_0^x |q_0(t)|g_0(z,t)\,\mathrm{d}t \leqslant \left(1 - \mathrm{e}^{-\phi(x)} - \mathrm{e}^{-\phi(x)}\frac{1}{|z|}\int_0^x |q_0(t)|\,\mathrm{d}t\right).$$

Using this estimate in (15) gives inequality (13) for $k = 0$ with $\eta_0 = 1$. Thus, using the triangle inequality, we also see that

$$|s_0(\lambda,x)| \leqslant \frac{\mathrm{e}^{\|q_0\|_1}\,\mathrm{e}^{\mathrm{Im}(z)x}}{|z|}.$$

Now assume that

$$\left|s_0^{(\ell,0)}(\lambda,x)\right| \leqslant \frac{\beta_\ell\,\mathrm{e}^{\mathrm{Im}(z)x}}{|z|^{\ell+1}}$$

for $\ell = 0,\ldots,k-1$ and certain constants β_ℓ which may depend on $\|q_0\|_1$. Then define

$$g_k(z,t) = \mathrm{e}^{-t\,\mathrm{Im}(z)}\left|s_0^{(k,0)}(\lambda,t) - s_{0,0}^{(k,0)}(\lambda,t)\right|.$$

The kth λ-derivative of the variation of constants formula and (11) give

$$g_k(z,t) \leqslant \sum_{j=0}^k \binom{k}{j}\frac{c_j}{|z|^{j+1}}\int_0^t |q_0(u)|\left|\mathrm{e}^{-u\,\mathrm{Im}(z)}s_0^{(k-j,0)}(\lambda,u)\right|\mathrm{d}u.$$

Using the induction hypothesis for the terms with $j > 0$ and the triangle inequality in the term with $j = 0$ yields

$$\begin{aligned} g_k(z,t) &\leqslant \frac{1}{|z|}\int_0^t |q_0(u)|\left(g_k(z,u) + \mathrm{e}^{-u\,\mathrm{Im}(z)}\left|s_{0,0}^{(k,0)}(\lambda,u)\right|\right)\mathrm{d}u + \sum_{j=1}^k \binom{k}{j}\frac{c_j\beta_{k-j}}{|z|^{k+2}}\int_0^t |q_0(u)|\,\mathrm{d}u \\ &\leqslant \frac{1}{|z|}\int_0^t |q_0(u)|g_k(z,u)\,\mathrm{d}u + \frac{\eta_k}{|z|^{k+2}}\int_0^t |q_0(u)|\,\mathrm{d}u \end{aligned}$$

where

$$\eta_k = c_k + \sum_{j=1}^k \binom{k}{j}c_j\beta_{k-j}.$$

Proceeding as before we arrive at the estimate

$$g_k(z,x) \leqslant \frac{\eta_k}{|z|^{k+1}}(\mathrm{e}^{\phi(x)} - 1)$$

which is inequality (13) for k. The induction is complete after realizing that (13) implies that

$$\left|s_0^{(k,0)}(\lambda,x)\right| \leqslant \frac{\beta_k\,\mathrm{e}^{\mathrm{Im}(z)x}}{|z|^{k+1}}$$

when one chooses $\beta_k = c_k + \eta_k\exp(\|q_0\|_1)$.

The variation of constants formula implies also

$$s_0^{(k,1)}(\lambda, x) = s_{0,0}^{(k,1)}(\lambda, x) + \sum_{j=0}^{k} \binom{k}{j} \int_0^x s_{0,0}^{(j,1)}(\lambda, x-u) q_0(u) s_0^{(k-j,1)}(\lambda, u)\, \mathrm{d}u.$$

The triangle inequality and previous estimates give

$$\left|s_0^{(k,1)}(\lambda, x) - s_{0,0}^{(k,1)}(\lambda, x)\right| \leqslant \mathrm{e}^{\mathrm{Im}(z)x} \sum_{j=0}^{k} \binom{k}{j} \int_0^x |q_0(u)| \frac{c_j}{|z|^{k+1}} (\eta_{k-j}(\mathrm{e}^{\phi(t)} - 1) + c_{k-j})\, \mathrm{d}u.$$

Since $c_{k-j} \leqslant \eta_{k-j}$ we obtain inequality (14) if we choose $\tilde{\eta}_k = \sum_{j=0}^{k} \binom{k}{j} c_j \eta_{k-j}$. □

Remark 4.2.

(1) As a corollary to this proof, there exist constants β_k depending only on $\|q_0\|_1$ such that

$$\left|s_0^{(k,0)}(\lambda, x)\right| \leqslant \frac{\beta_k\, \mathrm{e}^{\mathrm{Im}(z)x}}{|z|^{k+1}}.$$

(2) The eigenvalues $\lambda_j(q_0)$, $\mu_j(q_0)$, $\lambda_j(q)$ and $\mu_j(q)$ lie in a horizontal strip in the complex plane. For $\lambda = z^2$ in this strip, $\mathrm{Im}(z) \geqslant 0$ is bounded above, and hence

$$\left|s_0^{(k,0)}(\lambda, x)\right| \leqslant \frac{C_k}{|z|^{k+1}}. \tag{16}$$

We shall use this fact in several places later in this paper.

(3) Using (12) and (14) one may prove similarly that

$$\left|s_0^{(k,1)}(\lambda, x)\right| \leqslant \frac{\tilde{C}_k}{|z|^{k}}. \tag{17}$$

Lemma 4.3. *Fix $k \in \mathbb{N}_0$. There is a positive number M_k depending only on $\|q_0\|_1$ such that*

$$\left|s_0^{(k,0)}(\lambda, x) - s_0^{(k,0)}(\lambda_j(q_0), x)\right| \leqslant \frac{M_k}{j^{k+2}} |\lambda - \lambda_j(q_0)|$$

and

$$\left|s_0^{(k,1)}(\lambda, x) - s_0^{(k,1)}(\lambda_j(q_0), x)\right| \leqslant \frac{M_k}{j^{k+1}} |\lambda - \lambda_j(q_0)|$$

for all $x \in [0, 1]$ and all $j \in \mathbb{N}$ provided that $|\lambda - \lambda_j(q_0)| \leqslant 1$.

Proof. Evidently,

$$s_0^{(k,0)}(\lambda, x) - s_0^{(k,0)}(\lambda_j(q_0), x) = \int_{\lambda_j(q_0)}^{\lambda} s_0^{(k+1,0)}(\mu, x)\, \mathrm{d}\mu.$$

Since $\mathrm{Im}\sqrt{\mu}$ is bounded we may now use (16) with k replaced by $k+1$ to obtain

$$\left|\int_{\lambda_j(q_0)}^{\lambda} s_0^{(k+1,0)}(\mu, x)\, \mathrm{d}\mu\right| \leqslant |\lambda - \lambda_j(q_0)| \frac{\tilde{M}_k}{|\sqrt{\lambda_j(q_0)}|^{k+2}}$$

for some constant $\tilde{M}_k$, since $|\lambda - \lambda_j(q_0)| \leqslant 1$. The first result of lemma 4.3 is then immediate. The second result follows similarly from (17). □

5. Riesz bases of generalized eigenfunctions

Let L be the operator defined by the boundary value problem

$$-y'' + q_0 y = \lambda y, \qquad y(0) = 0, \qquad y(1) = 0$$

and suppose that λ is an eigenvalue of L with algebraic multiplicity ν. Then there are indices $\kappa, \ldots, \kappa + \nu - 1$ such that $\lambda = \lambda_\kappa(q_0) = \cdots = \lambda_{\kappa+\nu-1}(q_0)$. Define

$$\varphi_{\kappa+j}(x) = \sum_{k=0}^{j} \frac{\gamma_{\kappa+j-k}}{k!} s_0^{(k,0)}(\lambda, x), \qquad j = 0, \ldots, \nu - 1$$

where the $\gamma_{\kappa+j-k}$ are to be determined. Note that φ_κ is an eigenfunction associated with λ and that

$$(L - \lambda)\varphi_{\kappa+j} = \varphi_{\kappa+j-1}, \qquad j = 1, \ldots, \nu - 1,$$

i.e., the $\varphi_{\kappa+j}$ form a Jordan chain of generalized eigenvectors for λ (recall that the geometric multiplicity of λ equals one).

If λ is an eigenvalue of L then $\overline{\lambda}$ is an eigenvalue of L^*, the adjoint of L, which is the operator associated with the boundary value problem

$$-y'' + \overline{q_0} y = \lambda y, \qquad y(0) = 0, \qquad y(1) = 0.$$

If λ has algebraic multiplicity ν as an eigenvalue of L then $\overline{\lambda}$ also has algebraic multiplicity ν as an eigenvalue of L^*. Define

$$\psi_{\kappa+j}(x) = \frac{\kappa\pi}{(\nu - 1 - j)!} \overline{s_0^{(\nu-1-j,0)}(\lambda, x)}, \qquad j = 0, \ldots, \nu - 1. \tag{18}$$

Then,

$$(L^* - \overline{\lambda})\psi_{\kappa+j} = \psi_{\kappa+j+1}, \qquad j = 0, \ldots, \nu - 2,$$

and $\psi_{\kappa+\nu-1}$ is an eigenfunction of L^* associated with $\overline{\lambda}$, i.e., the $\psi_{\kappa+j}$ form a Jordan chain of generalized eigenvectors for $\overline{\lambda}$. (Note, however, that the order is reversed when compared to the $\varphi_{\kappa+j}$.)

It is well known that if $\lambda_1 \neq \lambda_2$ then the algebraic eigenspace of L associated with λ_1 and the algebraic eigenspace of L^* associated with $\overline{\lambda_2}$ are orthogonal.

Let $A(\kappa)$ be the $\nu \times \nu$-matrix with entries $A(\kappa)_{j+1,k+1} = (\psi_{\kappa+j}, \varphi_{\kappa+k})$. Since

$$(\psi_{\kappa+j}, \varphi_{\kappa+k}) = (\psi_{\kappa+j}, (L - \lambda)\varphi_{\kappa+k+1}) = ((L^* - \overline{\lambda})\psi_{\kappa+j}, \varphi_{\kappa+k+1}) = (\psi_{\kappa+j+1}, \varphi_{\kappa+k+1})$$

when $0 \leqslant j, k \leqslant \nu - 2$ we find that the entries in the diagonals ($j - k$ is constant) of $A(\kappa)$ are constant. Since

$$(\psi_{\kappa+\nu-1}, \varphi_{\kappa+k}) = (0, \varphi_{\kappa+k+1}) = 0$$

when $0 \leqslant k \leqslant \nu - 2$ we see that $A(\kappa)$ is upper triangular. The ψ_j, $j \in \mathbb{N}$ are complete and so, since we have shown that $(\psi_j, \varphi_\kappa) = 0$ if $j \neq \kappa$, we conclude that $(\psi_\kappa, \varphi_\kappa) \neq 0$. In fact, we may choose the coefficients $\gamma_{\kappa+j}$, $j = 0, \ldots, \nu - 1$ in such a way that $A(\kappa)$ becomes the identity matrix. For this choice of the coefficients, the ψ_j and the φ_k form biorthogonal sequences, i.e., $(\psi_j, \varphi_k) = \delta_{j,k}$. In fact, due to our normalizations, both $\{\psi_j : j \in \mathbb{N}\}$ and $\{\varphi_k : k \in \mathbb{N}\}$ form Riesz bases of $L^2([0, 1])$ (see [11]).

For all sufficiently large j the eigenvalues $\lambda_j(q_0)$ are simple. In these cases, $\varphi_j(x) = \gamma_j s_0(\lambda_j(q_0), x)$. Now from lemma 4.1 we can see that

$$s_0(\lambda_j(q_0), x) = \frac{\sin(\sqrt{\lambda_j(q_0)}x)}{\sqrt{\lambda_j(q_0)}} + O(|\lambda_j(q_0)|^{-1}), \tag{19}$$

where we have used the fact that the term $\exp\left(\int_0^x |q_0(t)/z|\,\mathrm{d}t\right) - 1$ in (13) is $O(1/|z|)$. Hence, $1 = (\varphi_j, \psi_j) = O(\gamma_j/j)$ and so the constant γ_j in the expression $\varphi_j(x) = \gamma_j s_0(\lambda_j(q_0), x)$ must be $O(j)$.

We now turn to the problem with eigenvalues $\mu_j(q_0)$. In complete analogy with the results already obtained, given the boundary value problem

$$-y'' + q_0 y = \lambda y, \qquad y(0) = 0, \qquad y'(1) = 0$$

we obtain biorthogonal sequences $k \mapsto \theta_k$ and $j \mapsto \omega_j$ of (generalized) eigenfunctions of the associated operator and its adjoint, respectively.

The following lemma gives uniform bounds on the eigen- and associated functions of the first boundary value problem and on the integrals of the eigen- and associated functions of the second boundary value problem.

Lemma 5.1. *There is a positive constant B_1 depending only on q_0 such that*

$$|\varphi_j(t)| \leqslant B_1 \qquad \text{and} \qquad \left|\int_0^t \theta_j(s)\,\mathrm{d}s\right| \leqslant B_1/j$$

for all $t \in [0, 1]$ and all $j \in \mathbb{N}$.

Proof. The bound on φ_j follows from (19) and the observation that γ_j is $O(j)$. We only need consider (19) since the eigenvalues of the problem are eventually simple.

For the second bound, we can simply replace $\lambda_j(q_0)$ by $\mu_j(q_0)$ in (19) and obtain, for some constant Γ_j,

$$\theta_j(x) = \Gamma_j s_0(\mu_j(q_0), x) = \Gamma_j \frac{\sin(\sqrt{\mu_j(q_0)}x)}{\sqrt{\mu_j(q_0)}} + \Gamma_j(s_0(\mu_j(q_0), x) - s_{0,0}(\mu_j(q_0), x)). \tag{20}$$

The term $s_0(\mu_j(q_0), x) - s_{0,0}(\mu_j(q_0), x)$ is $O(|\mu_j(q_0)|^{-1}) = O(j^{-2})$ as we observed previously, by using (13). The factor Γ_j must then be $O(j)$ as $1 = (\theta_j, \omega_j) = O(\Gamma_j/j)$. Integrating both sides, we obtain

$$\int_0^x \theta_j(t)\,\mathrm{d}t = \frac{\Gamma_j}{\mu_j(q_0)}(1 - \cos(\sqrt{\mu_j(q_0)}x)) + O(1/j) \leqslant O(1/j).$$

This completes the proof. □

6. An interpolation error estimate

The last technical result which we shall require in the proof of theorem 2.1 is the following.

Lemma 6.1. *Let S be an analytic function on a disc with centre z_0 and radius 2. Let $z_1, \ldots, z_\nu$ be ν points (not necessarily distinct) in a disc of radius ε $(0 < \varepsilon < 1/2)$ centred at z_0. Let p be the unique polynomial of degree at most $\nu - 1$ which interpolates S and its derivatives in the usual way at the points $z_1, \ldots, z_\nu$: namely, if the value of z_j appears m_j times in the list $z_1, \ldots, z_\nu$ then $p^{(n)}(z_j) = S^{(n)}(z_j)$ for $n = 0, \ldots, m_j - 1$. Let $R = S - p$. Then for each $0 \leqslant j \leqslant \nu - 1$,*

$$|R^{(j)}(z_0)| \leqslant B_2 \varepsilon^{\nu - j} \sup_{|\zeta - z_0| = 1} |S(\zeta)| \tag{21}$$

where the constant B_2 depends only on ν and not on ε, S or the positions of the points $z_1, \ldots, z_\nu$.

Proof. Let $\omega(z) = \prod_{j=1}^{\nu}(z - z_j)$ and let Γ be the circle of centre z_0, radius 1. Using the Hermite formula for the difference $R(z) = S(z) - p(z)$ (see, e.g., [10, p 69]) immediately gives

$$R(z) = \frac{\omega(z)}{2\pi \mathrm{i}} \int_\Gamma \frac{S(\zeta)}{\omega(\zeta)(\zeta - z)} \,\mathrm{d}\zeta. \tag{22}$$

On the circle Γ, we have $|\omega(\zeta)| \geqslant 2^{-\nu}$ since all the points z_j are at a distance at least $1/2$ from Γ. We also have $|\zeta - z_0| = 1$ when ζ lies on Γ. Furthermore, since all the points z_j are at a distance less than ε from z_0 it follows that $|\omega(z_0)| \leqslant \varepsilon^\nu$. Hence,

$$|R(z_0)| \leqslant (2\varepsilon)^\nu \sup_{\zeta\in\Gamma} |S(\zeta)|,$$

which proves the inequality (21) for $j = 0$.

Differentiating (22) yields

$$R'(z) = \frac{\omega'(z)}{2\pi \mathrm{i}} \int_\Gamma \frac{S(\zeta)}{\omega(\zeta)(\zeta - z)} \,\mathrm{d}\zeta + \frac{\omega(z)}{2\pi \mathrm{i}} \int_\Gamma \frac{S(\zeta)}{\omega(\zeta)(\zeta - z)^2} \,\mathrm{d}\zeta.$$

The first term is now $O(\nu\varepsilon^{\nu-1})$ while the second is bounded by the same bound as the first: in fact,

$$|R'(z_0)| \leqslant \varepsilon^{\nu-1}(\nu + \varepsilon)2^\nu \sup_{\zeta\in\Gamma} |S(\zeta)|.$$

The higher derivatives are dealt with analogously. $\square$

7. Proof of theorem 2.1

In view of theorem 3.3, we plan to estimate the sup-norm of

$$\tilde{K}_0(x, t) = \frac{1}{2} \int_{x-t-1}^{x+t-1} (f'(s) + g(s)) \,\mathrm{d}s,$$

i.e., the suprema of $|f(t)|$ and $\left|\int_0^t g \,\mathrm{d}s\right|$. The factor $\exp(\|q\|_2 + \|q_0\|_2)$ appearing in theorem 3.3 contributes the factor of $\exp(\|q\|_2)$ appearing in theorem 2.1; the remaining constants will depend only on q_0, as will become apparent in the proof.

Note that $f = \sum_{j=1}^{\infty} \alpha_j \varphi_j$ where $\alpha_j = (f, \psi_j)$. We will show below that

$$|\alpha_j| \leqslant 2M(1 + \|f\|_2)\frac{a_j}{j} \tag{23}$$

where $a_j = |\lambda_j(q) - \lambda_j(q_0)|$ and where M is a constant depending only on q_0. Since the φ_k form a Riesz basis we have the existence of a positive number R such that

$$\|f\|_2^2 \leqslant R^2 \sum_{j=1}^{\infty} |\alpha_j|^2.$$

Thus,

$$\begin{aligned}\|f\|_2^2 &\leqslant R^2 \sum_{j=1}^{\infty} 4M^2(1 + \|f\|_2)^2 \frac{a_j^2}{j^2} \leqslant 8R^2M^2\big(1 + \|f\|_2^2\big) \sum_{j=1}^{\infty} \frac{a_j^2}{j^2} \\ &\leqslant 8R^2M^2\big(1 + \|f\|_2^2\big)\left(\frac{\pi^2\varepsilon^2}{6} + \frac{\|a\|_2^2}{N^2}\right),\end{aligned}$$

where we have used the assumption $a_j \leqslant \varepsilon$ for $j = 1, \ldots, N$ from the theorem. The inequality in theorem 2.1 is claimed for sufficiently small ε and sufficiently large N. Thus, we may assume without loss of generality that

$$8R^2M^2\left(\frac{\pi^2\varepsilon^2}{6} + \frac{\|a\|_2^2}{N^2}\right) \leqslant \frac{1}{2}.$$

We then get $\|f\|_2^2 \leqslant \left(1 + \|f\|_2^2\right)/2$, whence $\|f\|_2 \leqslant 1$. From (23), it follows that

$$|\alpha_j| \leqslant 4M\frac{a_j}{j}.$$

Therefore, using lemma 5.1, and defining $\hat{M}$ to be the greater of the constant M in (23) and the constant B_1 appearing in lemma 5.1,

$$|f(t)| \leqslant \sum_{j=1}^{\infty} |\alpha_j||\varphi_j(t)| \leqslant 4\hat{M}^2 \sum_{j=1}^{\infty} \frac{a_j}{j} \leqslant 4\hat{M}^2(1 + \log N)\varepsilon + \frac{4\hat{M}^2\|a\|_2}{N^{1/2}}.$$

It remains to show the validity of inequality (23). First note that equation (1) for $x = 1$ gives

$$s(\lambda, 1) = s_0(\lambda, 1) + \int_0^1 f(t)s_0(\lambda, t)\,\mathrm{d}t \tag{24}$$

where we recall that $f(t) = K(1, t)$. Moreover, this equation may differentiated with respect to λ arbitrarily many times. Now let

$$\lambda = \lambda_\kappa(q_0) = \cdots = \lambda_{\kappa+\nu-1}(q_0)$$

be an eigenvalue of multiplicity ν. We deal with the cases $\nu = 1$ and $\nu > 1$ separately.

For the case of a simple eigenvalue ($\nu = 1$) we have, from (18) and from (24) evaluated at $\lambda = \lambda_k(q_0)$,

$$\alpha_\kappa = (f, \psi_\kappa) = \kappa\pi \int_0^1 f(t)s_0(\lambda_\kappa(q_0), t)\,\mathrm{d}t = \kappa\pi s(\lambda_\kappa(q_0), 1).$$

Likewise if we evaluate (24) at $\lambda = \lambda_\kappa(q)$ we obtain

$$0 = s_0(\lambda_\kappa(q), 1) + \int_0^1 f(t)s_0(\lambda_\kappa(q), t)\,\mathrm{d}t.$$

Subtracting,

$$\alpha_\kappa = -\kappa\pi s_0(\lambda_\kappa(q), 1) + \kappa\pi \int_0^1 f(t)(s_0(\lambda_\kappa(q_0), t) - s_0(\lambda_\kappa(q), t))\,\mathrm{d}t$$

which we can also write as

$$\alpha_\kappa = \kappa\pi(s_0(\lambda_\kappa(q_0), 1) - s_0(\lambda_\kappa(q), 1)) + \kappa\pi \int_0^1 f(t)(s_0(\lambda_\kappa(q_0), t) - s_0(\lambda_\kappa(q), t))\,\mathrm{d}t$$

since $s_0(\lambda_\kappa(q_0, 1) = 0$. It now follows from elementary estimates that

$$|\alpha_\kappa| \leqslant \kappa\pi(1 + \|f\|_1)\|s_0(\lambda_\kappa(q_0, \cdot) - s_0(\lambda_\kappa(q), \cdot)\|_\infty.$$

Using lemma 4.3 with $k = 0$ and $j = \kappa$, we thus obtain

$$|\alpha_\kappa| \leqslant \kappa\pi(1 + \|f\|_1)\frac{M_0}{\kappa^2}a_\kappa.$$

The term $\|f\|_1$ is bounded above by $\|f\|_2$ since the problem is posed in $L^2(0, 1)$, and so (23) follows with a suitable choice of M (e.g. $\pi M_0/2$).

Next consider the case of a multiple eigenvalue ($\nu > 1$). Since the q_0 problem has at most finitely many multiple eigenvalues, we may assume without loss of generality that the eigenvalue is one of the first N which are approximated with accuracy at least ε. We may thus assume that there are ν eigenvalues $\lambda_\kappa(q), \ldots, \lambda_{\kappa+\nu-1}(q)$, counted according to algebraic multiplicity and therefore not necessarily distinct, in a disc of centre $\lambda_\kappa(q_0)$ and radius ε. For each t, let $p(\lambda, t)$ and $p_0(\lambda, t)$ be, respectively, the unique polynomials of degree at most $(\nu - 1)$ interpolating $s(\lambda, t)$ and $s_0(\lambda, t)$ at the points $\lambda_\kappa(q), \ldots, \lambda_{\kappa+\nu-1}(q)$. From (24), we know that

$$p(\lambda, 1) = p_0(\lambda, 1) + \int_0^1 f(t) p_0(\lambda, t)\,\mathrm{d}t$$

for $\lambda = \lambda_\kappa(q), \ldots, \lambda_{\kappa+\nu-1}(q)$ and hence for all λ, since both sides of the equation are polynomials of degree at most $\nu - 1$. We can therefore differentiate this formula $\nu - j - 1$ times, for $j = 0, \ldots, \nu - 1$, and obtain

$$p^{(\nu-1-j,0)}(\lambda, 1) = p_0^{(\nu-1-j,0)}(\lambda, 1) + \int_0^1 f(t) p_0^{(\nu-1-j,0)}(\lambda, t)\,\mathrm{d}t. \tag{25}$$

We now observe that since $s(\lambda, 1) = 0$ at all of the points $\lambda_\kappa(q), \ldots, \lambda_{\kappa+\nu-1}(q)$, the function $p(\lambda, 1)$ is identically zero. Thus, the left-hand side of (25) is identically zero, and in particular

$$0 = p_0^{(\nu-1-j,0)}(\lambda_\kappa(q_0), 1) + \int_0^1 f(t) p_0^{(\nu-1-j,0)}(\lambda_\kappa(q_0), t)\,\mathrm{d}t. \tag{26}$$

Now the coefficients $\alpha_{\kappa+j} = (f, \psi_{\kappa+j})$ in the expansion of f is given, from (18), by

$$\alpha_{\kappa+j} = (f, \psi_{\kappa+j}) = \frac{\kappa\pi}{(\nu - 1 - j)!} \int_0^1 f(t) s_0^{(\nu-1-j,0)}(\lambda_\kappa(q_0), t)\,\mathrm{d}t, \qquad j = 0, 1, \ldots, \nu - 1.$$

By virtue of (26) and the fact that $s_0^{(\nu-1-j,0)}(\lambda_j(q_0), 1) = 0$, we can write this equation in terms of $R(\lambda, t) := s_0(\lambda, t) - p_0(\lambda, t)$ as

$$\alpha_{\kappa+j} = \frac{\kappa\pi}{(\nu - 1 - j)!} \int_0^1 f(t) R^{(\nu-1-j,0)}(\lambda_\kappa(q_0), t)\,\mathrm{d}t + R^{(\nu-1-j,0)}(\lambda_\kappa(q_0), 1).$$

An application of lemma 6.1 allows us to deduce that

$$|\alpha_{\kappa+j}| \leqslant \frac{\kappa\pi}{(\nu - 1 - j)!} \varepsilon^{j+1} B_2 2^\nu \sup_{|\lambda-\lambda_\kappa(q_0)|=1} \sup_{t\in[0,1]} |s_0(\lambda, t)|(1 + \|f\|_1).$$

The term $\sup_{t\in[0,1]} |s_0(\lambda, t)|$ is bounded by (16) since $\mathrm{Im}\sqrt{\lambda}$ is bounded. Since $\|f\|_1 \leqslant \|f\|_2$, we obtain an estimate of the required form (23).

The estimate for $\left|\int_0^t g\,\mathrm{d}s\right|$ is obtained in a very similar way. We point out the main differences. Instead of equation (24), we need

$$s'(\lambda, 1) = s_0'(\lambda, 1) + \int_0^1 g(t) s_0(\lambda, t)\,\mathrm{d}t$$

which follows from equation (1) after using that $K(1, 1) = 0$ (by the equality of the mean values of q and q_0). Next, we observe that

$$s'(\mu_j(q_0), 1) = \int_0^1 g(t) s_0(\mu_j(q_0), t)\,\mathrm{d}t = \frac{(g, \omega_j)}{j\pi},$$

since $\omega_j = j\pi\overline{s_0(\mu_j(q_0), \cdot)}$. Also,

$$0 = s_0'(\mu_j(q), 1) + \int_0^1 g(t) s_0(\mu_j(q), t)\,\mathrm{d}t.$$

Since $\beta_j = (g, \omega_j)$, we obtain

$$\frac{\beta_j}{j\pi} = \int_0^1 g(t)[s_0(\mu_j(q_0), t) - s_0(\mu_j(q), t)]\,\mathrm{d}t + [s_0'(\mu_j(q_0), 1) - s_0'(\mu_j(q), 1)].$$

Recalling that s_0' is $s_0^{(0,1)}$, we may appeal to lemma 4.3 and thus obtain the estimate

$$|\beta_j| \leqslant Cj\left[\frac{b_j}{j^2}\|g\|_2 + \frac{b_j}{j}\right]$$

where $b_j = |\mu_j(q) - \mu_j(q_0)|$. Bearing in mind that $b_j \leqslant \varepsilon$ for $j = 1, \ldots, N$, we get

$$\|g\|_2^2 \leqslant R^2 \sum_{j=1}^{\infty} |\beta_j|^2 \leqslant 2R^2C^2\big\{\|b\|_2^2 + \big(\varepsilon^2\pi^2/6 + \|b\|_2^2/N^2\big)\|g\|_2^2\big\}.$$

It then follows that $\|g\|_2 \leqslant \tilde{C}\|b\|_2$ for some constant $\tilde{C}$, and hence for some positive constant $\tilde{M}$,

$$|\beta_j| \leqslant 4\tilde{M}b_j.$$

Finally, using lemma 5.1, and taking $\mathcal{M}$ to be the greater of $\tilde{M}$ and the constant B_1 appearing in lemma 5.1,

$$\left|\int_0^t g(s)\,\mathrm{d}s\right| = |(\chi_{[0,t]}, g)| \leqslant \sum_{j=1}^{\infty} |\beta_j||(\chi_{[0,t]}, \phi_j)| \leqslant 4\mathcal{M}^2 \sum_{j=1}^{\infty} \frac{b_j}{j}$$

$$\leqslant 4\mathcal{M}^2(1 + \log N)\varepsilon + \frac{4\mathcal{M}^2\|b\|_2}{N^{1/2}}.$$

This completes the proof.

8. Extensions and numerical results

We consider the possibility of improving the result in theorem 2.1 in two ways: by strengthening the norm and by improving the factor of $1/\sqrt{N}$ in the error bound to something smaller.

Strengthening the norm can be easily achieved if one is prepared to make *a priori* assumptions about the boundedness of $q - q_0$ in some stronger Sobolev space. We have the following result.

Theorem 8.1. *Suppose that q and q_0 are complex-valued functions in $L^2([0, 1])$ with the same mean value. Suppose also that $q - q_0$ lies in a bounded set in the Sobolev space $H^n([0, 1])$, $n \geqslant 0$. Let $a_j = |\lambda_j(q) - \lambda_j(q_0)|$ and $b_j = |\mu_j(q) - \mu_j(q_0)|$. Let $\varepsilon_0 \geqslant 0$ and $N_0 \in \mathbb{N}$ be fixed. Then for each $-1 \leqslant r \leqslant n$ there exists a constant C depending only on ε_0, N_0, r and q_0 such that the following statement is true.*

If $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $N \geqslant N_0$ and $\max\{a_1, \ldots, a_N, b_1, \ldots, b_N\} \leqslant \varepsilon$, then

$$\|q - q_0\|_{H^r} \leqslant C\left[\varepsilon \log N + \frac{\|a\|_2 + \|b\|_2}{\sqrt{N}}\right]^{(n-r)/(n+1)}.$$

Proof. The hypothesis that $q - q_0$ is bounded in $H^n([0, 1])$ means that q lies in a bounded set in $L^2([0, 1])$ determined by q_0, since $q_0 \in L^2([0, 1])$. Thus, the term $\exp(\|q\|_2)$ appearing in theorem 2.1) can be absorbed into the constant C. The result is then immediate from standard results in interpolation space theory, and in particular the inequality

$$\|f\|_{H^{(1-\theta)r+\theta s}} \leqslant C\|f\|_{H^r}^{1-\theta}\|f\|_{H^s}^{\theta},$$

for $0 \leqslant \theta \leqslant 1$ (see, e.g., [9]). □

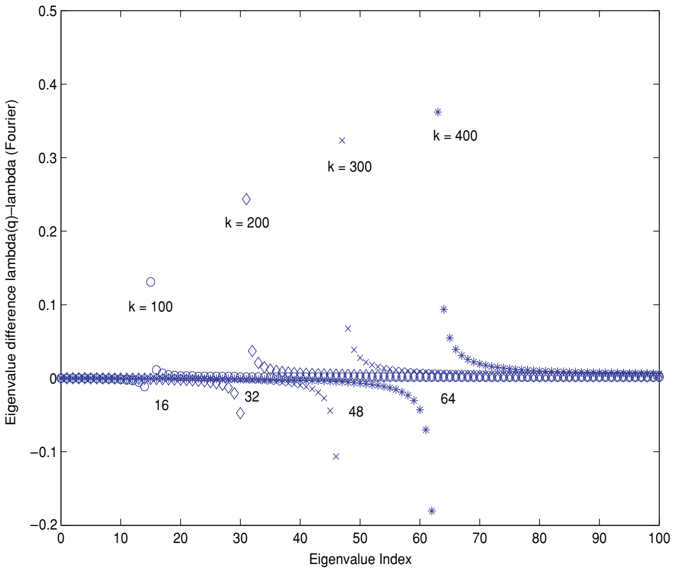

Figure 1. Differences between Fourier eigenvalues and eigenvalues for $q(x) = \sin(kx)$ for different values of k.

Note that q and q_0 are not each required to be in H^n for this result: it is enough that their difference possess the necessary smoothness. If one is using the technique of Rundell and Sacks [15] for solving the inverse problem, for instance, then $q - q_0$ will generally be smoother than q_0, and this improved error bound is then available.

The question of whether or not the factor of $1/\sqrt{N}$ is best possible is more difficult to address. Consider the example

$$q_0(x) = 0,$$

for which all the eigenvalues are known, and the sequence of potentials $q(x) = \sin(kx)$, $k \in \mathbb{N}$, for which

$$\sup_{x\in[0,1]} \left| \int_0^x (q(t) - q_0(t))\,\mathrm{d}t \right| = \frac{1}{k}.$$

Fix $\varepsilon > 0$ and let N be (as a function of k) the number of eigenvalues approximated to within ε. If our error bound is tight then we should have

$$\frac{1}{k} \geqslant C\left[\varepsilon \log(N) + \frac{\|a\|_2 + \|b\|_2}{\sqrt{N}}\right]$$

for some positive constant k and therefore, in particular,

$$\frac{1}{k} \geqslant \frac{C}{\sqrt{N}},$$

yielding $N \geqslant Ck^2$. Is this seen in practice? The results in figure 1 suggest that this is not so. It seems that one has only $N \geqslant Ck$, which would indicate that the term $\frac{1}{\sqrt{N}}$ in the error bound ought to be $\frac{1}{N}$.

However, in order to obtain such a bound one would need to have more subtle eigenvalue asymptotics than the ones we have used, which are $\sqrt{\lambda_j} = j\pi + \alpha_j$, $\sqrt{\mu_j} = (j - 1/2)\pi + \beta_j$, with $(\alpha_j), (\beta_j) \in \ell^2$. In view of McLaughlin's diffeomorphic relationship between these remainder sequences in ℓ^2 and the potentials in L^2, better asymptotics would actually depend on further smoothness assumptions on the potential.

Appendix. Riesz bases

Let H be a separable Hilbert space. The set $\{f_n : n \in \mathbb{N}\} \subset H$ is called a Riesz basis of H if its closed linear span equals H and if there exist positive constants r and R such that

$$r^2 \sum_{n=1}^{N} |c_n|^2 \leqslant \left\| \sum_{n=1}^{N} c_n f_n \right\|^2 \leqslant R^2 \sum_{n=1}^{N} |c_n|^2$$

for every sequence $c : n \mapsto c_n$ in $\ell^2(\mathbb{N})$.

If $\{f_n : n \in \mathbb{N}\}$ is a Riesz basis of H then there exists a unique Riesz basis $\{g_n : n \in \mathbb{N}\}$ such that $(f_n, g_k) = \delta_{n,k}$ for all $n, k \in \mathbb{N}$. The sequences f_n and g_n are then called biorthogonal.

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