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THE INVERSE RESONANCE PROBLEM FOR JACOBI OPERATORS

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Abstract

It is proved in this paper that super-exponentially decaying, possibly non-selfadjoint perturbations of the free Jacobi operator are uniquely determined by the location of all their eigenvalues and resonances.

1. *Introduction*

It is well known that the one-dimensional Schrödinger equation

$$-y'' + qy = \lambda y, \qquad x \in [0, \infty)$$

has essential spectrum on $[0, \infty)$ together with possible negative eigenvalues when q is real-valued with $q(x) \to 0$ as $x \to \infty$. In addition, it may possess resonance poles whose presence is reflected in the properties of the spectral measure of the associated operator in $L^2[0, \infty)$ (and, as such, may be observable in laboratory experiments). The spectral points of the operator may be characterized in terms of the limiting behavior of the so-called *Titchmarsh–Weyl m-function* as the now complex-valued spectral parameter approaches the real line. The Titchmarsh–Weyl function is itself defined in terms of the Jost function, and the resonance poles are the zeros of the analytic continuation of the Jost function to the second Riemann sheet, provided that such a continuation exists. Poles that are close to the real line can be manifested also by a local change in the spectral measure which gives rise to the phenomenon of spectral concentration. These notions of spectral concentration and resonance poles have been the subject of much study (see [**12**, **19**, **1**, **5**, **14**]), as has the region to which analytic continuation is possible [**2**, **3**]. When q has super-exponential decay, it is known that such continuation is possible to all of the second Riemann sheet.

The inverse problem in the presence of resonances has also been studied. The first result goes back to Marchenko, who showed that when $(1+x)q(x) \in L^1[0, \infty)$, then q is uniquely determined by the scattering phase, eigenvalues and norming constants [**13**], these quantities being determined at least for compactly supported potentials by the Jost function. Further, in some recent work [**6**] a more-general approach to this problem has been proposed. The formulation of that method allows even complex-valued potentials to be considered, and it is applicable whenever the concept of a resonance makes sense. However, the method requires that rather

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precise information be available on the asymptotic distribution of the resonance poles; this information is available only when q has compact support or is $\exp(-x^2)$, and this leads to some technical problems in the analysis. The example contained in [**6**] shows that if the potential q has compact support, then the eigenvalues and resonance poles uniquely determine q. In [**4**] this result has been extended to a class of perturbations of algebro-geometric potentials.

In this paper we turn our attention to related problems, but this time focusing on Jacobi matrices. There are, of course, many papers that deal with the problem of the recovery of the Jacobi matrix, given the eigenvalues (see, for example, [**9**, **18**, **7**, **15**]). Further, there has been some discussion of the recovery of the coefficients of the matrix, given the scattering data [**8**], which should be viewed as a discrete analogue of the famous Marchenko result [**13**] for differential equations. In addition, the non-selfadjoint Jacobi matrix has been treated, and it has been shown that its coefficients can be recovered from the generalized spectral function [**10**, **11**].

In this work we extend our approach, developed in [**6**], and we show that, given the eigenvalues and resonances of the associated operator, we can determine the coefficients of the Jacobi matrix up to certain similarity transformations. This result for Jacobi matrices is more embracing than the result given in [**6**] for differential equations, because in that work the authors need good asymptotic information on the location of the eigenvalues for each specific problem. However, in the discrete setting we are able to prove a theorem that covers all coefficients that are eventually super-exponentially small.

Section 2 contains the basic definitions needed to define the Jacobi operator and the Titchmarsh–Weyl m-function. It also discusses the required similarity transformation. Section 3 is devoted to the main theorem of the paper, while Section 4 contains a discussion of the inverse problem when super-exponential coefficients are present.

2. *Preliminaries*

2.1. *Jacobi operators*

Throughout this paper, we assume that

$$(a_1, a_2, \ldots), \qquad (b_1, b_2, \ldots), \qquad \text{and} \qquad (c_1, c_2, \ldots)$$

are bounded complex-valued sequences such that the products

$$\prod_{n=1}^{\infty} a_n \qquad \text{and} \qquad \prod_{n=1}^{\infty} c_n$$

converge absolutely to nonzero numbers. In particular, none of the numbers a_n or c_n is equal to zero. Set $a_0 = 1$. We consider here the Jacobi operator given by

$$(Jf)(n) = a_{n-1} f(n-1) + b_n f(n) + c_n f(n+1), \qquad n \in \mathbb{N},$$

and the difference equation

$$(Jf)(n) = \mu f(n), \qquad n \in \mathbb{N}, \tag{2.1}$$

where μ is a complex parameter.

The case where all a_n and c_n are equal to 1 and all b_n are equal to 0 is called the *free Jacobi operator*.

2.2. *Similarity of Jacobi operators*

Let $u : \mathbb{N}_0 \longrightarrow \mathbb{C} - \{0\} : n \mapsto u_n$ be a sequence such that $u_0 = u_1 = 1$ and

$$\sum_{n=1}^{\infty} \left| \frac{u_{n+1}}{u_n} - 1 \right| < \infty.$$

Under this assumption, the sequences $n \mapsto u_n$ and $n \mapsto 1/u_n$ are convergent, and hence bounded.

Given such a sequence $n \mapsto u_n$, define

$$\tilde{a}_n = \frac{u_n}{u_{n+1}} a_n, \qquad \tilde{b}_n = b_n, \qquad \text{and} \qquad \tilde{c}_n = \frac{u_{n+1}}{u_n} c_n.$$

Note that $a_n c_n = \tilde{a}_n \tilde{c}_n$, and that the products $\prod_{n=1}^{\infty} \tilde{a}_n$ and $\prod_{n=1}^{\infty} \tilde{c}_n$ are absolutely convergent.

Conversely, given sequences $n \mapsto a_n$, $n \mapsto \tilde{a}_n$, $n \mapsto c_n$, and $n \mapsto \tilde{c}_n$ such that the corresponding infinite products converge absolutely and $a_n c_n = \tilde{a}_n \tilde{c}_n$ for all $n \in \mathbb{N}$, define $u_0 = u_1 = 1$ and

$$u_{n+1} = \prod_{k=1}^{n} \frac{\tilde{c}_k}{c_k} = \prod_{k=1}^{n} \frac{a_k}{\tilde{a}_k}$$

to obtain a sequence $n \mapsto u_n$ with the properties mentioned above.

The operator J associated with the sequences $n \mapsto a_n$, $n \mapsto b_n$, and $n \mapsto c_n$ and the operator $\tilde{J}$ associated with the sequences $n \mapsto \tilde{a}_n$, $n \mapsto \tilde{b}_n$, and $n \mapsto \tilde{c}_n$ are similar in the sense that

$$(Jf)(n) = u_n^{-1} \tilde{J}(uf)(n) \qquad \text{for all } n \in \mathbb{N}.$$

(More formally, $J \circ \iota$ is similar to $\tilde{J} \circ \iota$ where $\iota : \ell^2(\mathbb{N}) \longrightarrow \{f \in \ell^2(\mathbb{N}_0) : f(0) = 0\}$ is the isomorphism that assigns the vector $(0, f(1), f(2), \ldots)$ to $(f(1), f(2), \ldots)$.) In particular, any Jacobi operator under consideration is similar to one where $a_n = c_n$ and $\mathrm{Re}(a_n) \geqslant 0$ for all $n \in \mathbb{N}$.

2.3. *The m-function*

Let $c(\mu, \cdot)$ and $s(\mu, \cdot)$ denote those solutions of (2.1) satisfying the initial conditions $c(\mu, 0) = s(\mu, 1) = 1$ and $c(\mu, 1) = s(\mu, 0) = 0$.

Under the given circumstances, the equation $Jy = \mu y$ has at most one linearly independent square summable solution. This solution is called a *Weyl solution*. The *Weyl m-function* is then defined for any μ for which a Weyl solution exists, as the coefficient m for which $\Psi(\mu, \cdot) = c(\mu, \cdot) + ms(\mu, \cdot)$ is a Weyl solution (m may be infinity if $s(\mu, \cdot)$ is square summable). Note that

$$m(\mu) = \frac{\Psi(\mu, 1)}{\Psi(\mu, 0)}.$$

We also define $M(z) = m(z + 1/z)$ when $0 < |z| < 1$. We are interested in potentials for which M may be extended meromorphically to the entire complex plane.

If J and $\tilde{J}$ are similar, the similarity being established by a sequence $n \mapsto u_n$, and if $n \mapsto \Psi(\mu, n)$ is a Weyl solution of $Jy = \mu y$, then $n \mapsto u_n \Psi(\mu, n)$ is a Weyl solution of $\tilde{J}y = \mu y$, and vice versa. Moreover, J and $\tilde{J}$ have the same m-function.

One can therefore expect only to retrieve a Jacobi operator from its m-function up to similarity.

Let Σ denote a fixed open sector of the complex plane whose vertex is at the origin, and let $\mathcal{L}_\Sigma$ denote the set of those Jacobi expressions satisfying the following conditions: (i) $a_0 = 1$, (ii) $a_n = c_n \neq 0$ for $n \in \mathbb{N}$, (iii) $\sum_{n=1}^{\infty} 1/|a_n| = \infty$, and (iv) the intersection of Σ and the closed convex hull of the set $\{a_{n-1}+b_n+a_n-ra_n : n \in \mathbb{N}, r \geqslant 0\}$ is bounded. The following theorem is proved in [**16**].

THEOREM 2.1. *Let J and $\tilde{J}$ be two Jacobi expressions in $\mathcal{L}_\Sigma$, and let m and $\tilde{m}$ be the associated m-functions. Let $\mathcal{R}$ be a ray in Σ emanating from the origin. Then the following statement holds: $m(\mu) - \tilde{m}(\mu) = O(\mu^{-2N-1})$ on $\mathcal{R}$ if and only if $b_n = \tilde{b}_n$ and $a_{n-1}^2 = \tilde{a}_{n-1}^2$ for $n \in \{1, \ldots, N\}$.*

Since we may assume that $\mathrm{Re}(a_n) \geqslant 0$, we see that all the Jacobi matrices under consideration here are contained in $\mathcal{L}_\Sigma$ if $\Sigma = \{\lambda : |\arg(\lambda)| < \alpha\}$ and $\alpha < \pi$.

3. *The main theorem*

DEFINITION 3.1. Let $\mathcal{C}$ be the family of Jacobi operators J satisfying the conditions mentioned in Section 2.1 for which there exists a function $\psi : \mathbb{C} \times \mathbb{N}_0 \longrightarrow \mathbb{C}$ with the following properties.

(1) For every nonzero complex number z, the functions $\psi(z, \cdot)$ and $\psi(1/z, \cdot)$ are nontrivial solutions of the difference equation $Jy = (z + 1/z)y$.

(2) There exists a nonzero number p such that

$$\psi(z,0)\psi(1/z,1) - \psi(1/z,0)\psi(z,1) = p\left(\frac{1}{z} - z\right)$$

for all $z \in \mathbb{C} - \{0\}$.

(3) $\psi(z, \cdot)$ is square summable for all z in some nonempty open subset of the unit disk $|z| < 1$.

(4) $\psi(\cdot, 0)$ and $\psi(\cdot, 1)$ are entire functions, and $\psi(\cdot, 0)$ has growth order zero.

(5) There exist a number A and a sequence of circles $\gamma_n : t \mapsto r_n \exp(it)$ such that r_n tends to infinity and

$$\left|\frac{\psi(z,1)}{\psi(z,0)}\right| \leqslant A|z|$$

for all z on the given circles.

THEOREM 3.1. *Assume that J, a Jacobi operator associated with the sequences $n \mapsto a_n$, $n \mapsto b_n$ and $n \mapsto c_n$, is in $\mathcal{C}$, and let ψ be the function from Definition 3.1 establishing that fact. Then the zeros of $\psi(\cdot, 0)$ and their multiplicities determine uniquely the quantities b_n and $a_n c_n$ for all $n \in \mathbb{N}$.*

Proof. It is well known that the Titchmarsh–Weyl m-function determines the b_n and the $a_n c_n$ in the selfadjoint case (where $a_n = c_n > 0$). By Theorem 2.1, this is also true in the present case, taking into account the similarity transformations mentioned above. Since, of course, M determines m, we have only to show that the given information suffices to determine M.

Next note that, without loss of generality, we may assume that $\psi(0,0)=1$. It follows from condition (3) that

$$M(z)=\frac{\psi(z,1)}{\psi(z,0)}.$$

Condition (4) implies that M is meromorphic, and that its poles are the zeros of $\psi(\cdot,0)$. We denote the poles of M by the pairwise distinct numbers $z_1,z_2,\dots$, and we use $n_1,n_2,\dots$ for their respective multiplicities. The poles are labelled such that $|z_1|\leqslant|z_2|\leqslant\dots$.

Let $h_z(\mu)=(z/\mu)^2/(z-\mu)$. Also define $\gamma_n(t)=r_n\exp(it)$ for $t\in[0,2\pi]$ and $B_n=\{z:|z|<r_n\}$. Note that $z=0$ is not a pole of M. Therefore, by the residue theorem,

$$\frac{1}{2\pi i}\int_{\gamma_n}h_z(\mu)M(\mu)d\mu=-M(z)+M(0)+M'(0)z+\sum_{z_j\in B_n}\operatorname{res}_{z_j}(h_zM)$$

if $0\neq|z|<r_n$ and if z is none of the poles of M. According to condition (5), the integral on the left tends to zero as n tends to infinity, proving firstly the convergence of the series and secondly that

$$M(z)=M(0)+M'(0)z+\sum_{j=1}^{\infty}\operatorname{res}_{z_j}(h_zM). \tag{3.1}$$

Next we will determine the residues of h_zM at the poles of M. To do this, let

$$f_j(\mu)=\frac{(\mu-z_j)^{n_j}}{\psi(\mu,0)}.$$

Then

$$\begin{aligned}\operatorname{res}_{z_j}(h_zM)&=\frac{1}{(n_j-1)!}(\psi(\cdot,1)h_zf_j)^{(n_j-1)}(z_j)\\&=\sum_{r=0}^{n_j-1}\sum_{s=0}^{r}\alpha_{r,s}\psi^{(r)}(z_j,1)h_z^{(s)}(z_j)f_j^{(n_j-1-r-s)}(z_j),\end{aligned}$$

where the $\alpha_{r,s}$ are certain rational numbers. Therefore each of the residues of h_zM may be computed once we know the function $\psi(\cdot,0)$ (and hence the functions f_j) and the numbers $\psi^{(r)}(z_j,1)$ for $r=0,\dots,n_j-1$.

Since $\psi(0,0)=1$, and since $\psi(\cdot,0)$ has growth order zero, Hadamard's factorization theorem shows that

$$\psi(z,0)=\prod_{k=1}^{\infty}(1-z/z_k)^{n_k}.$$

Next, letting $\psi(1/z,0)=\zeta_0(z)$, $\psi(1/z,1)=\zeta_1(z)$ and $W(z)=1/z-z$, we see from condition (2) that

$$\psi(z,0)\zeta_1(z)-\zeta_0(z)\psi(z,1)=pW(z). \tag{3.2}$$

Taking $r\leqslant n_j-1$ derivatives of this equation and evaluating at z_j gives

$$\sum_{\ell=0}^{r}\binom{r}{\ell}\psi^{(\ell)}(z_j,1)\zeta_0^{(r-\ell)}(z_j)=-pW^{(r)}(z_j).$$

Assume now that $z_j\neq1/z_j$. Then $\zeta_0(z_j)\neq0$ because $\psi(z_j,\cdot)$ and $\psi(1/z_j,\cdot)$ are linearly independent. Since ζ_0 and its derivatives are known functions, we know their values at z_j, and therefore we may compute the numbers $\psi(z_j,1)$, $\psi'(z_j,1),\dots,\psi^{(n_j-1)}(z_j,1)$ recursively. Note that each one equals p times a known quantity independent of p. If $z_j=1/z_j$, then the first derivative of equation (3.2) gives

$$2\psi'(z_j,0)\zeta_1(z_j)=-2p.$$

This proves that necessarily $n_j=1$ and $\psi(z_j,1)=\zeta_1(z_j)=-p/\psi'(z_j,0)$, again a multiple of p.

Since the function $z\mapsto h_z^{(s)}(z_j)$ is of order z^2 as z tends to zero, regardless of s, the same is therefore true of the function $z\mapsto\sum_{j=1}^{\infty}\operatorname{res}_{z_j}(h_zM)$. Therefore we now find that $M(z)=M(0)+M'(0)z+pz^2G(z)$, where G is analytic at zero and does not depend on p.

Since the asymptotic behavior of the m-function is $m(z+1/z)=z+O(z^2)$ as z tends to zero (see [**16**] for a proof in the non-selfadjoint setting) we see, furthermore, that $M(0)=0$ and $M'(0)=1$. Thus the theorem is proved once we show that the number p is also determined from the zeros of $\psi(\cdot,0)$. Since $M(z)=z+pz^2G(z)$, condition (2) implies that

$$1+\frac{p}{z}G(1/z)=z^2+pz^3G(z)+\frac{p(1-z^2)}{\psi(z,0)\psi(1/z,0)}.$$

Hence, if $\psi(\cdot,0)$ is not bounded, then there is a sequence $n\mapsto w_n$ converging to zero such that $1+pG(1/w_n)/w_n$ tends to zero; that is,

$$-p^{-1}=\lim_{n\to\infty}G(1/w_n)/w_n.$$

If, on the other hand, $\psi(\cdot,0)$ is bounded, it must be constant and hence identically equal to 1. In this case $M(z)=z$, so that the value of p is not even needed. □

4. *Super-exponentially decaying perturbations of the free Jacobi operator*

The free Jacobi equation

$$y(n-1)+y(n+1)=(z+1/z)y(n),\qquad n\in\mathbb{N},$$

has the solution

$$s_0(z,m,n)=\frac{z}{1-z^2}(z^{m-n}-z^{n-m}).$$

LEMMA 4.1. *The following statements hold.*

(1) *If* $1/Z\leqslant|z|\leqslant Z$ *where* $Z\geqslant 2$, *then*

$$|s_0(z,m,n)|\leqslant\tfrac{4}{3}Z^{3|m-n|-3}.$$

(2) *If* $|z|\leqslant 1/2$, *then*

$$|s_0(z,m,n)|\leqslant\tfrac{4}{3}|z|^{-|m-n|+1}.$$

(3) *If* $|z|\geqslant 2$, *then*

$$|s_0(z,m,n)|\leqslant\tfrac{4}{3}|z|^{|m-n|-1}.$$

Proof. Since $s_0(z,n,n) = 0$ and $s_0(z,m,n) = -s_0(z,n,m)$, we may assume that $m > n$. Note that

$$s_0(z,m,n) = -\left(\frac{1}{z}\right)^{m-n-1} \sum_{k=0}^{m-n-1} z^{2k}.$$

To prove statement (1), we estimate both $|z|$ and $1/|z|$ by Z and use the fact that $Z^2/(Z^2-1) \leqslant 4/3$ if $Z \geqslant 2$. To prove statement (2), we estimate the sum by $4/3$, the value of the geometric series for $1/2$. To prove statement (3), note that $s_0(z,m,n) = s_0(1/z,m,n)$. $\square$

Define

$$K(z,m,n) = (a_m - 1)s_0(z,m+1,n) + b_m s_0(z,m,n) + (c_{m-1} - 1)s_0(z,m-1,n),$$
$$\psi_0(z,n) = \frac{z^n}{a_n},$$

and

$$\psi_{k+1}(z,n) = \frac{1}{a_n} \sum_{m=n+1}^{\infty} K(z,m,n)\psi_k(z,m).$$

Let $\alpha = \sup\{1/|a_n| : n \in \mathbb{N}\}$ and

$$\gamma_m = \max\{|a_m - 1|, |b_m|, |c_{m-1} - 1|\},$$

where we set $c_0 = 1$ (even though it is never needed). We will make the following assumption for the remainder of this section. There exist $C > 0$ and $\beta > 1$ such that $\gamma_m \leqslant C\exp(-m^\beta)$ for all $m \in \mathbb{N}$. Then $\|\gamma\|_1 \leqslant C$.

LEMMA 4.2. *Suppose that $Z \geqslant 2$, and define $N = \lfloor (5\log Z)^{1/(\beta-1)} \rfloor$. If $1/Z \leqslant |z| \leqslant Z$, then*

$$|\psi_k(z,n)| \leqslant \alpha(8\alpha C)^k \begin{cases} Z^{4N-3n} & \text{if } 0 \leqslant n \leqslant N-k, \\ Z^{-4n-k} & \text{if } n \geqslant N-k+1. \end{cases}$$

Proof. From Lemma 4.1, we find that

$$|K(z,m,n)| \leqslant 4\gamma_m Z^{3m-3n}$$

if $m > n$. Moreover, $\gamma_m \leqslant CZ^{-5m}$, provided that $m \geqslant N+1$. Note that

$$\sum_{m=k+1}^{\infty} a^{-m} \leqslant 2a^{-k-1} \leqslant a^{-k}$$

if $a > 2$.

We prove the lemma by induction on k. If $0 \leqslant n \leqslant N-1$, then

$$\begin{aligned} |\psi_1(z,n)| &\leqslant 4\alpha^2 \sum_{m=n+1}^{N} \gamma_m Z^{4m-3n} + 4\alpha^2 \sum_{m=N+1}^{\infty} \gamma_m Z^{4m-3n} \\ &\leqslant 4\alpha^2 C Z^{4N-3n} + 4\alpha^2 C Z^{-3n} \sum_{m=N+1}^{\infty} Z^{-m} \\ &\leqslant 8\alpha^2 C Z^{4N-3n}, \end{aligned}$$

where we used $|z| \leqslant Z$ and, for the second sum, $\gamma_m \leqslant CZ^{-5m}$.

If $n \geqslant N$, we obtain

$$\begin{aligned}|\psi_1(z,n)| &\leqslant 4\alpha^2 \sum_{m=n+1}^{\infty} \gamma_m Z^{4m-3n} \\ &\leqslant 4\alpha^2 C Z^{-3n} \sum_{m=n+1}^{\infty} Z^{-m} \\ &\leqslant 8\alpha^2 C Z^{-4n-1}.\end{aligned}$$

Assuming now the validity of the inequality in our statement for some $k \in \mathbb{N}$, we obtain, for $0 \leqslant n \leqslant N-k-1$:

$$\begin{aligned}|\psi_{k+1}(z,n)| &\leqslant 4\alpha^2(8\alpha C)^k \sum_{m=n+1}^{N-k} \gamma_m Z^{4N-3n} + 4\alpha^2 C(8\alpha C)^k \sum_{m=N-k+1}^{\infty} Z^{-m-3n-k} \\ &\leqslant 4\alpha^2 C(8\alpha C)^k Z^{4N-3n} + 4\alpha^2 C(8\alpha C)^k Z^{-3n-N} \\ &\leqslant \alpha(8\alpha C)^{k+1} Z^{4N-3n},\end{aligned}$$

where we used $\gamma_m \leqslant \|\gamma\|_1 \leqslant C$. For $n \geqslant N-k$, we find that

$$|\psi_{k+1}(z,n)| \leqslant 4\alpha^2(8\alpha C)^k \sum_{m=n+1}^{\infty} \gamma_m Z^{-m-3n-k} \leqslant \alpha(8\alpha C)^{k+1} Z^{-4n-k-1},$$

again using $\gamma_m \leqslant C$. □

LEMMA 4.3. *Let $a > 1$ and $N = \lfloor (\log a)^{1/(\beta-1)} \rfloor$. If $|z| \leqslant 1/2$, then*

$$|\psi_k(z,n)| \leqslant \alpha(8\alpha C)^k \begin{cases} |z|^n & \text{if } 0 \leqslant n \leqslant N-k, \\ |z|^n a^{-n-k} & \text{if } n \geqslant N-k+1. \end{cases}$$

Proof. The proof of this lemma is practically identical to the proof of the previous one, except that we now use the estimates

$$|K(z,m,n)| \leqslant 4\gamma_m |z|^{n-m} \qquad \text{for } m > n$$

and

$$\gamma_m \leqslant C a^{-m} \qquad \text{for } m \geqslant N+1.$$ □

THEOREM 4.4. *Suppose that there exist $C > 0$ and $\beta > 1$ such that*

$$\gamma_m = \max\{|a_m - 1|, |b_m|, |c_{m-1} - 1|\} \leqslant C \exp(-m^\beta)$$

for all $m \in \mathbb{N}$. Then the associated Jacobi operator is in the class $\mathcal{C}$ defined in Definition 3.1. Moreover, this operator is uniquely determined by the location of its eigenvalues and resonances (repeated according to their multiplicities) up to similarity.

Proof. Let ψ_k, $k \in \mathbb{N}_0$, be the functions defined above, and note that they are entire functions. We find from Lemma 4.2 that $\psi(z,n) = \sum_{k=0}^{\infty} \psi_k(z,n)$ is absolutely and uniformly convergent in the annulus $\{z : 1/Z \leqslant |z| \leqslant Z\}$ if $Z \geqslant \max\{2, 16\alpha C\}$, since

$$\sum_{k=N-n+1}^{\infty} |\psi_k(z,n)| \leqslant \alpha Z^{-4n} \sum_{k=1}^{\infty} (8\alpha C)^k Z^{-k}$$

where $N = \lfloor (5 \log Z)^{1/(\beta-1)} \rfloor$. Hence $\psi(\cdot,n)$ is an analytic function on $\mathbb{C} - \{0\}$.

Lemma 4.2 shows that zero is a removable singularity of $\psi(\cdot, n)$ since, choosing there $a \geqslant 16\alpha C$, we have

$$\sum_{k=N'-n+1}^{\infty} |\psi_k(z,n)| \leqslant \alpha(|z|/a)^n,$$

where $N' = \lfloor (\log 16\alpha C)^{1/(\beta-1)} \rfloor$.

Standard arguments now show that $\psi(z,\cdot)$ satisfies the Volterra equation

$$\psi(z,n) = \frac{z^n}{a_n} + \frac{1}{a_n} \sum_{m=n+1}^{\infty} K(z,m,n)\psi(z,m),$$

as well as the Jacobi equation

$$a_{n-1}\psi(z,n-1) + b_n\psi(z,n) + c_n\psi(z,n+1) = (z+1/z)\psi(z,n), \qquad n \in \mathbb{N}.$$

To see the latter, replace $\psi(z,n-1)$ by the series above, and use

$$K(z,m,n-1) + K(z,m,n+1) = (z+1/z)K(z,m,n).$$

After a little algebra, one finds that the equation is satisfied if $K(z,n,\cdot)$ satisfies the initial conditions $K(z,n,n-1) = -(z+1/z)(a_n-1) - b_n$ and $K(z,n,n) = c_{n-1} - a_n$. Solving the difference equation under these initial conditions gives the above expression for K. Thus ψ satisfies condition (1) of Definition 3.1.

Define $p_1 = 1$ and, recursively, $p_{n+1} = c_n p_n / a_{n-1}$. If we denote the determinant of the matrix

$$\begin{pmatrix} f(n) & g(n) \\ p_{n+1}f(n+1) & p_{n+1}g(n+1) \end{pmatrix}$$

by $[f,g](n)$, then $[y_1,y_2](n)$ is independent of n when y_1 and y_2 denote two solutions of $Jy = (z+1/z)y$. Choosing $y_1 = \psi(z,\cdot)$ and $y_2 = \psi(1/z,\cdot)$, we find that

$$p\left(\frac{1}{z} - z\right) = p_{n+1}(\psi(z,n)\psi(1/z,n+1) - \psi(1/z,n)\psi(z,n+1)),$$

for all $n \in \mathbb{N}$.

Assume that $Z \geqslant \max\{2, 16\alpha C\}$ and $1 < |z| \leqslant Z$. We obtain from Lemma 4.2 that both $|\psi(z,m)|$ and $|\psi(1/z,m)|$ are bounded by $2\alpha Z^m$, provided that $m > N = \lfloor (5\log Z)^{1/(\beta-1)} \rfloor$. Therefore, if $n \geqslant N$, then

$$\left|\psi(z,n) - \frac{z^n}{a_n}\right| \leqslant \alpha \sum_{m=n+1}^{\infty} 4\gamma_m Z^{3m-3n} 2\alpha Z^m \leqslant 8\alpha^2 C Z^{-4n},$$

using $\gamma_m \leqslant CZ^{-5m}$. With these estimates, it follows easily that

$$\lim_{n\to\infty} (\psi(z,n)\psi(1/z,n+1) - \psi(1/z,n)\psi(z,n+1)) = \frac{1}{z} - z$$

since $\lim_{n\to\infty} a_n a_{n+1} = 1$. This implies, finally, that

$$p = \lim_{n\to\infty} p_{n+1} = \prod_{n=1}^{\infty} \frac{c_n}{a_n}.$$

This proves the validity of condition (2).

Lemma 4.3 shows that $\psi(z,\cdot)$ is square summable if $0 < |z| < 1/2$. Hence $\psi(z,\cdot)$ is the Weyl solution for $\mu = z + 1/z$. This proves condition (3).

To establish condition (4), remember that we have shown before that the functions $\psi(\cdot, n)$ are entire. Let z be sufficiently large, and let $Z = |z|$. Then Lemma 4.2 gives

$$
\begin{aligned}
|\psi(z,0)| &\leqslant \frac{1}{a_n} + \sum_{k=1}^{N} \alpha(8\alpha C)^k |z|^{4N} + \sum_{k=N+1}^{\infty} \alpha(8\alpha C)^k |z|^{-k} \\
&\leqslant 1 + \alpha + \alpha N (8\alpha C)^N |z|^{4N},
\end{aligned}
$$

since the last sum converges to a number less than α, once $|z| > 16\alpha C$. The growth of this latter expression is equal to the growth of

$$
|z|^{4N} = \exp(4N \log|z|) \leqslant \exp\left((5 \log|z|)^{\beta/(\beta-1)}\right),
$$

which has growth order zero.

By a theorem of Wiman [**17**], the minimum modulus of an entire function of growth order less than 1/2 is unbounded. Hence there exists a sequence $n \mapsto r_n$ such that r_n tends to infinity and $\min\{|\psi(z,0)| : |z| = r_n, n \in \mathbb{N}\} \geqslant 1$. For sufficiently large z, we also have $|\psi(1/z,0)| \geqslant 1/2$ and $|M(1/z)| \leqslant 1$. Therefore, using the already established condition (2), we have

$$
|M(z)| = \left| M(1/z) - \frac{pW(z)}{\psi(z,0)\psi(1/z,0)} \right| \leqslant 1 + 2|pW(z)| \leqslant (1 + 4|p|)|z|
$$

on any of the above-mentioned circles with sufficiently large radius. This establishes condition (5).

The last statement now follows simply by an application of Theorem 3.1. □

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