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Stability for the inverse resonance problem for a Jacobi operator with complex potential

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Abstract

The potential of a discrete half-line Schrödinger operator is uniquely determined by the location of its Dirichlet eigenvalues and resonances. In a practical setting one can expect only to know a finite number of these. In this paper we give an estimate for the difference of two potentials (one of them finitely supported) for which eigenvalues and small resonances are the same but which may differ with respect to their large resonances.

1. Introduction

The famous Gelfand–Levitan theorem on the inverse spectral problem states that the (real-valued) potential q of the equation $-y'' + qy = \lambda y$ on $[0, \infty)$ is uniquely determined by the spectral function. Similarly, Marchenko's inverse scattering theorem states that q is uniquely determined by the scattering phase, the location of the eigenvalues and the norms of those eigenfunctions with a given asymptotic[3]. Korotyaev [5], Brown, Knowles and Weikard [1], and Brown and Weikard [3] pointed out situations where the potential is uniquely determined by just the location of all eigenvalues and resonances. Later Brown, Naboko and Weikard [2] established an analogous result for a certain class of Jacobi operators. (We mention in passing that the methods in [1–3] allow the treatment of complex-valued potentials.)

While the spectral function or the scattering phase cannot be obtained directly from laboratory measurements the eigenvalues and the resonances are fundamental objects in quantum physics. Eigenvalues and at least small resonances can be observed in the laboratory. Moreover, asymptotics of resonances suggest that large resonances are physically unimportant. Thus, in a practical setting, only finitely many data are given and the inverse problem is then expected to have infinitely many solutions. The usual philosophy in the numerical analysis literature in such circumstances is to construct recovery algorithms which select one of the infinitely many possible solutions. Numerical experiments are then carried out in which finite

[3] Both theorems require the condition $\int_0^\infty (1 + x)|q(x)|\,dx < \infty$ to be satisfied.

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spectral data are generated from some known potential, and the algorithm is declared to be good or bad according to how well it manages to recover the selected potential in some norm. This process is meaningless unless one can prove that all of the infinitely many solutions to the finite data inverse problems are 'close', in some suitable sense. The point of this paper is to establish such a result in the case of a discrete Schrödinger equation. We emphasize that the corresponding result for the continuous Schrödinger equation on a half-line appears to be an unsolved problem. To the best of our knowledge the present result provides the first proof of a stability result for an inverse resonance problem.

The analogous result for the classical problem of $-y'' + qy = \lambda y$ on the compact interval $[0, 1]$ was treated in [6], where we established an error bound for $\sup\left\{\left|\int_0^x (q - q_0)\,\mathrm{d}t\right| : x \in [0, 1]\right\}$ for the case when the first N Dirichlet–Dirichlet eigenvalues and the first N Dirichlet–Neumann eigenvalues for q and q_0, respectively, coincide to within an error of ε.

Another result, close in spirit to the present one, is that of Hitrik [4], which concerns an inverse scattering problem in $L^2(\mathbb{R})$ when finitely many values of the reflection coefficient are known.

We denote the set of complex-valued sequences defined on $\mathbb{N}_0$ and $\mathbb{N}$ by $\mathbb{C}^{\mathbb{N}_0}$ and $\mathbb{C}^{\mathbb{N}}$, respectively. Let Q be a bounded sequence in $\mathbb{C}^{\mathbb{N}}$. An operator

$$\mathbf{J}: \quad \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}}$$

such that

$$(\mathbf{J}y)(n) = y(n-1) + Q_n y(n) + y(n+1), \qquad n \in \mathbb{N}$$

is called a discrete Schrödinger or Jacobi operator. The sequence Q is called the potential associated with $\mathbf{J}$. We are interested in the equation $\mathbf{J}y = \lambda\hat{y}$, where $\hat{y}$ denotes the restriction of $y \in \mathbb{C}^{\mathbb{N}_0}$ to $\mathbb{N}$. In the following our notation will not distinguish anymore between y and $\hat{y}$ as the meaning is always clear from the context.

Definition 1. *Let $\mathcal{C}$ be the family of Jacobi operators $\mathbf{J}$ with bounded potentials for which there exists a function $\psi : \mathbb{C} \times \mathbb{N}_0 \to \mathbb{C}$ with the following properties.*

(1) For every nonzero complex number z the functions $\psi(z, \cdot)$ and $\psi(1/z, \cdot)$ are nontrivial solutions of the difference equation $\mathbf{J}y = (z + 1/z)y$.

(2) For some $p \neq 0$ and all $z \in \mathbb{C} - \{0\}$:

$$W(z) = \psi(z, 0)\psi(1/z, 1) - \psi(1/z, 0)\psi(z, 1) = p(1/z - z).$$

(3) $\psi(z, \cdot)$ is square summable for all z in some nonempty open subset of the unit disk $|z| < 1$.

(4) $\psi(\cdot, 0)$ and $\psi(\cdot, 1)$ are entire functions and $\psi(\cdot, 0)$ has growth order zero.

(5) There is a number A and a sequence of circles $\gamma_n : t \mapsto r_n \exp(\mathrm{i}t)$ such that r_n tends to infinity and

$$\left|\frac{\psi(z, 1)}{\psi(z, 0)}\right| \leqslant A|z|$$

for all z on the given circles.

Without loss of generality, we assume from now on that $\psi(0, 0) \neq 0$ and, in fact, $\psi(0, 0) = 1$. For, if $\psi \in \mathcal{C}$ and $\psi(\cdot, 0)$ has a zero at zero of order k then the function $(z, n) \mapsto z^{-k}\psi(z, n)$ is also in $\mathcal{C}$. The only fact whose validity is not obvious is that $z \mapsto z^{-k}\psi(z, 1)$ is still entire. This follows since $\psi(z, 1)/\psi(z, 0)$ is the Titchmarsh–Weyl m-function for $\lambda = z + 1/z$ which tends to zero as z tends to zero[4] so that $\psi(\cdot, 1)$ has a zero at zero of order $k + 1$ at least.

[4] See, e.g., [7, theorem 4.2] taking into account that the definition of m is slightly different there.

If $\psi(z,\cdot)$ is square summable it is called the Jost solution of the difference equation $\mathbf{J}y = (z+1/z)y$. The function $\psi(\cdot,0)$ is called the Jost function. The following theorem is a special case of theorem 3.1 proved in [2].

Theorem 1.1. *Assume that* $\mathbf{J}$*, a Jacobi operator with bounded potential* Q*, is in* $\mathcal{C}$ *and let* ψ *be the function from definition 1 establishing that fact. Then the zeros of* $\psi(\cdot,0)$ *and their multiplicities determine uniquely the quantities* Q_n *for all* $n \in \mathbb{N}$*.*

In this paper we require the potential Q of the Jacobi operator $\mathbf{J}$ not only to be bounded but to be super-exponentially decaying. More precisely, we make the following assumption throughout the remainder of the paper for any potential under consideration.

Hypothesis 1. *There are numbers* $C > 0$ *and* $\beta > 1$ *such that* $|Q_m| \leqslant C\exp(-m^\beta)$ *for all* $m \in \mathbb{N}$*.*

In this circumstance $\mathbf{J}$ is in the class $\mathcal{C}$, i.e., there is a function $\psi : \mathbb{C}\times\mathbb{N}_0 \to \mathbb{C}$ so that, as a corollary, the zeros of $\psi(\cdot,0)$ determine Q uniquely. This follows, as a special case, from [2, theorem 4.4]. In appendix A, we give an independent proof which constructs the function ψ and shows that $p = 1$ in condition (2) of definition 1. In view of condition (3) of definition 1 and the proof of theorem A.1, a value of $\lambda = z + 1/z$ is an eigenvalue or a resonance if $\psi(z,0) = 0$ and either $|z| < 1$ or $|z| \geqslant 1$, respectively. The zeros of $\psi(\cdot,0)$ are denoted by z_n. They are repeated according to their multiplicity and ordered so that $|z_n|$ is monotone nondecreasing. Since $\psi(0,0) = 1$ and $\psi(\cdot,0)$ has growth order zero, Hadamard's factorization theorem gives us now that

$$\psi(z,0) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right).$$

Let $K(n,m)$ denote the Taylor coefficients of $\psi(z,n) - z^n$, i.e., let $\psi(z,n) = z^n + \sum_{m=0}^{\infty} K(n,m)z^m$ and define, for $n, m \geqslant 1$,

$$(\Delta K)(n,m) = K(n-1,m) + K(n+1,m) - K(n,m+1) - K(n,m-1).$$

Lemma 1.2. *Let* ψ *be the function constructed in appendix A. If* $|z| \leqslant 1/2$*, then* $|\psi(z,n) - z^n| \leqslant |z|^n(\mathrm{e}^{2|z|\beta(n)} - 1)$*, where* $\beta(n) = \sum_{m=n+1}^{\infty}|Q_m|$*. Moreover, the Taylor coefficients* $K(n,m)$ *have the following properties:*

(1) if $m \leqslant n$ *then* $K(n,m) = 0$*;*
(2) $K(n,n+1) = -\sum_{m=n+1}^{\infty} Q_m$*, and*
(3) $(\Delta K)(n,m) + Q_n K(n,m) = 0$ *for* $1 \leqslant n < m$*.*

Proof. We use the notation introduced in appendix A. Note that for $|z| < 1$ we may estimate $\sum_{k=0}^{m-n-1}|z|^{2k+1}$ by $|z|/(1-|z|^2)$. Since $1 - |z|^2 \geqslant 1/2$ when $|z| \leqslant 1/2$, one proves by induction that

$$\left|\left(T_z^k\,\mathbf{e}\right)(n)\right| \leqslant \frac{(2|z|\beta(n))^k}{k!}$$

which gives the stated estimate.

Since $\varphi(\cdot,n)$ is entire and $\psi(z,n) = z^n\varphi(z,n)$ it is clear that $K(n,m) = 0$ if $m < n$. Also $z \mapsto T_z$ is continuous and $T_0 = 0$. Thus, letting z tend to zero in $T_z\varphi(z,\cdot) = \varphi(z,\cdot) - \mathbf{e}$ shows that $\varphi(0,n) = 1$ and $K(n,n) = 0$ for all n. This proves (1).

Because of this we have now

$$K(n, n+1) = \dot{\varphi}(0, n) = \lim_{z\to 0} \frac{\varphi(z, n) - 1}{z} = \lim_{z\to 0} \frac{1}{z}(T_z\varphi(z, \cdot))(n)$$
$$= -\lim_{z\to 0} \sum_{m=n+1}^{\infty} Q_m \varphi(z, m) \sum_{k=0}^{m-n-1} z^{2k} = -\sum_{m=n+1}^{\infty} Q_m.$$

This proves (2).

The difference equation satisfied by $\psi(z, \cdot)$ shows next that

$$0 = (Q_n + K(n-1, n) - K(n, n+1))z^n + \sum_{m=n+1}^{\infty} \{K(n-1, m) + K(n+1, m) + Q_n K(n, m)$$
$$- K(n, m-1) - K(n, m+1)\}z^m$$

for all $n \in \mathbb{N}$ which proves (3). □

Lemma 1.3. *Suppose that Q is finitely supported, more precisely assume that, for some N, the potential Q satisfies $Q_n = 0$ for $n > N$ but $Q_N \neq 0$ (we include also the case $Q = 0$ whence $N = 0$). Then $\psi(\cdot, n)$ is a polynomial of degree at most $2N - n - 1$ when $n < N$. When $n \geqslant N$ then $\psi(z, n) = z^n$.*

Proof. We know from lemma 1.2 that $\psi(z, n) = z^n$ when $n \geqslant N$ and $|z| \leqslant 1/2$ (and hence for all z). We need to prove that $K(n, m) = 0$ for $m \geqslant 2N - n$. This is certainly true when $n \geqslant N$. Employing the identity $(\Delta K)(n, m) + Q_n K(n, m) = 0$ which holds for $1 \leqslant n < m$ and induction prove our claim. □

In section 2 we prove that a potential Q is pointwise small if it has no eigenvalues and if all resonances are large, equivalently, if all zeros of $\psi(\cdot, 0)$ are large. In section 3 we compare two potentials Q and $\tilde{Q}$ where $\tilde{Q}$ is finitely supported. We show there that, if all the eigenvalues and resonances of $\tilde{Q}$ are also eigenvalues and resonances of Q and if all other resonances of Q are comparatively large, then $Q - \tilde{Q}$ is pointwise small. We illustrate these results in some basic examples presented in appendix B.

2. Potentials with only very large resonances

In this section we are interested in the case where there are no eigenvalues and only very large resonances. More precisely, we require subsequently that $\sum_{n=1}^{\infty} 1/|z_n| = \varepsilon \leqslant 1/4$. Recall that the reciprocals of the zeros of an entire function of growth order zero are always summable. In view of lemma 1.2 the key to bounding the potential Q is to bound the Taylor coefficients $K(n, m)$. The following lemma is a first step in this direction.

Lemma 2.1. *Suppose that $\sum_{n=1}^{\infty} 1/|z_n| = \varepsilon$. Then the Taylor coefficients of $\psi(z, 0) - 1$ satisfy*

$$|K(0, m)| \leqslant \left(\frac{e\varepsilon}{m}\right)^m \leqslant 3\varepsilon^m.$$

Proof. By Cauchy's formula for Taylor coefficients, we have

$$K(0, m) = \frac{1}{2\pi \mathrm{i}} \int_{|z|=R} \frac{\psi(z, 0)}{z^{m+1}} \,\mathrm{d}z$$

for any $R > 0$. Thus

$$|K(0, m)| \leqslant R^{-m} \sup\{|\psi(z, 0)| : |z| = R\}.$$

But

$$|\psi(z,0)| \leqslant \prod_{n=1}^{\infty}\left(1+\frac{|z|}{|z_n|}\right) \leqslant \prod_{n=1}^{\infty}\exp(|z|/|z_n|) \leqslant \exp(|z|\varepsilon).$$

The choice $R = m/\varepsilon$ gives the best estimate on $K(0,m)$. □

In order to obtain estimates on $K(n,m)$ for $n > 0$ we proceed by induction. The induction step is provided by the following lemma.

Lemma 2.2. *Suppose that* $|K(n,m)| \leqslant a(4/3)^n\varepsilon^{m+n}$ *for some* $a \in (0,3]$, $\varepsilon \in (0,1/4]$, $n \in \mathbb{N}_0$ *and all* $m \geqslant n+1$. *Then* $|K(n+1,m)| \leqslant a(4/3)^{n+1}\varepsilon^{m+n+1}$ *for all* $m \geqslant n+2$.

Proof. Define the potential $\hat{Q}$ by $\hat{Q}_m = Q_{m+n}$. The Jost solution $\hat{\psi}$ associated with $\hat{Q}$ is given by $\hat{\psi}(z,k) = z^{-n}\psi(z,k+n)$. This, in turn, means that $\hat{K}(k,m) = K(k+n,m+n)$. Thus, by assumption, $|\hat{K}(0,m)| \leqslant a(4/3)^n\varepsilon^{m+2n}$.

When we insert the Taylor expansions of $\hat{\psi}(\cdot,0)$ and $\hat{\psi}(\cdot,1)$ into

$$\hat{\psi}(z,0)\hat{\psi}(1/z,1) - \hat{\psi}(1/z,0)\hat{\psi}(z,1) = 1/z - z$$

we obtain that the relationship

$$\hat{K}(1,m) = \hat{K}(0,m+1) + \sum_{j=2}^{\infty}(\hat{K}(0,j+m) - \hat{K}(0,j-m))\hat{K}(1,j)$$

must necessarily hold for all $m \in \mathbb{N}$. Ignoring the equation for $m = 1$ and defining $x_m = \hat{K}(1,m+1)$, $h_m = \hat{K}(0,m+2)$, and $F_{m,j} = \hat{K}(0,j+m+2) - \hat{K}(0,j-m)$ we arrive at the system

$$x = h + Fx. \tag{1}$$

Since the $K(1,m)$ are Taylor coefficients they must be superexponentially decaying as m tends to infinity. Therefore we seek our solution x in $H_\varepsilon = \{y \in \mathbb{C}^{\mathbb{N}} : \varepsilon^{-m}|y_m| \text{ is bounded}\}$. The function $y \mapsto \|y\| = \sup\{\varepsilon^{-m}|y_m| : m \in \mathbb{N}\}$ provides a norm for H_ε. One now checks easily that F is a bounded linear operator from H_ε to itself and that $\|F\| \leqslant a\varepsilon^2(1+\varepsilon^2)/(1-\varepsilon^2) \leqslant 1/4$. Also $\|h\| \leqslant a(4/3)^n\varepsilon^{2n+2}$. Thus, system (1) has a unique solution x for which $\|x\| \leqslant a(4/3)^{n+1}\varepsilon^{2n+2}$. This gives now the desired estimate on $K(n+1,m) = \hat{K}(1,m-n) = x_{m-n-1}$. □

Theorem 2.3. *Suppose Q satisfies hypothesis 1. Let z_n denote the zeros of the associated Jost function and assume that* $\sum_{n=1}^{\infty} 1/|z_n| = \varepsilon \leqslant 1/4$. *Then*

$$|K(n,m)| \leqslant 3(4/3)^n\varepsilon^{m+n}.$$

Moreover, the potential Q satisfies

$$|Q_n| \leqslant 3(4/3)^n\varepsilon^{2n-1}.$$

Proof. The first statement is proved by induction. The validity of the statement for $n = 0$ follows from lemma 2.1 and the induction step is provided by lemma 2.2 with $a = 3$.

Since, by lemma 1.2, $\sum_{m=n+1}^{\infty} Q_m = -K(n,n+1)$ the second statement follows from the first with the aid of the triangle inequality. □

3. Perturbation of a finitely supported potential

Suppose that $\tilde{Q}$ is a potential with support in $\{1, 2, \ldots, N\}$ and that Q is a potential which satisfies hypothesis 1. We denote the associated Jost solutions by $\tilde{\psi}$ and ψ, respectively. According to lemmas 1.2 and 1.3, we have

$$\tilde{\psi}(z, n) = z^n + \sum_{j=n+1}^{2N-n-1} T(n, j) z^j$$

for appropriate coefficients $T(n, j)$. We also assume that the zeros of $\tilde{\psi}(\cdot, 0)$ are also zeros of $\psi(\cdot, 0)$ but that the latter function has possibly other very large zeros. More precisely, we assume that[5]

$$\psi(z, 0) = \tilde{\psi}(z, 0) \prod_{n=1}^{\infty} (1 - z/z_n),$$

where $\sum_{n=1}^{\infty} |z_n|^{-1} = \varepsilon$. We then define quantities $K(n, m)$ by setting

$$\psi(z, n) = \tilde{\psi}(z, n) + \sum_{m=n+1}^{\infty} K(n, m) z^m.$$

In the following we mimic the approach of section 2, getting first an estimate on the $K(0, m)$ from the main hypothesis and then on the $K(1, m)$ by using the Wronskian relationship. Again an induction argument allows us to get the estimates on the $K(n, m)$ for $n \geqslant 2$. Naturally, more details will be involved.

Lemma 3.1. *There exists a constant M_0 depending only on $\tilde{Q}$ such that*

$$|K(0, m)| \leqslant M_0 \begin{cases} \varepsilon & \text{if} \quad 1 \leqslant m \leqslant 2N \\ \varepsilon^{m+1-2N} & \text{if} \quad m \geqslant 2N. \end{cases}$$

Proof. When $c_m, m \geqslant 1$, denotes the mth Taylor coefficient of $\prod_{n=1}^{\infty}(1 - z/z_n)$ then, as in lemma 2.1, $|c_m| \leqslant 3\varepsilon^m$. Note that

$$K(0, m) = c_m + \sum_{j=1}^{2N-1} T(0, j) c_{m-j}$$

when we agree to set $c_m = 0$ for $m \leqslant 0$. Our claim follows now from the triangle inequality. □

We now set out to estimate the coefficients $K(1, m)$. Since

$$\psi(z, 0)\psi(1/z, 1) - \psi(1/z, 0)\psi(z, 1) = \frac{1}{z} - z = \tilde{\psi}(z, 0)\tilde{\psi}(1/z, 1) - \tilde{\psi}(1/z, 0)\tilde{\psi}(z, 1) \quad (2)$$

we may compare coefficients in the resulting Laurent series. Comparing the coefficients of z^0 gives a triviality but the coefficients for z^m, where $m \neq 0$ give conditions the $K(1, m)$ have to obey. In fact, the coefficients of z^m and z^{-m} give identical conditions, namely

$$\begin{aligned} 0 = K(1, m) - K(0, m+1) - \sum_{j=2}^{\infty} (T(0, j+m) - T(0, j-m) + K(0, j+m) \\ - K(0, j-m)) K(1, j) + \sum_{j=1}^{\infty} (T(1, j+m) - T(1, j-m)) K(0, j) \end{aligned} \quad (3)$$

[5] We can allow for $\psi(\cdot, 0)$ to have only finitely many zeros by setting $1/z_n = 0$ for sufficiently large n.

if we agree to set $T(j, k) = 0$ unless $k \in \{1 + j, \dots, 2N - 1 - j\}$. We will consider these equations only for $m \geqslant 2$ (the equation for $m = 1$ will be discarded) and shift indices accordingly.

Again, the $K(1, m)$ must be superexponentially decaying as m tends to infinity. Therefore the sequence $m \mapsto K(1, m+1)$ must be an element of $\mathbb{C}^{2N-2} \oplus H_\varepsilon$, where we think of $\mathbb{C}^{2N-2}$ as a normed space with the supremum norm and, as before, of H_ε as a vector space normed by $\|y\| = \sup\{\varepsilon^{-m}|y_m| : m \in \mathbb{N}\}$, where $0 < \varepsilon < 1$. According to this decomposition we may express equations (3) as

$$\begin{pmatrix} A_0 + A_1 & B \\ C & I + D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}.$$

More explicitly, we have $x_m = K(1, m+1)$, $y_m = K(1, 2N - 1 + m)$,

$$\begin{aligned}
A_{0;m,j} &= \delta_{m,j} + T(0, j-m) - T(0, j+m+2), \\
A_{1;m,j} &= K(0, j-m) - K(0, j+m+2), \\
B_{m,j} &= T(0, 2N-2+j-m) + K(0, 2N-2+j-m) - K(0, 2N+j+m), \\
C_{m,j} &= -K(0, 2N+j+m), \\
D_{m,j} &= T(0, j-m) + K(0, j-m) - K(0, 4N-2+m+j),
\end{aligned}$$

and

$$g_m = K(0, m+2) + \sum_{j=2}^{2N-2} T(1, j)K(0, j+m+1) - \sum_{j=m+2}^{2N-2} T(1, j)K(0, j-m-1),$$

$$h_m = K(0, m+2N) + \sum_{j=2}^{2N-2} T(1, j)K(0, 2N-1+m+j).$$

The following two lemmas can now be derived easily when the $K(0, m)$ satisfy the estimate stated in lemma 3.1.

Lemma 3.2. *There is a positive constant M', depending only on $\tilde{Q}$, such that $\|g\| \leqslant M'\varepsilon$ and $\|h\| \leqslant M'\varepsilon$.*

Lemma 3.3. *The operators $B : H_\varepsilon \to \mathbb{C}^{2N-2}$, $C : \mathbb{C}^{2N-2} \to H_\varepsilon$, and $D : H_\varepsilon \to H_\varepsilon$ are all bounded. In fact, if $0 < \varepsilon < 1$ there is a constant M'', depending only on $\tilde{Q}$, such that $\|B\| \leqslant M''\varepsilon/(1-\varepsilon)$, $\|C\| \leqslant M''\varepsilon^2/(1-\varepsilon)$ and $\|D\| \leqslant M''\varepsilon/(1-\varepsilon)$.*

Note that for $N = 1$ we deal only with the equation $(I + D)y = h$ which shows that $\|y\| \leqslant 2M'\varepsilon$, provided ε is small enough so that $M''\varepsilon/(1-\varepsilon) \leqslant 1/2$. In the general case, we have the equation $(I + D)y = h - Cx$. Assuming again that $M''\varepsilon/(1-\varepsilon) \leqslant 1/2$ we obtain $y = (I + D)^{-1}(h - Cx)$. Using this result, the equation $(A_0 + A_1)x + By = h$ becomes

$$(A_0 + A_1 - B(I + D)^{-1}C)x = g - B(I + D)^{-1}h.$$

Here the right-hand side is a vector whose norm is of order ε. We will show below that the matrix A_0 is invertible. The norm of its inverse depends only on the coefficients $T(0, m)$, i.e., on the potential $\tilde{Q}$. Each entry of A_1 and hence the norm of A_1 is of order ε. Therefore $\|x\|$ is of order ε, i.e., each one of the coefficients $K(1, 2) = x_1, \dots, K(1, 2N-1) = x_{2N-2}$ is of order ε. This, in turn, implies that the norm of $y = (I + D)^{-1}(h - Cx)$ is also of order ε, i.e., $K(1, 2N - 1 + m) = y_m$ is of order ε^{m+1} for all $m \in \mathbb{N}$.

To prove the invertibility of A_0 consider the case when the $K(0, m)$ are all zero. In this case our system becomes

$$\begin{pmatrix} A_0 & B_0 \\ 0 & I + D_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where B_0 and D_0 are obtained from B and D, respectively, by setting $K(0, \cdot) = 0$. Again, the norm of D_0 is small so that we must have $y = 0$ which implies that also $A_0 x = 0$. Let x be any solution of this equation. Then $\psi(z, 1) - \tilde{\psi}(z, 1) = \sum_{m=1}^{2N-2} x_m z^{m+1}$ has at least a double zero at zero and equation (2) becomes

$$z^{-2}\psi(z, 0) \sum_{m=0}^{2N-3} x_{m+1} z^{-m} = z^2 \psi(1/z, 0) \sum_{m=0}^{2N-3} x_{m+1} z^m.$$

Since $\psi(\cdot, 0)$ is a polynomial of degree $2N - 1$ with $\psi(0, 0) = 1$, this shows that x_{2N-2} must be zero, so that the $\psi(\cdot, 1) - \tilde{\psi}(\cdot, 1)$ has at most $2N - 4$ non-vanishing zeros. Now note that $\psi(\cdot, 0)$ has at least $2N - 3$ zeros different from 0 and ± 1 and that, if z_0 is one of those, then $\psi(1/z_0, 0) \neq 0$. All of these must also be zeros of $\psi(\cdot, 1) - \tilde{\psi}(\cdot, 1)$. Hence $\psi(\cdot, 1) = \tilde{\psi}(\cdot, 1)$, i.e., $x = 0$ is the only solution of $A_0 x = 0$.

Thus we have the validity of the following lemma.

Lemma 3.4. *Suppose that*

$$|K(0, m)| \leqslant M_0 \begin{cases} \varepsilon & \text{if} \quad 1 \leqslant m \leqslant 2N \\ \varepsilon^{m+1-2N} & \text{if} \quad m \geqslant 2N. \end{cases}$$

Then there are constants $\varepsilon_1 \leqslant 1/2$ and $M_1 \geqslant M_0$, depending only on $\tilde{Q}$, such that the Taylor coefficients $K(1, m)$ of $\psi(\cdot, 1) - \tilde{\psi}(\cdot, 1)$ satisfy

$$|K(1, m)| \leqslant M_1 \begin{cases} \varepsilon & \text{if} \quad 2 \leqslant m \leqslant 2N - 1 \\ \varepsilon^{m+2-2N} & \text{if} \quad m \geqslant 2N - 1 \end{cases}$$

provided that $\varepsilon \leqslant \varepsilon_1$.

To obtain estimates on $K(n, m)$ for $2 \leqslant n \leqslant N$ we proceed again by induction as in the proof of lemma 2.2. This gives the existence of an $\varepsilon_n \leqslant \varepsilon_{n-1}$ and an $M_n \geqslant M_{n-1}$ such that

$$|K(n, m)| \leqslant M_n \begin{cases} \varepsilon & \text{if} \quad n + 1 \leqslant m \leqslant 2N - n \\ \varepsilon^{m+n+1-2N} & \text{if} \quad m \geqslant 2N - n \end{cases}$$

as long as $\varepsilon \leqslant \varepsilon_n$. The numbers M_n and ε_n depend only on $\tilde{Q}$.

Next we define $\hat{Q}$ by $\hat{Q}_m = Q_{N+m}$ and let $\hat{\psi}$ be the associated Jost solution and $\hat{K}(n, m)$ its Taylor coefficients. Then, as before, $\hat{K}(k, m) = K(N + k, N + m)$ and, in particular, $\hat{K}(0, m) = K(N, N + m)$ and $|\hat{K}(0, m)| \leqslant M_N \varepsilon^{m+1}$. If $\varepsilon \leqslant \varepsilon_{N+1} = \min\{\varepsilon_N, 3/M_N\}$ we get from lemma 2.2, using $a = \varepsilon M_N$, that $|\hat{K}(n, m)| \leqslant M_N (4/3)^n \varepsilon^{m+n+1}$, i.e.,

$$|K(n, m)| = |\hat{K}(n - N, m - N)| \leqslant M_N (4/3)^{n-N} \varepsilon^{m+n+1-2N}.$$

Since $Q_n - \tilde{Q}_n = K(n, n + 1) - K(n - 1, n)$ we obtain now our final result.

Theorem 3.5. *Suppose that $\tilde{Q}$ is a potential with support in $\{1, 2, \ldots, N\}$ and that Q is a potential which satisfies hypothesis 1 and that their Jost functions satisfy*

$$\psi(z, 0) = \tilde{\psi}(z, 0) \prod_{n=1}^{\infty} (1 - z/z_n),$$

where $\sum_{n=1}^{\infty} |z_n|^{-1} = \varepsilon$. *Then there exist constants* $\mu > 0$ *and* $M > 1$, *depending only on* $\tilde{Q}$, *such that*

$$|Q_n - \tilde{Q}_n| \leqslant M \begin{cases} \varepsilon & \text{if} \quad n \leqslant N \\ (4/3)^{n-N-1} \varepsilon^{2(n-N)} & \text{if} \quad n \geqslant N+1 \end{cases}$$

provided $\varepsilon \leqslant \mu$.

Proof. Choose $\mu = \varepsilon_{N+1}$ and $M = 2M_N$. □

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Appendix A. Construction of the Jost solutions

In [2, section 4], Jost solutions are constructed for a class of Jacobi operators which are more general than those considered here. This culminates in theorem 4.4 which implies that the Jacobi operator $\mathbf{J}$ with a potential satisfying hypothesis 1 is in the class $\mathcal{C}$. Below we give an independent proof of this fact, which is shorter, less technical, and gives additional information on the function ψ it constructs. Some additional properties of ψ were deduced in lemma 1.2.

Theorem A.1. *Let* $\mathbf{J}$ *be a discrete Schrödinger operator with potential* Q. *Suppose that there are numbers* $C > 0$ *and* $\beta > 1$ *such that* $|Q_m| \leqslant C \exp(-m^\beta)$ *for all* $m \in \mathbb{N}$. *Then* $\mathbf{J}$ *is an element of the class* $\mathcal{C}$ *defined in definition 1*.

Proof. Fix $R \geqslant 2$ and let $\Omega = \{z \in \mathbb{C} : |z| < R\}$. Since $\sum_{k=0}^{m-n-1} |z|^{2k+1} \leqslant \frac{4}{3} R^{2m-2n-1} \leqslant R^{2m-2n}$ whenever $z \in \Omega$ and since $\alpha(R) = \sum_{m=1}^{\infty} |Q_m| R^{2m}$ is finite, due to our decay condition on the sequence Q, we may define the bounded operator

$$T_z : \ell^\infty(\mathbb{N}_0) \to \ell^\infty(\mathbb{N}_0) : (T_z y)(n) = -\sum_{m=n+1}^{\infty} \sum_{k=0}^{m-n-1} Q_m z^{2k+1} y(m).$$

Let $\mathbf{e}(n) = 1$ for all $n \in \mathbb{N}_0$. Then $T_z^k\, \mathbf{e} \in \ell^\infty(\mathbb{N}_0)$ for all $k \in \mathbb{N}_0$. Note that

$$|(T_z \mathbf{e})(n)| \leqslant R^{-2n} \sum_{m=n+1}^{\infty} |Q_m| R^{2m} \leqslant \alpha(R) R^{-2n}.$$

Defining $\beta(n) = \sum_{m=n+1}^{\infty} |Q_m|$ one may show by induction that

$$\left|\left(T_z^k\, \mathbf{e}\right)(n)\right| \leqslant \alpha(R) R^{-2n} \frac{\beta(n)^{k-1}}{(k-1)!}$$

for all $k \in \mathbb{N}$. This induction proof uses that $\sum_{m=n+1}^{\infty} |Q_m| \beta(m)^{k-1} \leqslant \beta(n)^k / k$ which follows after a summation by parts using that the sequence $\beta(m)$ is non-negative and non-increasing.

Thus $j \mapsto \sum_{k=0}^{j} \left(T_z^k\, \mathbf{e}\right)(n)$ is a Cauchy sequence whose limit we will denote by $\varphi(z, n)$. Since $|\varphi(z, n) - 1| \leqslant \alpha(R) R^{-2n}\, \mathrm{e}^{\beta(n)}$, it follows that $\varphi(z, \cdot) \in \ell^\infty(\mathbb{N}_0)$. Since R may be arbitrarily large, $\varphi(z, n)$ is defined for every $z \in \mathbb{C}$.

Next note that, by the continuity of T_z,

$$(T_z\varphi(z,\cdot))(n) = \varphi(z,n) - 1$$

and that therefore

$$\varphi(z,n+1) - \varphi(z,n) = (T_z\varphi(z,\cdot))(n+1) - (T_z\varphi(z,\cdot))(n) = \sum_{m=n+1}^{\infty} Q_m z^{2m-2n-1}\varphi(z,m).$$

A simple computation shows now that the sequence $n \mapsto z^n\varphi(z,n)$ solves the Jacobi equation $\mathbf{J}y = (z+1/z)y$, i.e., the function ψ defined by $\psi(z,n) = z^n\varphi(z,n)$ satisfies property (1) of definition 1.

The estimate $|\varphi(z,n) - 1| \leqslant \alpha(R)R^{-2n}\,\mathrm{e}^{\beta(n)}$ also shows that $\lim_{n\to\infty}\varphi(z,n) = 1$. This and the fact that the Wronskian $W(z)$ of $\psi(z,\cdot)$ and $\psi(1/z,\cdot)$ is constant show that

$$\begin{aligned}\psi(z,n)&\psi(1/z,n+1) - \psi(1/z,n)\psi(z,n+1)\\ &= \frac{1}{z}\varphi(z,n)\varphi(1/z,n+1) - z\varphi(1/z,n)\varphi(z,n+1)\end{aligned}$$

converges to $1/z - z$ as n tends to infinity. This proves property (2) of definition 1 with $p = 1$.

Because the sequence $\varphi(z,\cdot)$ is bounded and since $n \mapsto z^n$ is square summable when $|z| < 1$ we have that $\psi(z,\cdot)$ is square summable for $|z| < 1$. This proves property (3) of definition 1.

Next, the fact that the series defining $T_z^k\,\mathbf{e}$ and $\varphi(z,n)$ converge absolutely and uniformly in compact subsets of Ω shows that the functions $\varphi(\cdot,n)$ as well as the functions $\psi(\cdot,n)$ are entire. To estimate their growth note that

$$|\psi(z,n)| \leqslant \alpha(R)R^{-n}\,\mathrm{e}^{\beta(n)} \leqslant \alpha(R)\,\mathrm{e}^{\|Q\|_1}$$

when $|z| = R$. The growth of $\psi(\cdot,n)$ is therefore determined by the growth of α as function of R. Define $N = \lfloor(3\log R)^{1/(\beta-1)}\rfloor$. Then $|Q_m| \leqslant CR^{-3m}$ when $m \geqslant N+1$ so that

$$\begin{aligned}\alpha(R) &= \sum_{m=1}^{N}|Q_m|R^{2m} + \sum_{m=N+1}^{\infty}|Q_m|R^{2m} \leqslant N\|Q\|_\infty R^{2N} + C\sum_{m=N+1}^{\infty}R^{-m}\\ &\leqslant N\|Q\|_\infty \exp((3\log R)^{\beta/(\beta-1)}) + \frac{CR}{R-1}.\end{aligned}$$

Since this has growth order zero we proved property (4) of definition 1.

To prove property (5) we use the already established condition (2). We find

$$\left|\frac{\psi(z,1)}{\psi(z,0)}\right| = \left|\frac{\psi(1/z,1)}{\psi(1/z,0)} - \frac{W(z)}{\psi(z,0)\psi(1/z,0)}\right|. \tag{A.1}$$

Thus we need lower bounds on $|\psi(z,0)|$ and $|\psi(1/z,0)|$ and an upper bound on $|\psi(1/z,1)|$ at least for z lying on certain circles. By a theorem of Wiman [8] the minimum modulus of an entire function of growth order less than $1/2$ is unbounded. Hence there exists a sequence of circles with radius r_n such that r_n tends to infinity and $\min\{|\psi(z,0)| : |z| = r_n, n \in \mathbb{N}\} \geqslant 1$. To obtain the other estimates note that

$$\|T_{1/z}\| \leqslant \|Q\|_1\frac{|1/z|}{1-|1/z|^2} \leqslant \frac{1}{3}$$

when z is sufficiently large. Since $\varphi(z,\cdot) - 1 = T_z\varphi(z,\cdot)$ we find

$$\|\varphi(1/z,\cdot)\|_\infty - 1 \leqslant \|\varphi(1/z,\cdot) - \mathbf{e}\|_\infty \leqslant \tfrac{1}{3}\|\varphi(1/z,\cdot)\|_\infty$$

which implies $\|\varphi(1/z,\cdot)\|_\infty \leqslant 3/2$ and $\|\varphi(1/z,\cdot)-\mathbf{e}\|_\infty \leqslant 1/2$. Therefore $|\psi(1/z,0)| \geqslant 1/2$ and $|\psi(1/z,1)| \leqslant 3/(2|z|)$ always assuming that z is sufficiently large. From (A.1)

$$\left|\frac{\psi(z,1)}{\psi(z,0)}\right| \leqslant \frac{5}{|z|} + 2|z| \leqslant 3|z|$$

on any of the above mentioned circles with sufficiently large radius. This establishes property (5). □

Appendix B. Examples

We illustrate here our results with the most basic examples.

Example B.1 (single site potentials). If $Q_n = 0$ for $n \geqslant 2$, we find $\psi(z, 0) = 1 - Q_1 z$. This potential has one resonance or eigenvalue determined by $z_1 = 1/Q_1$. If this resonance is very large then the potential is very small: $\sum_{n=1}^{\infty} 1/|z_n| = 1/|z_1| = \varepsilon$ implies $|Q_1| = \varepsilon$.

Example B.2 (double site potentials). If $Q_n = 0$ for $n \geqslant 3$, we find

$$\psi(z, 0) = 1 - (Q_1 + Q_2)z + Q_1 Q_2 z^2 - Q_2 z^3 = (1 - z/z_1)(1 - z/z_2)(1 - z/z_3).$$

Thus

$$Q_2 = \frac{1}{z_1 z_2 z_3}, \qquad Q_1 Q_2 = \frac{z_1 + z_2 + z_3}{z_1 z_2 z_3}, \qquad Q_1 + Q_2 = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}.$$

Note that this implies that z_1, z_2, and z_3 cannot be chosen independently.

Assuming that all three resonances are very large, i.e., $\sum_{n=1}^{3} 1/|z_n| = \varepsilon$, we find $Q_2 = O(\varepsilon^3)$ and $Q_1 = O(\varepsilon)$ as ε tends to zero since $1/|z_j| \leqslant \varepsilon$. This illustrates theorem 2.3.

We may also compare Q with a single-site potential $\tilde{Q}$ with a resonance or eigenvalue determined by $z_1 = a$. If Q has that same resonance or eigenvalue and if z_2 and z_3, determining the other two, are very large then we have $\varepsilon = 1/|z_2| + 1/|z_3|$ implying $Q_2 = O(\varepsilon^2)$ and $Q_1 - \tilde{Q}_1 = Q_1 - 1/a = O(\varepsilon)$ as ε tends to zero. This illustrates theorem 3.5.

Example B.3 (triple site potentials). If $Q_n = 0$ for $n \geqslant 4$ we find

$$\begin{aligned}\psi(z, 0) = 1 &- (Q_1 + Q_2 + Q_3)z + (Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3)z^2 - (Q_2 + Q_3 + Q_1 Q_2 Q_3)z^3 \\ &+ (Q_1 Q_3 + Q_2 Q_3)z^4 - Q_3 z^5 \\ = (1 &- z/z_1)(1 - z/z_2)(1 - z/z_3)(1 - z/z_4)(1 - z/z_5).\end{aligned}$$

Again, assume first that all resonances are very large for another illustration of theorem 2.3. Then $1/|z_j| \leqslant \varepsilon = \sum_{n=1}^{5} 1/|z_n|$ for $1 \leqslant j \leqslant 5$. This shows firstly that $Q_3 = O(\varepsilon^5)$ and $Q_1 + Q_2 = O(\varepsilon)$. With these estimates established we get $Q_1 Q_2 = O(\varepsilon^2)$ and $Q_1 Q_2 Q_3 = O(\varepsilon^7)$. Putting all this together we get $Q_2 = O(\varepsilon^3)$ and $Q_1 = O(\varepsilon)$.

One can also assume that only $z_2, \ldots, z_5$ are large and compare with a single-site potential $\tilde{Q}$ for which $\tilde{Q}_1 = 1/z_1$. Or one can assume that only z_4 and z_5 are large and compare with a double-site potential $\tilde{Q}$ with eigenvalues and/or resonances determined by z_1, z_2, and z_3. In accordance with theorem 3.5 one finds that $Q_3 = O(\varepsilon^4)$ in the former case and $Q_3 = O(\varepsilon^2)$ in the latter case. After a bit of already tedious algebra one can also establish the predicted estimates on $Q_1 - \tilde{Q}_1$ and $Q_2 - \tilde{Q}_2$ in either of these cases.

References

[1] Brown B M, Knowles I and Weikard R 2003 On the inverse resonance problem *J. London Math. Soc. (2)* **68** 383–401

[2] Brown B M, Naboko S and Weikard R 2005 The inverse resonance problem for Jacobi operators *Bull. London Math. Soc.* **37** 727–37

[3] Brown B M and Weikard R 2004 The inverse resonance problem for perturbations of algebro-geometric potentials *Inverse Problems* **20** 481–94

[4] Hitrik M 2000 Stability of an inverse problem in potential scattering on the real line *Commun. Part. Differ. Equ.* **25** 925–55

[5] Korotyaev E 2004 Inverse resonance scattering on the half line *Asymptot. Anal.* **37** 215–26

[6] Marletta M and Weikard R 2005 Weak stability for an inverse Sturm–Liouville problem with finite spectral data and complex potential *Inverse Problems* **21** 1275–90

[7] Weikard R 2004 A local Borg–Marchenko theorem for difference equations with complex coefficients *Partial Differential Equations and Inverse Problems (Contemporary Mathematics* vol 362*)* (Providence, RI: American Mathematical Society) pp 403–10

[8] Wiman A 1915 Über eine Eigenschaft der ganzen Funktionen von Höhe null *Math. Ann.* **76** 197–211