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Treibich-Verdier potentials and the stationary (m)KDV hierarchy

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1 Introduction

This paper represents a contribution to the problem of finding a characterization of all elliptic finite-gap solutions of the stationary (modified) Korteweg-de Vries ((m)KdV) hierarchy, a problem posed, e.g., in [39], p. 152. This theme dates back to a 1940 paper of Ince [31] who studied the Lamé potential

$$q(x) = -s(s+1)\wp(x+\omega_3), \quad s \in \mathbb{N},\ x \in \mathbb{R} \tag{1.1}$$

in connection with the second-order ordinary differential equation

$$\psi''(E,x) + [q(x) - E]\psi(E,x) = 0, \quad E \in \mathbb{C}. \tag{1.2}$$

Here $\wp(x) \equiv \wp(x;\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods (f.p.) $2\omega_1$, $2\omega_3$, $\operatorname{Im}(\omega_3/\omega_1) \neq 0$ (see [1], Ch. 18). In the special case where ω_1 is real and ω_3 is purely imaginary the potential q is real-valued and Ince's striking result [31], in modern spectral theoretic terminology, yields the fact that the self-adjoint operator L associated with the differential expression $\frac{d^2}{dx^2} + q$ in $L^2(\mathbb{R})$ has a finite-gap spectrum of the type

$$\sigma(L) = (-\infty, E_{2s}] \cup \bigcup_{m=1}^{s} [E_{2m-1}, E_{2(m-1)}], \quad E_{2s} < E_{2s-1} < \cdots < E_0. \tag{1.3}$$

In obvious notation, any potential q that amounts to a finite-gap spectrum of the type (1.3) is called a finite-gap potential. (The proper extension of this notion to complex-valued meromorphic q on the basis of elementary algebro-geometrical concepts will be given in Sect. 2.) Prior to Ince's investigations, Hermite (see, e.g., [30], pp. 118–122, 266–418, 475–478 and the quotations in [47] on pages 570-573) devoted a series of papers to Lamé's equation (1.2) with $q(x) = -s(s+1)\wp(x)$. In particular, Hermite proved the existence of a pair of solutions of (1.2) whose product is a polynomial of degree s with respect to the spectral parameter E. A combination of this fact and the approach by Its and Matveev [34] (or alternatively, combining Hermite's result with the recursion formalism briefly summarized in Sect. 2) then proves that $-s(s+1)\wp(x+\omega_3)$ (in the real-valued case) is a finite-gap potential with spectrum of the type (1.3). This remarkable result of Hermite on certain products of solutions of Lamé's equation is also recorded in the monographs by Halphen [29], pp. 494–531 and Burkhardt [9], pp. 343–353.

Subsequent work by Novikov [38], Dubrovin [12], Its and Matveev [34], and McKean and van Moerbeke [36] then showed that every finite-gap potential q satisfies appropriate higher-order stationary KdV equations. Moreover, the KdV flow $q_t = \frac{1}{4}q_{xxx} + \frac{3}{2}qq_x$ with initial condition $q(x,0) = -6\wp(x)$ was explicitly integrated by Dubrovin and Novikov [14] (see also [16], [17], [18], [33]) and found to be of the type

$$q(x,t) = -2\sum_{j=1}^{3} \wp(x - x_j(t)) \tag{1.4}$$

for an appropriate expression of $\{x_j(t)\}_{j=1}^3$. In their 1977 seminal paper [2], Airault, McKean and Moser gave the first systematic study of the isospectral torus $I_{\mathbb{E}}(q_0)$ of real-valued smooth potentials q_0 of the form

$$q_0(x) = -2\sum_{j=1}^{M} \wp(x - x_j) \tag{1.5}$$

with a finite-gap spectrum as in (1.3). Among a variety of results they proved that any element of $I_{\mathbb{E}}(q_0)$ is an elliptic function of the type (1.5) (for different sets $\{x_j\}_{j=1}^M$) with M constant throughout $I_{\mathbb{E}}(q_0)$ and $g := \dim I_{\mathbb{E}}(q_0) \leq M$. The next breakthrough occurred in 1988 when Verdier [46] published new explicit examples of elliptic finite-gap potentials. Verdier's examples inspired Belokolos and Enol'skii [5] and Smirnov [41] and subsequently Taimanov [42] and Kostov and Enol'skii [35] to find further such examples by employing the reduction process of Abelian integrals to elliptic integrals (see, e.g., [6]). Finally, this development culminated in the recent result of Treibich and Verdier [44], [45] that a general complex-valued potential of the form

$$q(z) = -\sum_{j=1}^{4} d_j \wp(z - \omega_j), \quad z \in \mathbb{C} \tag{1.6}$$

($\omega_2 = \omega_1 + \omega_3$, $\omega_4 = 0$) is a finite-gap potential if and only if $d_j/2$ are triangular numbers, i.e., if and only if

$$d_j = s_j(s_j + 1) \text{ for some } s_j \in \mathbb{Z},\ 1 \leq j \leq 4. \tag{1.7}$$

The methods of Treibich and Verdier are based on the notion of hyperelliptic tangent covers of the torus $\mathbb{C}/\Lambda$ (Λ the period lattice generated by $2\omega_1, 2\omega_3$).

Motivated by the results above and by the fact that a complete characterization of all elliptic finite-gap solutions of the stationary KdV hierarchy is still open, we started to develop our own approach toward a solution of this problem. In contrast to all current approaches in this area, our methods to characterize elliptic finite-gap solutions of the (m)KdV hierarchy rely on entirely different ideas based on a systematic use of a powerful theorem of Picard (see Theorem 2.2) concerning ordinary differential equations with elliptic coefficients in combination with explicit realizations of the isospectral manifold corresponding to a given (elliptic) finite-gap base potential q_0 (see, e.g., [10], [20], [25]). This approach immediately recovers and extends the results of [5], [41], [43], [44], [45], and, in particular, yields a complete characterization of all even (i.e., $q(z) = q(-z)$) elliptic finite-gap potentials [27]. Moreover, it leads to a natural conjecture on the structure of general elliptic finite-gap solutions of the KdV hierarchy (see Sect. 2).

In this paper we shall discuss in detail the case of Treibich-Verdier potentials (1.6). In addition to providing an elementary alternative argument of the finite-gap result (1.6), (1.7) of Treibich and Verdier, we shall derive a new and effective algorithm to compute the (arithmetic) genus g of the underlying (possibly singular) hyperelliptic curve K_g

$$K_g : y^2 = \prod_{m=0}^{2g}(E - E_m) \tag{1.8}$$

associated with (1.6) in Sect. 3. More precisely, Theorem 3.5 (ii), our principal new result, reduces the computation of g and the location of the (finite) branch points resp. singular points $(E_m, 0)$ of K_g to the study of certain linear algebraic eigenvalue problems involving Jacobi (tri-diagonal) matrices. (As shown in [28], appropriate modifications of the approach in Sect. 3 extend to the far more complex case of all even elliptic finite-gap solutions of the stationary KdV hierarchy.) In Sect. 4 we carry over the results of Sect. 3 to analogous elliptic finite-gap solutions of the stationary mKdV hierarchy (for simplicity, these stationary mKdV solutions will still be called Treibich-Verdier potentials). To the best of our knowledge, these (m)KdV findings in Sect. 4 are new. In Sect. 2 we briefly review the essentials of the (m)KdV hierarchy, its connection with finite-gap potentials, and Picard's theorem as needed in Sects. 3 and 4.

The far simpler case of Lamé -Ince potentials (1.1) (and its generalization to the mKdV hierarchy) has been dealt with in [26].

2 The (m)KdV hierarchy and Picard's theorem

Since most of the material in this section has been presented in some detail elsewhere (see, e.g., [3], [11], Ch. 12, [21], [26]) we confine ourselves here to a very brief account.

The KdV hierarchy is defined as follows. Consider the recursion relation

$$\hat{f}_{j+1,x} = \frac{1}{4}\hat{f}_{j,xxx} + q\hat{f}_{j,x} + \frac{1}{2}q_x\hat{f}_j, \quad 0 \leq j \leq n,\ \hat{f}_0 = 1, \tag{2.1}$$

i.e., explicitly,

$$\hat{f}_0 = 1,\ \hat{f}_1 = \frac{1}{2}q + c_1,\ \hat{f}_2 = \frac{1}{8}q_{xx} + \frac{3}{8}q^2 + \frac{c_1}{2}q + c_2, \text{ etc}, \tag{2.2}$$

where c_j, $j \in \mathbb{N}$ are integration constants. Using the convention that the corresponding homogeneous quantities, defined by $c_\ell \equiv 0$, $\ell \in \mathbb{N}$ are denoted by $f_j := \hat{f}_j(c_\ell \equiv 0)$, the KdV hierarchy is then defined as the sequence of evolution equations

$$\mathrm{KdV}_n(q) := q_t - 2f_{n+1,x} = 0, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

Explicitly, one obtains

$$\mathrm{KdV}_0(q) = q_t - q_x,\ \mathrm{KdV}_1(q) = q_t - \frac{1}{4}q_{xxx} - \frac{3}{2}qq_x, \text{ etc.} \tag{2.4}$$

with $\mathrm{KdV}_1(.)$ the usual KdV functional. The inhomogeneous version of (2.3) then reads

$$q_t - 2\hat{f}_{n+1,x} = q_t - 2\sum_{j=0}^{n} c_{n-j}f_{j+1,x} = 0,\ c_0 = 1. \tag{2.5}$$

The special case of the n-th-order stationary KdV equation characterized by $q_t = 0$ is then given by

$$f_{n+1,x} = 0 \text{ respectively } \hat{f}_{n+1,x} = \sum_{j=0}^{n} c_{n-j}f_{j+1,x} = 0. \tag{2.6}$$

Next, introducing the polynomial in $E \in \mathbb{C}$

$$\hat{F}_n(E,x,t) = \sum_{j=0}^{n} E^j\hat{f}_{n-j}(x,t), \tag{2.7}$$

(2.5) becomes

$$q_t = \frac{1}{2}\hat{F}_{n,xxx} + 2(q-E)\hat{F}_{n,x} + q_x\hat{F}_n \tag{2.8}$$

and the stationary (inhomogeneous) KdV hierarchy then reads

$$\frac{1}{2}\hat{F}_{n,xxx} + 2(q-E)\hat{F}_{n,x} + q_x\hat{F}_n = 0. \tag{2.9}$$

Integrating (2.9) times $\hat{F}_n$ once results in

$$\frac{1}{4}\hat{F}_{n,x}^2 - \frac{1}{2}\hat{F}_{n,xx}\hat{F}_n - (q-E)\hat{F}_n^2 = \hat{R}_{2n+1}(E), \tag{2.10}$$

where the integration constant $\hat{R}_{2n+1}(E)$ is easily seen to be a polynomial in E of degree $2n+1$ with leading coefficient 1, i.e.,

$$\hat{R}_{2n+1}(E) = \prod_{m=0}^{2n}(E - E_m), \ \{E_m\}_{m=0}^{2n} \subset \mathbb{C}. \tag{2.11}$$

This naturally leads to an underlying (possibly singular) hyperelliptic curve K_n of (arithmetic) genus n of the type

$$K_n : y^2 = \hat{R}_{2n+1}(E) = \prod_{m=0}^{2n}(E - E_m). \tag{2.12}$$

In the self-adjoint case, where $\{E_m\}_{m=0}^{2n} \subset \mathbb{R}$, $E_{2n} < E_{2n-1} < \cdots < E_0$, the zeros E_m, $0 \leq m \leq 2n$ of $\hat{R}_{2n+1}(.)$ are precisely the spectral band edges in the sense of (1.3).

Finally, introducing the differential expressions (Lax pair)

$$L(t) = \frac{d^2}{dx^2} + q(x,t), \tag{2.13}$$

$$\hat{P}_{2n+1}(t) = \sum_{j=0}^{n}\Big[-\frac{1}{2}\hat{f}_{j,x}(x,t) + \hat{f}_j(x,t)\frac{d}{dx}\Big]L(t)^{n-j}, \quad n \in \mathbb{N} \cup \{0\}, \tag{2.14}$$

one can show that

$$[\hat{P}_{2n+1}, L] = 2\hat{f}_{n+1,x} \tag{2.15}$$

($[.,.]$ the commutator symbol) and hence $q_t = 0$ is equivalent to the commutativity of $\hat{P}_{2n+1}$ and L, i.e.,

$$[\hat{P}_{2n+1}, L] = 0. \tag{2.16}$$

A well-known result of Burchnall and Chaundy [7], [8] then implies that $\hat{P}_{2n+1}$ and L satisfy an algebraic relation of the form

$$\hat{P}_{2n+1}^2 = \hat{R}_{2n+1}(L) = \prod_{m=0}^{2n}(L - E_m) \tag{2.17}$$

illustrating once again the importance of the curve K_n in (2.12).

The mKdV hierarchy is obtained as follows. Consider the recursion relation

$$\hat{g}_{j+1,x} = \frac{1}{4}\hat{g}_{j,xxx} - \phi^2\hat{g}_{j,x} - \phi_x\Big[\int^x dx'\phi\hat{g}_{j,x'} - c_j\Big],$$
$$0 \leq j \leq n, \ \hat{g}_0 = 1, \tag{2.18}$$

where c_j, $j \in \mathbb{N}$ are the integration constants from (2.2) and $c_0 = 1$. Also, since $\phi\hat{g}_{j,x}$ turns out to be the derivative of a certain differential polynomial in ϕ, the integral in (2.18) is understood to be homogeneous. One computes explicitly,

$$\hat{g}_0 = 1, \ \hat{g}_1 = \phi + c_1, \ \hat{g}_2 = \frac{1}{4}\phi_{xx} - \frac{1}{2}\phi^3 + c_1\phi + c_2, \ \text{etc.} \tag{2.19}$$

By $g_j := \hat{g}_j(c_\ell \equiv 0)$, $\ell \in \mathbb{N}$ we denote the homogeneous versions of $\hat{g}_j$ as before in the context of $\hat{f}_j$. The mKdV hierarchy is then defined as the sequence of evolution equations

$$\mathrm{mKdV}_n(\phi) := \phi_t - g_{n+1,x} = 0, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.20}$$

Explicitly,

$$\mathrm{mKdV}_0(\phi) = \phi_t - \phi_x, \ \mathrm{mKdV}_1(\phi) = \phi_t - \frac{1}{4}\phi_{xxx} + \frac{3}{2}\phi^2\phi_x, \ \text{ etc.} \tag{2.21}$$

and we emphasize the symmetry $\phi \to -\phi$ of solutions of the mKdV hierarchy. The special case of the n-th-order stationary mKdV equations characterized by $\phi_t = 0$ then reads

$$g_{n+1,x} = 0 \text{ respectively } \hat{g}_{n+1,x} = \sum_{j=0}^{n} c_{n-j} g_{j+1,x} = 0, \ c_0 = 1. \tag{2.22}$$

Miura's identity [37] then connects the two hierarchies

$$\mathrm{KdV}_n(\mp\phi_x - \phi^2) = [-2\phi \mp \partial_x]\mathrm{mKdV}_n(\phi), \quad n \in \mathbb{N} \cup \{0\}. \tag{2.23}$$

Introducing the Lax pair

$$\begin{aligned} \mathcal{M}(t) &= \begin{pmatrix} 0 & \frac{d}{dx} + \phi(x,t) \\ \frac{d}{dx} - \phi(x,t) & 0 \end{pmatrix}, \\ \hat{Q}_{2n+1}(t) &= \begin{pmatrix} \hat{P}_{2n+1}(t) & 0 \\ 0 & \hat{\tilde{P}}_{2n+1}(t) \end{pmatrix}, \end{aligned} \tag{2.24}$$

where $\hat{P}_{2n+1}(t)$ respectively $\hat{\tilde{P}}_{2n+1}(t)$ are defined as in (2.14), with q respectively $\tilde{q}$ given by

$$q = -\phi_x - \phi^2, \ \tilde{q} = \phi_x - \phi^2, \tag{2.25}$$

one verifies that

$$[\hat{Q}_{2n+1}, \mathcal{M}] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \hat{g}_{n+1,x}. \tag{2.26}$$

The analogs of (2.12) and (2.17) in the stationary mKdV case are finally given by

$$y^2 = \prod_{m=0}^{2n}(w - E_m^{1/2})(w + E_m^{1/2}) = \prod_{m=0}^{2n}(w^2 - E_m) \tag{2.27}$$

and

$$\hat{Q}_{2n+1}^2 = \prod_{m=0}^{2n}(\mathcal{M} - E_m^{1/2})(\mathcal{M} + E_m^{1/2}) = \prod_{m=0}^{2n}(\mathcal{M}^2 - E_m). \tag{2.28}$$

This leads to

Definition 2.1 *Any solution q (respectively ϕ) of one of the stationary equations (2.6) (respectively (2.22)) is called an* **(algebro-geometric) finite-gap potential** *associated with the KdV (respectively mKdV) hierarchy.*

The potentials q (respectively ϕ) then can be expressed in terms of the Riemann theta function associated with the (possibly singular) hyperelliptic curve K_n of (arithmetic) genus n as pioneered by Its and Matveev [34] (see also [23] and the references therein).

In the particular case where q is an elliptic function (see, e.g., (1.1), (1.6)), the following theorem of Picard plays a crucial role in its analysis. (For brevity, we only state it in the second-order case.)

Theorem 2.2 *(Picard, see, e.g., [32], p. 375–376) Consider the differential equation*

$$\psi''(z) + Q(z)\psi(z) = 0, \quad z \in \mathbb{C} \tag{2.29}$$

with Q an elliptic function with f.p. $2\omega_1$, $2\omega_3$. Suppose the general solution of (2.29) is meromorphic. Then there exists at least one solution ψ_1 which is elliptic of the second kind, i.e., ψ_1 is meromorphic and

$$\psi_1(z + 2\omega_j) = \rho_j \psi_1(z), \quad j = 1, 3 \tag{2.30}$$

for some constants ρ_1, $\rho_3 \in \mathbb{C}$. If in addition the characteristic equation corresponding to the substitution $z + 2\omega_1$ (or $z + 2\omega_3$) (see [32], p. 376 and 358) has distinct roots, then there exists a fundamental system of elliptic functions of the second kind of (2.29).

By the theory of elliptic functions, ψ_1 is elliptic of the second kind if and only if it is of the form

$$\psi_1(z) = Ce^{\lambda z} \prod_{j=1}^{m} [\sigma(z - a_j)/\sigma(z - b_j)] \tag{2.31}$$

for suitable $m \in \mathbb{N}$ and constants C, λ, a_j, b_j, $1 \leq j \leq m$. (Here $\sigma(z)$ is the Weierstrass σ-function associated with Λ, see [1], Ch. 18.)

Theorem 2.2 motivates

Definition 2.3 *Let q be an elliptic function. Then q is called a* **Picard potential** *if and only if*

$$\psi'' + q\psi = E\psi \tag{2.32}$$

has a meromorphic fundamental system of solutions for each $E \in \mathbb{C}$.

It can be shown [27] that q is a Picard potential whenever (2.32) has a meromorphic fundamental system of solutions for a sufficiently large but finite number of distinct values of E.

The connection between Picard potentials and elliptic finite-gap potentials is now the following: By the Its-Matveev formula [34] for q and the corresponding Baker-Akhiezer function in terms of the associated Riemann theta function one proves

Theorem 2.4 *Every elliptic finite-gap potential q is Picard (in the sense of Definition 2.3).*

(For further details see, e.g., [13], Ch. III, [40], Thm. 6.10, [28].) Naturally, one is led to conjecture that the converse of Theorem 2.4 is also true. Hence it seems appropriate to formulate the following

Conjecture. *An elliptic potential q is finite-gap if and only if it is a Picard potential.*

By a systematic use of Picard's Theorem 2.2 we have proven this conjecture in [28]. This covers and extends in particular the cases of Lamé-Ince and Treibich-Verdier potentials (1.1) and (1.6) (and all other examples in [5] and [41]). Moreover, it should be stressed at this point that this characterization of elliptic finite-gap potentials as Picard potentials yields the most effective criterion to date for determining whether or not a given elliptic potential is actually finite-gap.

A key element in proving this conjecture turned out to be the following characterization of general periodic finite-gap potentials (not necessarily elliptic) and their associated diagonal Green's function.

Theorem 2.5 *([28]) Assume that $q(x)$ is a periodic continuous function of period $\Omega > 0$ on $\mathbb{R}$ and that $L = \frac{d^2}{dx^2} + q(x)$ has two linearly independent Floquet solutions for all $E \in \mathbb{C}\backslash\{\tilde{E}_j\}_{j=0}^{\tilde{M}}$ for some $\tilde{M} \in \mathbb{N} \cup \{0\}$ and precisely one Floquet solution for each $E = \tilde{E}_j$ (assuming $\tilde{E}_j \neq \tilde{E}_{j'}$ for $j \neq j'$). Denote by $\tilde{d}(E)$ the algebraic multiplicity of E as an (anti)periodic eigenvalue and by $\tilde{p}(E)$ the minimal algebraic multiplicity of E as a Dirichlet eigenvalue on $[x_0, x_0 + \Omega]$ as x_0 varies in $\mathbb{R}$. Let $\tilde{q}(E) = \tilde{d}(E) - 2\tilde{p}(E)$. Then*

(i). $\tilde{q}(E)$ is positive on a finite set $\{\tilde{E}_0, \ldots, \tilde{E}_M\}$, $M \geq \tilde{M}$ and zero elsewhere. Let $\tilde{q}_j = \tilde{q}(\tilde{E}_j)$, $j = 0, \ldots, M$. Then $\sum_{j=0}^{M} \tilde{q}_j = 2n + 1$ for some nonnegative integer n, i.e., $\sum_{j=0}^{M} \tilde{q}_j$ is an odd positive integer. The Wronskian of two nontrivial Floquet solutions which are linearly independent on some punctured disk $0 < |E - \lambda| < \varepsilon$ tends to zero as E tends to λ if and only if $\lambda \in \{\tilde{E}_0, \ldots, \tilde{E}_M\}$.

(ii). The diagonal Green's function $G(E, x, x)$ associated with L is of the type

$$G(E,x,x) = \frac{-1}{2} \hat{F}_n(E,x)/[\hat{R}_{2n+1}(E)]^{1/2}, \tag{2.33}$$

where

$$\hat{F}_n(E,x) = \prod_{\ell=1}^{n} [E - \mu_\ell(x)], \tag{2.34}$$

$$\hat{R}_{2n+1}(E) = \prod_{j=0}^{M} (E - \tilde{E}_j)^{\tilde{q}_j}, \tag{2.35}$$

and where $\mu_\ell(x)$ denote (some of the) Dirichlet eigenvalues of L on the interval $[x, x + \Omega]$.

(iii). $q(x)$ is an algebro-geometric finite-gap potential associated with the compact (possibly singular) hyperelliptic curve of (arithmetic) genus n obtained upon one-point compactification of the curve

$$y^2 = \hat{R}_{2n+1}(E) = \prod_{j=0}^{M}(E - \tilde{E}_j)^{\tilde{q}_j}, \tag{2.36}$$

where $n = [(\sum_{j=0}^{M}\tilde{q}_j) - 1]/2$. *Equivalently, there exists an ordinary differential expression* $\hat{P}_{2n+1}$ *of order* $2n+1$, *i.e.,*

$$\hat{P}_{2n+1} = \sum_{\ell=0}^{2n+1} p_\ell(x)\frac{d^\ell}{dx^\ell}, \quad p_{2n+1}(x) = 1 \tag{2.37}$$

which commutes with L, and satisfies the Burchnall-Chaundy polynomial relation $\hat{P}_{2n+1}^2 = \hat{R}_{2n+1}(L)$.

The proof of Theorem 2.5 in [28] is based on well-known identities for the diagonal Green's function $G(E,x,x)$ in terms of the Floquet discriminant $\Delta(E)$ and a fundamental system of solutions of $L\psi(E,y) = E\psi(E,y)$ with respect to a reference point $x \in \mathbb{R}$, Hadamard-type factorizations of such solutions with respect to E, the nonlinear second-order differential equation satisfied by $G(E,x,x)$ respectively $\hat{F}_n(E,x)$ in (2.10), and the recursion formalism displayed in (2.1)–(2.17).

3 Treibich-Verdier potentials associated with the KdV hierarchy

In this section we study the Treibich-Verdier potentials

$$q(z) = -\sum_{j=1}^{4} s_j(s_j+1)\mathscr{P}(z-\omega_j), \quad s_j \in \mathbb{N}\cup\{0\},\ 1 \le j \le 4,\ z \in \mathbb{C}, \tag{3.1}$$

where $\omega_2 = \omega_1 + \omega_3$ and $\omega_4 = 0$, and the associated linear problem

$$\psi''(E,z) + [q(z) - E]\psi(E,z) = 0, \quad E \in \mathbb{C} \tag{3.2}$$

in detail. (In order to avoid the trivial case $q = 0$, we assume that at least one $s_j > 0$.)

Theorem 3.1 *The potential* $q(z) = -\sum_{j=1}^{4} d_j\mathscr{P}(z-\omega_j)$, $d_j \in \mathbb{C}$, *is a Picard potential if and only if each* $d_j/2$ *is a triangular number, i.e., if and only if*

$$d_j = s_j(s_j+1) \textit{ for some } s_j \in \mathbb{Z},\ 1 \le j \le 4. \tag{3.3}$$

Proof . We have to determine when equation (3.2) has only meromorphic solutions (see, e.g., [32], Ch. XVI for the standard Frobenius method in this context). For this it is necessary that any singular point of equation (3.2), i.e., any pole of q is a regular singular point and that the exponents relative to this singularity are unequal integers. Vicariously for any of these singularities we consider the half period ω_k (assuming $d_k = s_k(s_k+1) \neq 0, s_k \in \mathbb{C}$). The indicial equation associated with ω_k then reads

$$m(m-1)-s_k(s_k+1)=0 \tag{3.4}$$

and hence both of its roots, namely $m=s_k+1$ and $m=-s_k$ are unequal integers if and only if $s_k\in\mathbb{Z}$. This is therefore a necessary condition for q to be a Picard potential. In order to show that it is also a sufficient condition we assume now that $s_k\in\mathbb{Z}$. Indeed, without loss of generality, we may assume that $s_k\in\mathbb{N}\cup\{0\}$. The Frobenius method then shows that there is always one solution of the form

$$\psi_1(z)=(z-\omega_k)^{s_k+1}\sum_{m=0}^{\infty}\alpha_m(z-\omega_k)^m,\quad \alpha_0=1 \tag{3.5}$$

which is meromorphic (in fact, analytic) near ω_k. If all solutions are to be meromorphic near ω_k then there must be another one of the type

$$\psi_2(z)=(z-\omega_k)^{-s_k}\sum_{m=0}^{\infty}\beta_m(z-\omega_k)^m,\quad \beta_0=1. \tag{3.6}$$

Using formula 18.5.1 of [1] and taking into account that $\wp^{(n)}(\omega_k-\omega_j)=0$ for n odd and $j\neq k$, we can rewrite (3.2) in the form

$$(z-\omega_k)^2\psi''(E,z)+Q(E,z)\psi(E,z)=0, \tag{3.7}$$

where near ω_k the function $Q(E,z)$ is given by

$$\begin{aligned}Q(E,z)&=-s_k(s_k+1)\Big[1+\sum_{m=2}^{\infty}\gamma_m(z-\omega_k)^{2m}\Big]\\&\quad-\sum_{\substack{j=1\\ j\neq k}}^{4}s_j(s_j+1)\sum_{m=0}^{\infty}(2m!)^{-1}\wp^{(2m)}(\omega_k-\omega_j)(z-\omega_k)^{2m+2}-E(z-\omega_k)^2\\&=-s_k(s_k+1)+\sum_{m=1}^{\infty}q_{2m}(E)(z-\omega_k)^{2m}\end{aligned} \tag{3.8}$$

for certain constants γ_m (described in 18.5.2 and 18.5.3 of [1]) and $q_{2m}(E)$. Inserting the ansatz (3.6) into (3.7) results in

$$\begin{aligned}0&=f(-s_k)(z-\omega_k)^{-s_k}+\{f(1-s_k)\beta_1+G_1\}(z-\omega_k)^{1-s_k}\\&\quad+\ldots+\{f(m-s_k)\beta_m+G_m\}(z-\omega_k)^{m-s_k}+\ldots,\end{aligned} \tag{3.9}$$

where

$$f(m)=m(m-1)-s_k(s_k+1), \tag{3.10}$$

$$G_{2m}=q_2(E)\beta_{2m-2}+q_4(E)\beta_{2m-4}+\ldots+q_{2m}(E)\beta_0, \tag{3.11}$$

$$G_{2m+1}=q_2(E)\beta_{2m-1}+q_4(E)\beta_{2m-3}+\ldots+q_{2m}(E)\beta_1, \tag{3.12}$$

and, in particular, $G_1=0$.

Next we note that $f(-s_k)=0$ and that we may determine the β_m successively from the requirement that the coefficient of $(z-\omega_k)^{m-s_k}$ must be zero. This fails only if $f(m - s_k) = 0$, i.e, for $m = 2s_k + 1$. Thus we may determine $\beta_1, \ldots, \beta_{2s_k}$. The coefficient of $(z - \omega_k)^{s_k+1}$ is G_{2s_k+1} and it determines whether or not a second meromorphic solution exists. More precisely, if $G_{2s_k+1} = 0$, then the above procedure to determine the β_m can be carried on and yields the existence of a second meromorphic solution. If, however, $G_{2k+1} \neq 0$, then there does not exist a solution of the form (3.6) and the second solution of the differential equation (3.2) involves logarithmic terms and hence is not meromorphic.

Since $G_1 = 0$ we find $\beta_1 = 0$ and then, from (3.12), $G_3 = 0$. This, in turn, gives $\beta_3 = 0$ and this argument can be carried on to show that in this case $G_{2s_k+1} = 0$ which, according to the above argument, proves that all solutions of (3.2) are meromorphic. □

From now on we assume that q in (3.1) is a Picard potential, i.e., we suppose (without loss of generality) that $s_j \in \mathbb{N} \cup \{0\}$ for $j = 1, \ldots, 4$. Then any solution $\psi(E,z)$ of (3.2) behaves near any pole b of q either like

$$c(z - b)^{-s_k} + O((z - b)^{-s_k+1}) \tag{3.13}$$

or like

$$c(z - b)^{s_k+1} + O((z - b)^{s_k+2}) \tag{3.14}$$

for some $c \neq 0$. Near any point z_0 which is not a pole of q any solution $\psi(E,z)$ behaves like

$$c + d(z - z_0) + O((z - z_0)^2) \tag{3.15}$$

for some $c, d \in \mathbb{C}$ which are not both zero.

By Picard's Theorem 2.2 equation (3.2) has at least one solution $\psi(E,z)$ which is elliptic of the second kind and hence of the form

$$\psi(E,z) = e^{\lambda(E)z} \prod_{n=1}^{N(E)} \sigma(z - c_n(E))^{\ell_n(E)} \tag{3.16}$$

for suitable constants $N \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $c_1, \ldots, c_N \in \Delta$ with $c_m \neq c_n$ for $m \neq n$, and $\ell_1, \ldots, \ell_N \in \mathbb{Z}$ with $\sum_{n=1}^{N} \ell_n = 0$. (Here Δ denotes the fundamental period parallelogram (fpp) with vertices $2\omega_1$, $2\omega_2$, $2\omega_3$, 0.) Note that the c_n are the roots and poles of ψ in the fundamental period parallelogram. Since, by (3.13), (3.14), and (3.15), ψ may have zeros of order higher than one or poles only at half-periods (modΔ), the numbers ℓ_n can be different from zero or one only if c_n is a half-period, i.e., if $c_n \in \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Since a factor with $\ell_n = 0$ can be dropped from the product on the right hand side of (3.16) we agree from this point on that $\ell_n = 0$ may occur only if c_n is a half-period. Thus we may choose

$$c = (c_1, \ldots, c_N) = (c_1, c_2, \ldots, c_{N-4}, \omega_1, \omega_2, \omega_3, \omega_4) \tag{3.17}$$

and obtain by (3.13), (3.14), and (3.15)

$$\ell = (\ell_1, \ldots, \ell_N) = (1, 1, \ldots, 1, \ell_{N-3}, \ell_{N-2}, \ell_{N-1}, \ell_N), \tag{3.18}$$

where

$$\ell_{N-4+j} \in \{-s_j, s_j + 1\}, \quad 1 \leq j \leq 4. \tag{3.19}$$

Now note that

$$\psi'/\psi = \lambda + \sum_{n=1}^{N} \ell_n \zeta(z - c_n), \tag{3.20}$$

$$(\psi'/\psi)' = -\sum_{n=1}^{N} \ell_n \mathscr{P}(z - c_n), \tag{3.21}$$

$$(\psi'/\psi)^2 = \sum_{n=1}^{N} \left\{ \ell_n^2 \mathscr{P}(z - c_n) + 2\ell_n \left[\lambda + \sum_{\substack{m=1 \\ m \neq n}}^{N} \ell_m \zeta(c_n - c_m) \right] \zeta(z - c_n) \right\}$$
$$- \sum_{\substack{m=1 \\ m \neq r}}^{N} \ell_m(\ell_m + 2\ell_r) \mathscr{P}(c_r - c_m) \tag{3.22}$$

for any $r \in \{1, \dots, N\}$ for which $\ell_r \neq 0$. Therefore,

$$q(z) = -\sum_{j=1}^{4} s_j(s_j + 1)\mathscr{P}(z - \omega_j) = E - [\psi''/\psi] = E - (\psi'/\psi)' - (\psi'/\psi)^2$$
$$= E - \sum_{n=1}^{N} \left\{ \ell_n(\ell_n - 1)\mathscr{P}(z - c_n) + 2\ell_n \left[\lambda + \sum_{\substack{m=1 \\ m \neq n}}^{N} \ell_m \zeta(c_n - c_m) \right] \zeta(z - c_n) \right\}$$
$$+ \sum_{\substack{m=1 \\ m \neq r}}^{N} \ell_m(\ell_m + 2\ell_r) \mathscr{P}(c_r - c_m) \tag{3.23}$$

for any $r \in \{1, \dots, N\}$ for which $\ell_r \neq 0$. Hence ψ in (3.16) solves (3.2) if and only if

$$\ell_n(\ell_n - 1) = \begin{cases} s_j(s_j + 1) & \text{if } c_n = \omega_j \\ 0 & \text{otherwise,} \end{cases} \tag{3.24}$$

$$0 = \lambda + \sum_{\substack{m=1 \\ m \neq n}}^{N} \ell_m \zeta(c_n - c_m). \tag{3.25}$$

for each $n \in \{1, \dots, N\}$ with $\ell_n \neq 0$ and

$$E = -\sum_{\substack{m=1 \\ m \neq r}}^{N} \ell_m(\ell_m + 2\ell_r)\mathscr{P}(c_r - c_m). \tag{3.26}$$

for some (and hence all) $r \in \{1, \dots, N\}$ for which $\ell_r \neq 0$.

According to whether $\ell_{N-4+j} = -s_j$ or $s_j + 1$, $1 \leq j \leq 4$, we introduce the index sets $M_1(E)$, $M_2(E) \subseteq \{1, 2, 3, 4\}$ defined by

$$M_1(E) = \{j \in \{1,2,3,4\} : \ell_{N-4+j} = -s_j \neq 0\}, \tag{3.27}$$

$$M_2(E) = \{j \in \{1,2,3,4\} : \ell_{N-4+j} = s_j + 1\}. \tag{3.28}$$

We note that $M_1(E) \cap M_2(E) = \emptyset$. Hence we can rewrite (3.16) as

$$\psi(E,z) = \psi_a(z) = e^{\lambda_a z} \frac{\prod_{\ell=1}^{s} \sigma(z - a_\ell(E))}{\prod_{j=1}^{4} \sigma(z - \omega_j)^{s_j}}, \tag{3.29}$$

where $s = \sum_{j=1}^{4} s_j$, $a = a(E) = (a_1(E), ..., a_s(E))$. If $j \in M_2(E)$ then $2s_j + 1$ of the a_ℓ coincide with ω_j. Hence if $M_2(E)$ is empty then $N = s + 4$ and $a = (c_1, ..., c_{N-4})$. If $M_2(E) = \{j_1, ..., j_k\}$ is not empty, however, then $N = \hat{s} + 4$, where

$$\hat{s} = s - \sum_{j \in M_2(E)} (2s_j + 1) = \sum_{j \in M_1(E)} s_j - \sum_{j \in M_2(E)} (s_j + 1) \tag{3.30}$$

and $\hat{s} \geq 0$ since $\sum_{n=1}^{N} \ell_n = 0$. This implies

$$a = (c_1, ..., c_{\hat{s}}, \underbrace{\omega_{j_1}, \omega_{j_1}, \ldots, \omega_{j_1}}_{(2s_{j_1}+1)\text{ times}}, \ldots, \underbrace{\omega_{j_k}, \omega_{j_k}, \ldots, \omega_{j_k}}_{(2s_{j_k}+1)\text{ times}}). \tag{3.31}$$

In terms of $\{a_\ell\}_{\ell=1}^{s}$ and $\{\omega_j\}_{j=1}^{4}$ the equations (3.25) read

$$\begin{aligned} \lambda_a &+ \sum_{\substack{m=1 \\ m \neq n}}^{\hat{s}} \zeta(a_n - a_m) \\ &- \sum_{k \in M_1} s_k \zeta(a_n - \omega_k) + \sum_{k \in M_2} (s_k + 1)\zeta(a_n - \omega_k) = 0, \end{aligned} \tag{3.32}$$

for all $n = 1, ..., \hat{s}$ and

$$\begin{aligned} \lambda_a &+ \sum_{m=1}^{\hat{s}} \zeta(\omega_j - a_m) \\ &- \sum_{\substack{k \in M_1 \\ j \neq k}} s_k \zeta(\omega_j - \omega_k) + \sum_{\substack{k \in M_2 \\ j \neq k}} (s_k + 1)\zeta(\omega_j - \omega_k) = 0, \end{aligned} \tag{3.33}$$

for all $j \in M_1(E) \cup M_2(E)$. According to (3.26), the spectral parameter E is given by

$$\begin{aligned} E \quad = \quad & -3 \sum_{\substack{m=1 \\ m \neq r}}^{\hat{s}} \wp(a_r - a_m) + \sum_{j \in M_1} s_j (2 - s_j) \wp(a_r - \omega_j) \\ & - \sum_{j \in M_2} (s_j + 1)(s_j + 3) \wp(a_r - \omega_j) \end{aligned} \tag{3.34}$$

for any $r \in \{1, ..., \hat{s}\}$ or by

$$
\begin{aligned}
E \quad = \quad & \sum_{m=1}^{\hat{s}}(2s_k - 1)\mathscr{P}(\omega_k - a_m) - \sum_{\substack{j\in M_1 \\ j\neq k}} s_j(s_j + 2s_k)\mathscr{P}(\omega_k - \omega_j) \\
& - \sum_{\substack{j\in M_2 \\ j\neq k}} (s_j + 1)(s_j + 1 - 2s_k)\mathscr{P}(\omega_k - \omega_j),
\end{aligned} \tag{3.35}
$$

for any $k \in M_1 \cup M_2$.

Next we choose a $k_0 \in M_1$, i.e., k_0 is such that $\omega_{k_0} \notin \{a_\ell\}_{\ell=1}^{s}$. (This is always possible since otherwise $\hat{s}$ would be negative, which is a contradiction.) Then we get from (3.33)

$$
\lambda_a = \sum_{\ell=1}^{s} \zeta(a_\ell(E) - \omega_{k_0}) - \sum_{\substack{j=1 \\ j\neq k_0}}^{4} s_j\zeta(\omega_j - \omega_{k_0}) \tag{3.36}
$$

which implies

$$
\lambda_a + \lambda_{-a} = -2\sum_{j=1}^{3} s_j\zeta(\omega_j). \tag{3.37}
$$

Next we prove

Lemma 3.2 *If $\psi_a(z)$ in (3.29) solves (3.2), then so does $\psi_{-a}(z)$, where $-a(E) = (-a_1(E), \ldots, -a_s(E))$.*

Proof . Due to the reflection symmetry of q, i.e., $q(z) = q(-z)$, we infer that together with $\psi_a(z)$ also $\psi_a(-z)$ is a solution of (3.2). Using (3.37) we obtain

$$
\begin{aligned}
\psi_{-a}(z)\psi_a(-z)^{-1} \quad & = \quad (-1)^{s_1+s_2+s_3}\exp\left[\left(\lambda_a + \lambda_{-a} + 2\sum_{j=1}^{3} s_j\zeta(\omega_j)\right)z\right] \\
& = \quad (-1)^{s_1+s_2+s_3}.
\end{aligned} \tag{3.38}
$$

This shows that $\psi_{-a}(z)$ is a multiple of $\psi_a(-z)$ and hence is also a solution of (3.2). □

Moreover, $\psi_{\pm a}(E,z)$ are Floquet solutions of (3.2) since

$$
\begin{aligned}
& \psi_{\pm a}(z + 2\omega_k) \\
& = \exp\left\{ \pm \left[2\omega_k\lambda_a - 2\zeta(\omega_k)\sum_{\ell=1}^{s} a_\ell(E) + 2\zeta(\omega_k)\sum_{j=1}^{3} s_j\omega_j\right]\right\}\psi_{\pm a}(z)
\end{aligned} \tag{3.39}
$$

for $k = 1, 3$. In deriving (3.39) we made use of (3.37).

As described in Theorem 2.5, in order to show that $q(z)$ in (3.1) is a finite-gap potential and determine the (arithmetic) genus of the underlying hyperelliptic curve K_g, we need to find the number of E-values where the Wronskian

$$
W(E) := W(\psi_a(E), \psi_{-a}(E)) \tag{3.40}
$$

of the Floquet functions $\psi_a(E,z)$ and $\psi_{-a}(E,z)$ vanishes. (Note that for no value of E the functions $\psi_{\pm a}$ are identically equal to zero). These values (together with the point at infinity upon one-point compactification) then yield the location of the branch points resp. singular points of the two-sheeted Riemann surface K_g.

First we observe that

$$W(E) = \psi_a \psi_{-a} \left(\frac{\psi'_{-a}}{\psi_{-a}} - \frac{\psi'_a}{\psi_a} \right) \tag{3.41}$$

and

$$\psi_a \psi_{-a} = e^{(\lambda_a + \lambda_{-a})z} \frac{[\prod_{j \in M_2} \sigma(z-\omega_j)^{2s_j+1} \sigma(z+\omega_j)^{2s_j+1}][\prod_{\ell=1}^{\hat{s}} \sigma(z-a_\ell)\sigma(z+a_\ell)]}{\prod_{k=1}^4 \sigma(z-\omega_k)^{2s_k}}. \tag{3.42}$$

Since by (3.36)

$$\lambda_{-a} - \lambda_a = \sum_{\ell=1}^{s} [\zeta(\omega_{k_0} - a_\ell) - \zeta(\omega_{k_0} + a_\ell)], \tag{3.43}$$

we obtain

$$\begin{aligned} \frac{\psi'_{-a}}{\psi_{-a}} - \frac{\psi'_a}{\psi_a} &= \sum_{\ell=1}^{s} [\zeta(z+a_\ell) - \zeta(z-a_\ell) - \zeta(\omega_{k_0}+a_\ell) + \zeta(\omega_{k_0}-a_\ell)] \\ &= -\sum_{\ell=1}^{s} \frac{\sigma(2a_\ell)\sigma(z-\omega_{k_0})\sigma(z+\omega_{k_0})}{\sigma(a_\ell-\omega_{k_0})\sigma(a_\ell+\omega_{k_0})\sigma(z-a_\ell)\sigma(z+a_\ell)}. \end{aligned} \tag{3.44}$$

Thus

$$W(E) = f_1(z) \sum_{\ell=1}^{\hat{s}} g_\ell(z) + f_2(z) \sum_{j \in M_2} h_j(z), \tag{3.45}$$

where

$$f_1(z) = e^{(\lambda_a + \lambda_{-a})z} \frac{\sigma(z-\omega_{k_0})\sigma(z+\omega_{k_0}) \prod_{j \in M_2} \sigma(z-\omega_j)^{2s_j+1} \sigma(z+\omega_j)^{2s_j+1}}{\prod_{k=1}^4 \sigma(z-\omega_k)^{2s_k}}, \tag{3.46}$$

$$g_\ell(z) = \frac{-\sigma(2a_\ell)}{\sigma(a_\ell-\omega_{k_0})\sigma(a_\ell+\omega_{k_0})} \prod_{\substack{m=1 \\ m \neq \ell}}^{\hat{s}} \sigma(z-a_m)\sigma(z+a_m), \tag{3.47}$$

$$f_2(z) = e^{(\lambda_a + \lambda_{-a})z} \sigma(z-\omega_{k_0})\sigma(z+\omega_{k_0}) \prod_{m=1}^{\hat{s}} \sigma(z-a_m)\sigma(z+a_m), \tag{3.48}$$

$$h_j(z) = \frac{-(2s_j+1)\sigma(2\omega_j)\sigma(z-\omega_j)^{2s_j}\sigma(z+\omega_j)^{2s_j}}{\sigma(\omega_j-\omega_{k_0})\sigma(\omega_j+\omega_{k_0})} \cdot \frac{\prod_{\substack{k\in M_2\\ k\neq j}} \sigma(z-\omega_k)^{2s_k+1}\sigma(z+\omega_k)^{2s_k+1}}{\prod_{i=1}^4 \sigma(z-\omega_i)^{2s_i}}. \tag{3.49}$$

Since $W(E)$ in (3.45) is independent of $z \in \mathbb{C}$, we now compute it at the points $z = a_\ell$, $1 \leq \ell \leq \hat{s}$. This yields

$$W(E) = -f_1(a_\ell)\frac{\sigma(2a_\ell)}{\sigma(a_\ell-\omega_{k_0})\sigma(a_\ell+\omega_{k_0})}\prod_{\substack{m=1\\ m\neq\ell}}^{\hat{s}}\sigma(a_\ell-a_m)\sigma(a_\ell+a_m) \tag{3.50}$$

for each $\ell \in \{1,\dots,\hat{s}\}$. Equation (3.50) yields $W(E)=0$ if and only if $\sigma(a_\ell+a_m)=0$ for some $m \in \{1,\dots,\hat{s}\}$ different from ℓ. This implies $a_\ell = -a_m (\mathrm{mod}\Delta)$, since $a_\ell \neq \omega_j$, $1 \leq j \leq 4$ (because of $\ell \leq \hat{s}$) and $a_\ell \neq a_k$ for $\ell \neq k$ (because of (3.15)). Since this consideration is true for all $\ell \in \{1,\dots,\hat{s}\}$ we infer that $W(E)=0$ implies that the numbers $a_1,\dots,a_{\hat{s}}$ appear in pairs $(a_k, -a_k(\mathrm{mod}\Delta))$. Hence if $W(E)=0$ and $M_2(E)=\{j_1,\dots j_k\}$ then a can be written as

$$a = (a_1,a_2,\dots,a_d,-a_1,-a_2,\dots,-a_d,\underbrace{\omega_{j_1},\omega_{j_1},\dots,\omega_{j_1}}_{(2s_{j_1}+1)\,\text{times}},\dots,\underbrace{\omega_{j_k},\omega_{j_k},\dots,\omega_{j_k}}_{(2s_{j_k}+1)\,\text{times}}), \tag{3.51}$$

where

$$2d = \hat{s} = \#\{a_\ell : a_\ell \notin \{\omega_j\}_{j=1}^4\}. \tag{3.52}$$

In particular, $\hat{s}$ must be an even number.

This information will now be used to rewrite our solution ψ_a of (3.2) if $W(E)=0$. In fact, assuming $W(E)=0$, we may replace (3.29) by

$$\psi_a(z) = f(z)Q_d(\mathscr{P}(z)), \tag{3.53}$$

where

$$f(z) = F_a e^{\lambda_a(E)z}\sigma(z)^{\hat{s}}\prod_{j=1}^4\sigma(z-\omega_j)^{t_j}, \tag{3.54}$$

$$F_a = (-1)^d\prod_{m=1}^d\sigma(a_m)^2, \tag{3.55}$$

$$Q_d(\mathscr{P}(z)) = \prod_{m=1}^d[\mathscr{P}(z)-\mathscr{P}(a_m(E))] = \sum_{n=0}^d \mu_n(E)[\mathscr{P}(z)-e_2]^n, \tag{3.56}$$

$$t_j = \begin{cases} s_j+1 & \text{if } 2s_j+1 \text{ of the } a_\ell \text{ equal } \omega_j\\ -s_j & \text{if none of the } a_\ell \text{ equals } \omega_j.\end{cases} \tag{3.57}$$

Here we used identity 18.4.4 of [1],

$$\sigma(z-\alpha)\sigma(z+\alpha) = -\sigma(\alpha)^2\sigma(z)^2[\mathscr{P}(z) - \mathscr{P}(\alpha)] \tag{3.58}$$

to arrive at (3.53).

Because of the two possibilities to choose each of the four numbers t_j in (3.57) we will distinguish 16 different cases. Since we want d to be in $\mathbb{N} \cup \{0\}$, eight cases occur only when s is even and the other eight cases only when s is odd. For the same reason, Case 8 below actually never occurs. The following tables list the various possibilities.

Table 3.3. $s = s_1 + s_2 + s_3 + s_4$ even

	1	2	3	4	5	6	7	8
t_1	$-s_1$	s_1+1	s_1+1	s_1+1	$-s_1$	$-s_1$	$-s_1$	s_1+1
t_2	$-s_2$	s_2+1	$-s_2$	$-s_2$	s_2+1	s_2+1	$-s_2$	s_2+1
t_3	$-s_3$	$-s_3$	s_3+1	$-s_3$	s_3+1	$-s_3$	s_3+1	s_3+1
t_4	$-s_4$	$-s_4$	$-s_4$	s_4+1	$-s_4$	s_4+1	s_4+1	s_4+1
d	$\frac{s}{2}$	$\frac{s}{2}-s_1-s_2-1$	$\frac{s}{2}-s_1-s_3-1$	$\frac{s}{2}-s_1-s_4-1$	$\frac{s}{2}-s_2-s_3-1$	$\frac{s}{2}-s_2-s_4-1$	$\frac{s}{2}-s_3-s_4-1$	$-\frac{s}{2}-2$

Table 3.4. $s = s_1 + s_2 + s_3 + s_4$ odd

	9	10	11	12	13	14	15	16
t_1	$-s_1$	s_1+1	s_1+1	s_1+1	$-s_1$	$-s_1$	$-s_1$	s_1+1
t_2	s_2+1	$-s_2$	s_2+1	s_2+1	$-s_2$	$-s_2$	s_2+1	$-s_2$
t_3	s_3+1	s_3+1	$-s_3$	s_3+1	$-s_3$	s_3+1	$-s_3$	$-s_3$
t_4	s_4+1	s_4+1	s_4+1	$-s_4$	s_4+1	$-s_4$	$-s_4$	$-s_4$
d	$s_1-\frac{s+3}{2}$	$s_2-\frac{s+3}{2}$	$s_3-\frac{s+3}{2}$	$s_4-\frac{s+3}{2}$	$\frac{s-1}{2}-s_4$	$\frac{s-1}{2}-s_3$	$\frac{s-1}{2}-s_2$	$\frac{s-1}{2}-s_1$

For each of the above cases we now insert our ansatz (3.53) into (3.2). We will use the notation $f'(z) = \frac{d}{dz}f(z)$, $Q_d'(\mathscr{P}(z)) = \frac{d}{d\mathscr{P}}Q_d(\mathscr{P}(z))$ etc. and the identities

$$\mathscr{P}'^2 = 4(\mathscr{P}-e_2)^3 + 12e_2(\mathscr{P}-e_2)^2 + 4(e_2-e_1)(e_2-e_3)(\mathscr{P}-e_2), \tag{3.59}$$

$$\mathscr{P}'' = 6(\mathscr{P}-e_2)^2 + 12e_2(\mathscr{P}-e_2) + 6e_2^2 - \frac{g_2}{2}. \tag{3.60}$$

Here g_2 and g_3 are the invariants associated with $\mathscr{P}(z;\omega_1,\omega_3)$ and $e_j = \mathscr{P}(\omega_j)$, $j = 1,2,3$. We also make the convention that $\mu_{-1} = \mu_{d+1} = 0$. This results in

$$\begin{aligned}
&\psi'' + (q-E)\psi \\
&= f[(f''f^{-1} + q - E)Q_d + (2f'f^{-1}\mathscr{P}' + \mathscr{P}'')Q_d' + \mathscr{P}'^2 Q_d''] && (3.61)\\
&= f\sum_{n=0}^{d}[\alpha_n\mu_{n-1} + (\beta_n - E)\mu_n + \gamma_n\mu_{n+1}](\mathscr{P}-e_2)^n, && (3.62)
\end{aligned}$$

where

$$\alpha_n = (2d - 2n + 1 + 2t_4)(2d - 2n + 2), \tag{3.63}$$

$$\beta_n = \sum_{j=1}^{3} t_j e_j (4n - 4d - 2t_4 - t_j)$$

$$+e_2[4d^2 - 2d(8n+1) + 12n^2 + 4t_4(d-2n)], \tag{3.64}$$

$$\gamma_n = 2(e_2 - e_1)(e_2 - e_3)(n+1)(2n+1+2t_2). \tag{3.65}$$

Thus $\psi'' + (q - E)\psi = 0$ and $W(E) = 0$ are equivalent to the eigenvalue problem

$$J\underline{\mu} = E\underline{\mu}, \ \underline{\mu} = (\mu_d, \dots, \mu_0)^T, \tag{3.66}$$

where J is the $(d+1) \times (d+1)$ Jacobi matrix

$$J = \begin{pmatrix} \beta_d & \alpha_d & 0 & \cdots & \cdots & 0 \\ \gamma_{d-1} & \beta_{d-1} & \ddots & \ddots & & \vdots \\ 0 & \gamma_{d-2} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \gamma_1 & \beta_1 & \alpha_1 \\ 0 & \cdots & \cdots & 0 & \gamma_0 & \beta_0 \end{pmatrix} \tag{3.67}$$

with α_n, β_n, γ_n given by (3.63)–(3.65).

In order to determine the (maximal) number of (finite) branch points and singular points of a given Treibich-Verdier potential, i.e., the number of E-values such that $W(E)$ vanishes, we now distinguish a number of different cases. For brevity we assume that

$$s_1 \geq s_2 \geq s_3 \geq s_4 \geq 0. \tag{3.68}$$

In all other cases one only has to consider appropriate permutations of $\{s_1, s_2, s_3, s_4\}$.

(1) s even:
Then Case 1 always occurs while Case 8 never occurs. With regard to cases 2–7 we need to determine for which pairs $\{j_1, j_2\} \subset \{1,2,3,4\}$ the inequalities

$$0 \leq 2d = s - 2s_{j_1} - 2s_{j_2} - 2 = s_{j_3} + s_{j_4} - s_{j_1} - s_{j_2} - 2, \tag{3.69}$$

are valid. Here $j_1, j_2, j_3, j_4 \in \{1,2,3,4\}$ are pairwise distinct.

(1a) $s_2 + s_3 < s_1 + s_4$:
In this case the right-hand-side of (3.69) is nonnegative if $\{j_1, j_2\}$ equals $\{2,3\}$, $\{2,4\}$, or $\{3,4\}$ and negative in all other cases. Hence we have four eigenvalue problems of the type (3.66) all together comprised by the cases 1, 5, 6, and 7. The dimensions $(d+1)$ of the four Jacobi matrices add up to

$$\frac{s}{2} + 1 + \frac{s}{2} - s_2 - s_3 + \frac{s}{2} - s_2 - s_4 + \frac{s}{2} - s_3 - s_4 = 2s_1 + 1. \tag{3.70}$$

(1b) $s_1 = s_2 = s_3 = s_4$:
Now (3.69) never holds and only the first case may occur. One obtains $d = 4s_1/2$ and the Jacobi matrix in (3.66) has dimension $2s_1 + 1$.

(1c) $s_1 = s_2 > s_3 = s_4$:
Here (3.69) holds only for $\{j_1, j_2\} = \{3,4\}$. Thus there are only two eigenvalue problems of the type (3.66) comprised by the cases 1 and 7. The dimensions of the corresponding Jacobi matrices add up to

$$\frac{s}{2} + 1 + \frac{s}{2} - s_3 - s_4 = 2s_1 + 1. \tag{3.71}$$

(1d) $s_2 + s_3 = s_1 + s_4$ and $s_1 > s_2$:
In addition to the cases in (1c) we now also have the case $\{j_1, j_2\} = \{2, 4\}$. Hence there are three eigenvalue problems given by the cases 1, 6, and 7 and the dimensions of the corresponding Jacobi matrices add up to

$$\frac{s}{2} + 1 + \frac{s}{2} - s_3 - s_4 + \frac{s}{2} - s_2 - s_4 = 2s_1 + 1. \tag{3.72}$$

(1e) $s_2 + s_3 > s_1 + s_4$:
In this case the right-hand-side of (3.69) is nonnegative if $\{j_1, j_2\}$ equals $\{1, 4\}$, $\{2, 4\}$, or $\{3, 4\}$ and is negative in all other cases. Hence we have four eigenvalue problems comprised by the cases 1, 4, 6, and 7 and the dimensions of the corresponding Jacobi matrices add up to

$$\frac{s}{2} + 1 + \frac{s}{2} - s_1 - s_4 + \frac{s}{2} - s_2 - s_4 + \frac{s}{2} - s_3 - s_4 = s - 2s_4 + 1. \tag{3.73}$$

(2) s odd:
Again we assume that $j_1, j_2, j_3, j_4 \in \{1, 2, 3, 4\}$ are pairwise distinct. With regard to cases 9–12 we need to consider the inequalities

$$0 \leq 2d = s - 2s_{j_1} - 2s_{j_2} - 2s_{j_3} - 3 = s_{j_4} - s_{j_1} - s_{j_2} - s_{j_3} - 3 \tag{3.74}$$

for each $j_4 \in \{1, 2, 3, 4\}$. The right-hand-side of (3.74) is nonnegative if and only if $j_4 = 1$ and $s_1 - s_2 - s_3 - s_4 > 1$.

In cases 13–16 we have the inequalities

$$0 \leq 2d = s - 2s_{j_1} - 1 = s_{j_2} + s_{j_3} + s_{j_4} - s_{j_1} - 1 \tag{3.75}$$

for each $j_1 \in \{1, 2, 3, 4\}$. This holds for $j_1 = 2, 3, 4$ (note that now s is odd). For $j_1 = 1$ inequality (3.75) holds if and only if $s_1 - s_2 - s_3 - s_4 < 1$.
(2a) $s_1 - s_2 - s_3 - s_4 < 1$:
According to the above observations, inequality (3.75) holds for each $j_1 \in \{1, 2, 3, 4\}$ and (3.74) never holds. The dimensions of the corresponding four Jacobi matrices (the cases 13–16) then add up to

$$\frac{s+1}{2} - s_1 + \frac{s+1}{2} - s_2 + \frac{s+1}{2} - s_3 + \frac{s+1}{2} - s_4 = s + 2. \tag{3.76}$$

(2b) $s_1 - s_2 - s_3 - s_4 = 1$:
Now the right hand side of (3.75) is nonnegative only for $j_1 = 2, 3, 4$ and (3.74) never holds. The dimensions of the corresponding three Jacobi matrices (the cases 13–15) add up to

$$\frac{s+1}{2} - s_2 + \frac{s+1}{2} - s_3 + \frac{s+1}{2} - s_4 = 2s_1 + 1. \tag{3.77}$$

(2c) $s_1 - s_2 - s_3 - s_4 > 1$:
Here inequality (3.74) holds for $j_4 = 1$ and (3.75) holds for $j_1 = 2, 3, 4$. The

dimensions of the corresponding four Jacobi matrices (the cases 9 and 14–16) then add up to

$$\frac{s-1}{2} - s_2 - s_3 - s_4 + \frac{s+1}{2} - s_2 + \frac{s+1}{2} - s_3 + \frac{s+1}{2} - s_4 = 2s_1 + 1. \quad (3.78)$$

For generic values of the invariants g_2 and g_3 associated with the period lattice Λ (see [1], Ch. 18), the eigenvalues of the Jacobi matrices involved will all be simple and different from each other and hence the underlying hyperelliptic curve K_g of genus g determined from (3.70)–(3.73), (3.76)–(3.78) is nonsingular. However, for particular values of g_2, g_3 some eigenvalues can coincide rendering K_g to be a singular curve (see, e.g., Example 2.1 in [26] for $g_2 = 0$).

Thus we have proved the following theorem.

Theorem 3.5 *(i). The Treibich-Verdier potential $q(z) = -\sum_{j=1}^{4} s_j(s_j+1)\wp(z-\omega_j)$, $s_j \in \mathbb{C}$, $1 \le j \le 4$ is a finite-gap potential associated with the stationary KdV hierarchy, or equivalently, a Picard potential if and only if $s_j \in \mathbb{Z}$, $1 \le j \le 4$.*
(ii). If $s_j \in \mathbb{N} \cup \{0\}$ with $s_{j_1} \ge s_{j_2} \ge s_{j_3} \ge s_{j_4}$ ($(s_{j_1}, s_{j_2}, s_{j_3}, s_{j_4})$ a permutation of (s_1, s_2, s_3, s_4)), the (arithmetic) genus g of the underlying (possibly singular) hyperelliptic curve $K_g : y^2 = \prod_{m=0}^{2g}(E - E_m)$ is given by the following table

Table 3.6.

s		g
even	$s_{j_1} + s_{j_4} \ge s_{j_2} + s_{j_3}$	s_{j_1}
even	$s_{j_1} + s_{j_4} \le s_{j_2} + s_{j_3}$	$\frac{s}{2} - s_{j_4}$
odd	$s_{j_1} > s_{j_2} + s_{j_3} + s_{j_4}$	s_{j_1}
odd	$s_{j_1} < s_{j_2} + s_{j_3} + s_{j_4}$	$\frac{s+1}{2}$

where $s = \sum_{j=1}^{4} s_j$. The location E_m of the (finite) branch points resp. singular points $(E_m, 0)$ of K_g is determined from the associated tri-diagonal eigenvalue problem (3.63)–(3.67) and Tables 3.3 and 3.4 in cases (1a)–(1e), (2a)–(2c), respectively.

In this case q satisfies a stationary KdV equation of the type

$$\sum_{j=0}^{g} c_{g-j} \mathrm{KdV}_j(q) = 0, \quad c_0 = 1 \quad (3.79)$$

with c_{g-j} depending on g_2, g_3.

(iii). For generic values of g_2 and g_3 the curve K_g is nonsingular (i.e., $E_\ell \ne E_m$ for $m \ne \ell$).

Given (3.79), q then satisfies appropriate stationary KdV equations of all orders higher than g.

We now study the case when $q(z)$ is real, periodic, and nonsingular for $z \in \mathbb{R}$. Without loss of generality we may assume that ω_1 is real. Then q is nonsingular as a function on $\mathbb{R}$ if and only if $s_1 = s_4 = 0$, i.e., if and only if q is of the form

$$q(z) = -s_2(s_2+1)\wp(z-\omega_2) - s_3(s_3+1)\wp(z-\omega_3). \quad (3.80)$$

Moreover, $q(z)$, $z \in \mathbb{R}$ in (3.80) is real-valued if and only if $g_2^3 - 27g_3^2 > 0$. In this special case we shall now prove that K_g is always nonsingular.

Corollary 3.7 *Let q be given by (3.80) with $g_2^3 - 27g_3^2 > 0$. Then the underlying hyperelliptic curve is nonsingular and of genus $g = \max\{s_2, s_3\}$. Moreover, all (finite) branch points are located on the real axis, i.e., $\{E_m\}_{m=0}^{2g} \subset \mathbb{R}$.*

Proof . Assume without loss of generality that $s_2 \geq s_3$ (otherwise consider $\tilde{q}(z) = q(z + \omega)$ which has the same hyperelliptic curve associated with it). In this case we have always $t_2 = -s_2$. Since $g_2^3 - 27g_3^2 > 0$, $\{e_j\}_{j=1}^3 \subset \mathbb{R}$ and hence α_n, β_n, $\gamma_n \in \mathbb{R}$ (see (3.63)–(3.65)). In fact, a closer look reveals that

$$\alpha_n = (2d - 2n + 1 + 2t_4)(2d - 2n + 2) > 0, \tag{3.81}$$

since $1 \leq n \leq d$ and $t_4 \in \{0, 1\}$. Also

$$\gamma_n = 2(e_2 - e_1)(e_2 - e_3)(n + 1)(2n + 1 - 2s_2) > 0, \tag{3.82}$$

since $e_3 < e_2 < e_1$ and $0 \leq 2n \leq 2d - 2 \leq s_2 + s_3 - 2$ which implies $2n + 1 - 2s_2 \leq s_3 - s_2 - 1 < 0$. Hence $\gamma_n \alpha_{n+1} > 0$, $0 \leq n \leq d - 1$ and by a well-known result on Jacobi matrices (see, e.g., Theorem 8.10 in [19]) this implies that all eigenvalues of J in (3.67) are real and simple. In particular, all finite branch points are located on the real axis.

However, to find all branch points we have to consider several eigenvalue problems, i.e., several matrices J. We will now show that an eigenvalue cannot occur simultaneously in two or more of the matrices associated with the potential under consideration. Assume, on the contrary, that an eigenvalue E appears in two of these matrices, say in J_1 and J_2. Then the eigenvector associated with E must be the same for both J_1 and J_2 since otherwise there would be two linearly independent Floquet solutions at this particular value E while, by construction, all the eigenvalues of these matrices refer to points where only one Floquet solution exists. Hence zero is an eigenvalue of $J_1 - J_2$ and, in particular, J_1 and J_2 have the same size.

Suppose that J_1 is obtained by choosing $t_1, ..., t_4$ in (3.63)–(3.65) to be $t_{1,1}, ..., t_{1,4}$, respectively, while J_2 is obtained by choosing $t_{2,1}, ..., t_{2,4}$. Note that always $t_{1,2} = t_{2,2} = -s_2$ while $t_{k,1}, t_{k,4} \in \{0, 1\}$ and $t_{k,3} \in \{-s_3, s_3 + 1\}$ for $k = 1, 2$. Therefore all elements in the subdiagonal of $J_1 - J_2$ are equal to zero and hence $J_1 - J_2$ is upper triangular. This implies that at least one of the diagonal elements of $J_1 - J_2$ is equal to zero, since zero is an eigenvalue. From (3.64) we obtain that the diagonal elements of $J_1 - J_2$ are given by

$$\sum_{j=1}^{3} \{e_j(4n - 4d)(t_{1,j} - t_{2,j}) - 2e_j(t_{1,4}t_{1,j} - t_{2,4}t_{2,j}) - e_j(t_{1,j}^2 - t_{2,j}^2)\}$$
$$+4e_2(d - 2n)(t_{1,4} - t_{2,4}) \tag{3.83}$$

for $n = 0, ..., d$. Next recall that $-2d + s_2 = t_{k,1} + t_{k,3} + t_{k,4}$ is independent of $k \in \{1, 2\}$. A moments thought reveals that, if $s_3 > 0$, then $t_{1,3} = t_{2,3}$ and

$(t_{k,1}, t_{k,4}) \in \{(0,1),(1,0)\}$ for $k = 1, 2$. In the following we will discuss only this case noting that for $s_3 = 0$ similar arguments will work. (Moreover, $s_3 = 0$ yields a Lamé potential which was discussed in [26].) Thus we obtain that (3.83) equals

$$\pm e_1(4n - 4d - 1) \pm (2e_2t_2 + 2e_3t_3) \mp 4e_2(d - 2n), \tag{3.84}$$

where the upper sign is to be used in the case $(t_{1,1}, t_{1,4}) = (1,0)$, $(t_{2,1}, t_{2,4}) = (0,1)$ while the lower sign is to be used in the other case. Next note that $e_1 = -e_2 - e_3$ and $2d = -1 - t_2 - t_3$. Therefore (3.84) becomes

$$\pm (4n + 2t_2 + 1)(e_2 - e_3). \tag{3.85}$$

Since $e_2 \neq e_3$ and $4n+2t_2+1 \neq 0$ we obtain that none of the diagonal elements and hence none of the eigenvalues of $J_1 - J_2$ is equal to zero contradicting an earlier result. Thus our initial assumption that J_1 and J_2 have a common eigenvalue turned out to be wrong and there are exactly $2s_2 + 1$ different branch points associated with q. □

As a simple illustration of Table 3.6 (and Corollary 3.7) we consider the following potentials.

$$q_4(z) = -20\mathscr{P}(z - \omega_j) - 12\mathscr{P}(z - \omega_k), \tag{3.86}$$

$$\hat{q}_4(z) = -20\mathscr{P}(z - \omega_j) - 6\mathscr{P}(z - \omega_k) - 6\mathscr{P}(z - \omega_\ell), \tag{3.87}$$

$$q_5(z) = -30\mathscr{P}(z - \omega_j) - 2\mathscr{P}(z - \omega_k), \tag{3.88}$$

$$\hat{q}_5(z) = -12\mathscr{P}(z - \omega_j) - 12\mathscr{P}(z - \omega_k) - 6\mathscr{P}(z - \omega_\ell) - 2\mathscr{P}(z - \omega_m), \tag{3.89}$$

where $j, k, \ell, m \in \{1,2,3,4\}$ are mutually distinct. Then q_4 and $\hat{q}_4$ correspond to (arithmetic) genus $g = 4$ while q_5 and $\hat{q}_5$ correspond to $g = 5$. However, we note that all four potentials correspond to $M = 16$ in (1.5). In addition, it can be shown that q_5 and $\hat{q}_5$ are isospectral while q_4 and $\hat{q}_4$ are not.

Part (i) of Theorem 3.5 recovers a recent result of Treibich and Verdier [44], [45]. The algebraic eigenvalue problem (3.66), which yields an effective method to compute the (arithmetic) genus g of the underlying hyperelliptic curve K_g and the location of its (finite) branch points and singular points (band edges) in part (ii) of Theorem 3.5, is our principal new result. In particular, a suitable modification of this method extends to all even elliptic finite-gap potentials [27]. In the special case where $s_2 = 0$ (or $s_3 = 0$), Corollary 3.7 recovers the celebrated result of Ince [31] (extended to the complex Lamé-Ince potential $q(z) = -s_1(s_1 + 1)\mathscr{P}(z)$ as studied in [43]). A detailed treatment of this special case with the present methods can be found in [26].

The corresponding solutions $\psi_{\pm a(E_m)}(z)$ in (3.53) with $\mu_j(E_m)$ determined from the eigenvector $\underline{\mu}$ in (3.66), subject to the relevant cases 1–16, are the analogs of the so called Lamé polynomials (see, e.g., [4], Ch. IX, [47], Ch. XXIII) familiar in the special case of Lamé-Ince potentials.

4 Treibich-Verdier potentials associated with the mKdV hierarchy

In this final section we briefly indicate how to transfer the results of Sect. 3 to the stationary mKdV hierarchy. The results in this section are novel.

Assuming in accordance with (2.25) and (3.1) that

$$q(z) = -\sum_{j=1}^{4} s_j(s_j+1)\wp(z-\omega_j) = -\phi'(z) - \phi(z)^2,$$
$$s_j \in \mathbb{N} \cup \{0\},\ z \in \mathbb{C}, \tag{4.1}$$

we shall compute ϕ in the following. By Miura's identity (2.23) and the commutation results of [15], [22], [24], ϕ will then solve appropriate stationary mKdV equations of all orders greater than or equal to g, where g is determined in Theorem 3.5.

We start by recalling a few general facts from Floquet theory (see, e.g., [22], Appendix F). Let $\psi_\pm(E, z, z_0)$ denote the normalized Floquet solutions

$$\psi_\pm(E,z,z_0) = \psi_{\pm a}(E,z)/\psi_{\pm a}(E,z_0) \tag{4.2}$$

for some appropriate $z_0 \in \mathbb{C}$ with $\psi_{\pm a}$ given by (3.29). One then verifies the relations

$$W(\psi_-(E,\cdot,z_0),\ \psi_+(E,\cdot,z_0)) = 2i\phi_I(E,z_0), \tag{4.3}$$

where

$$\begin{aligned}\phi_\pm(E,z) &= \frac{d}{dz}\ln[\psi_\pm(E,z,z_0)]\\ &= -\frac{1}{2}\frac{d}{dz}\ln[\phi_I(E,z)] \pm i\phi_I(E,z)\end{aligned} \tag{4.4}$$

and

$$q(z) - E = -\phi'_\pm(E,z) - \phi_\pm(E,z)^2. \tag{4.5}$$

Equations (4.4) and (3.29) then yield

$$\phi_\pm(E,z) = \lambda_{\pm a}(E) + \sum_{\ell=1}^{s}\zeta(z \mp a_\ell(E)) - \sum_{j=1}^{4} s_j\zeta(z-\omega_j). \tag{4.6}$$

Using (3.36) and the addition formula for ζ-functions one then obtains

$$\phi_\pm(E,z) = \pm C_{a(E)} + \frac{1}{2}\sum_{\substack{\ell=1\\ a_\ell\neq 0}}^{s}\frac{\wp'(z)\pm\wp'(a_\ell(E))}{\wp(z)-\wp(a_\ell(E))} - \frac{1}{2}\sum_{j=1}^{3}\frac{s_j\wp'(z)}{\wp(z)-\wp(\omega_j)}, \tag{4.7}$$

where

$$C_{a(E)} = \begin{cases} \frac{1}{2}\sum\limits_{\substack{\ell=1\\ a_\ell\neq 0}}^{s}\frac{\wp'(a_\ell(E))}{\wp(a_\ell(E))-\wp(\omega_{k_0})} & \text{if } \omega_{k_0}\neq 0\\ 0 & \text{if } \omega_{k_0}=0.\end{cases} \tag{4.8}$$

(We recall that k_0 in (3.36) is chosen in such a way that ω_{k_0} is different from all the $a_\ell(E)$.)

ϕ_I, which plays an important role in Floquet theory (it is closely related to the Green's function of L) and in the context of complete integrability of the KdV hierarchy in the periodic case, is obtained in the following way. From (4.4), (3.43) and (4.6) or (4.7), respectively, we get

$$\begin{aligned} 2i\phi_I(E,z) &= \phi_+(E,z) - \phi_-(E,z) \\ &= \sum_{\ell=1}^{s}[\zeta(z - a_\ell(E)) - \zeta(z + a_\ell(E)) + \zeta(\omega_{k_0} + a_\ell(E)) - \zeta(\omega_{k_0} - a_\ell(E))] \\ &= 2C_{a(E)} + \sum_{\substack{\ell=1 \\ a_\ell\neq 0}}^{s} \frac{\mathscr{P}'(a_\ell(E))}{\mathscr{P}(z) - \mathscr{P}(a_\ell(E))}, \end{aligned} \tag{4.9}$$

where $C_{a(E)}$ is given by (4.8).

In analogy to the Treibich-Verdier potential $q(z) = -\sum_{j=1}^{4} s_j(s_j+1)\mathscr{P}(z-\omega_j)$, $s_j \in \mathbb{N}\cup\{0\}$, $1 \le j \le 4$ in the context of the stationary KdV hierarchy, we shall call $\phi_\pm(z) := \phi_\pm(0,z)$ in (4.7) a Treibich-Verdier potential associated with the stationary mKdV hierarchy. The commutation methods in [15], [22], [24] relating $L = \frac{d^2}{dx^2} + q$, $q = -\phi' - \phi^2$, $\tilde{L} = \frac{d^2}{dx^2} + \tilde{q}$, $\tilde{q} = \phi' - \phi^2$, and $\mathscr{M} = \begin{pmatrix} 0 & \frac{d}{dx}+\phi \\ \frac{d}{dx}-\phi & 0 \end{pmatrix}$ together with (2.23), Theorem 3.5, (4.5), and (4.7) then yield the following result.

Theorem 4.1 *The Treibich-Verdier potential*

$$\phi_\epsilon(z) = \pm\epsilon C_{a(0)} \pm \frac{1}{2}\sum_{\substack{\ell=1 \\ a_\ell\neq 0}}^{s} \frac{\mathscr{P}'(z) + \epsilon\mathscr{P}'(a_\ell(0))}{\mathscr{P}(z) - \mathscr{P}(a_\ell(0))} \mp \frac{1}{2}\sum_{j=1}^{3} \frac{s_j\mathscr{P}'(z)}{\mathscr{P}(z) - \mathscr{P}(\omega_j)}, \tag{4.10}$$

where $\epsilon \in \{+,-\}$ and $C_{a(0)}$ is defined in (4.8), is a finite-gap potential associated with the stationary mKdV hierarchy if and only if $a_\ell(0)$, $1 \le \ell \le s$ satisfy (3.32)–(3.35) for $E = 0$. The underlying hyperelliptic curve K_{2g} is of the form $y^2 = \prod_{m=0}^{2g}(w - E_m^{1/2})(w + E_m^{1/2}) = \prod_{m=0}^{2g}(w^2 - E_m)$ with g determined as in Theorem 3.5 (ii) and ϕ_ϵ satisfies a stationary mKdV equation of the type

$$\sum_{j=0}^{g} c_{g-j}\,\mathrm{mKdV}_j(\phi_\epsilon) = 0, \quad c_0 = 1 \tag{4.11}$$

with c_{g-j} as in (3.79).

As discussed in Sect. 3, the curve K_{2g} is nonsingular for generic values of g_2, g_3. Moreover, ϕ_ϵ automatically satisfies appropriate stationary mKdV equations of all orders higher than g.

Finally, using equation (4.6) for $E = 0$, one obtains for the finite-gap potential $\tilde{q}_\epsilon = q + 2\phi'_\epsilon$ in $\tilde{L}_\epsilon = \frac{d^2}{dx^2} + \tilde{q}_\epsilon$

$$\tilde{q}_\epsilon(z) = -\sum_{j=1}^{4} s_j(s_j - 1)\mathscr{P}(z - \omega_j) - 2\sum_{\ell=1}^{s} \mathscr{P}(z - \epsilon a_\ell(0)). \tag{4.12}$$

$\tilde{q}_\epsilon(z)$ is isospectral to $q(z) = -\sum_{j=1}^{4} s_j(s_j+1)\mathscr{P}(z - \omega_j)$, i.e., it corresponds to the same hyperelliptic curve $K_g : y^2 = \prod_{m=0}^{2g}(E - E_m)$.

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