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ON PICARD POTENTIALS

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1. Introduction. This paper represents a further contribution to the problem of characterizing all elliptic finite-gap solutions of the stationary Korteweg-de Vries (KdV) hierarchy, a problem posed, e.g., in [22, p. 152]. This theme dates back to a 1940 paper of Ince [15] who studied the Lamé potential

$$q(x) = -s(s+1)\mathcal{P}(x+\omega_3), \quad s \in \mathbb{N},\ x \in \mathbb{R} \tag{1}$$

in connection with the second-order ordinary differential equation

$$\psi''(E,x) + [q(x) - E]\psi(E,x) = 0, \quad E \in \mathbb{C}. \tag{2}$$

Here $\mathcal{P}(x) \equiv \mathcal{P}(x;\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods (f.p.) $2\omega_1$, $2\omega_3$, $\Im(\omega_3/\omega_1) \neq 0$ (see [1], Ch. 18). In the special case where ω_1 is real and ω_3 is purely imaginary the potential q is real-valued and Ince's striking result [15], in modern spectral theoretic terminology, yields the fact that the self-adjoint operator L associated with the differential expression $\frac{d^2}{dx^2} + q$ in $L^2(\mathbb{R})$ has a finite-gap (or finite-band) spectrum of the type

$$\sigma(L) = (-\infty, E_{2s}] \bigcup_{m=1}^{s} [E_{2m-1}, E_{2(m-1)}], \quad E_{2s} < E_{2s-1} < \cdots < E_0. \tag{3}$$

In obvious notation, any potential q that amounts to a finite-gap spectrum of the type (3) is called a finite-gap potential. The proper extension of this notion to complex-valued meromorphic q on the basis of elementary algebro-geometrical concepts is obtained as follows: The starting point is the definition of the so called KdV hierarchy. Let L be the second-order differential expression

$$L = \frac{\partial^2}{\partial x^2} + q,$$

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where the potential q depends on $x \in \mathbb{R}$ and, in addition, on a (deformation) parameter $t \in \mathbb{R}$. It is well known (see, e.g., Wilson [30]) that the coefficients p_j in

$$P_{2n+1} = \frac{\partial^{2n+1}}{\partial x^{2n+1}} + p_{2n-1}\frac{\partial^{2n-1}}{\partial x^{2n-1}} + \cdots + p_0$$

can be chosen in such a way that P_{2n+1} and L are almost commuting, i.e., that their commutator $[P_{2n+1}, L]$ is a multiplication operator. More specifically, the p_j have to be certain polynomials in q and its x-derivatives. (P_{2n+1}, L) is then called a Lax pair and the equation $q_t = [P_{2n+1}, L]$ is a nonlinear evolution equation for q. The collection of all these equations for all possible choices of P_{2n+1} and all nonnegative integers n is called the KdV hierarchy. In particular, the choice

$$P_3 = \frac{\partial^3}{\partial x^3} + \frac{3}{2}q\frac{\partial}{\partial x} + \frac{3}{4}q_x,$$

yields the usual KdV equation. Novikov [21], Dubrovin [9], Its and Matveev [17], and McKean and van Moerbeke [20] then showed that a real-valued smooth potential q is a finite-gap potential if and only if it satisfies appropriate higher-order stationary KdV equations. Moreover, it should be emphasized that the stationary KdV hierarchy, characterized by $q_t = 0$, or equivalently, by $[P_{2n+1}, L] = 0$, is intimately related to the question of commutativity of ordinary differential expressions. In this context a classical result by Burchnall and Chaundy [6], [7] implies that P_{2n+1} and L satisfy an algebraic relation of the form

$$P_{2n+1}^2 = R_{2n+1}(L) = \prod_{m=0}^{2n}(L - E_m).$$

The locations E_m of the finite branch points of the associated compact hyperelliptic Riemann surface $y^2 = R_{2n+1}(z)$ are precisely the band edges, i.e., the end points of the spectral bands, of the operator L whenever the potential q is real-valued and smooth.

Because of these facts, it is common to call q a finite-gap potential if it satisfies one (and hence infinitely-many) equation(s) of the stationary KdV hierarchy, or equivalently, if there is a hyperelliptic curve which has finitely many branch points respectively singular points associated with it in the manner described above. These characterizations extend to complex-valued, quasi-periodic, and meromorphic potentials q, where the branch points and singular points are not necessarily located on the real axis. One calls q an n-gap potential if the underlying Riemann surface has (arithmetic) genus n. (See, e.g., [11] for further details on finite-gap potentials and the associated isospectral manifolds.)

A second important fact for real-valued periodic and smooth potentials q is that the band edges of the operator L are precisely those points E where equation (2) has only one Floquet solution (up to constant multiples). In the main part of the paper, i.e., in Sections 2–5, we will therefore call q a finite-gap potential if and only if there is at most a finite number of values of the spectral parameter such that there exists only one Floquet solution (up to constant multiples). In our paper [14] we proved the equivalence of this latter notion of finite-gap potential with the standard one.

In their 1977 seminal paper [3], Airault, McKean and Moser presented the first systematic study of the isospectral torus $I_{\mathbb{R}}(q_0)$ of real-valued smooth potentials q_0 of the form

$$q_0(x) = -2\sum_{j=1}^{M} \mathcal{P}(x - x_j) \tag{4}$$

with a finite-gap spectrum as in (3). Among a variety of results they proved that any element of $I_{\mathbb{R}}(q_0)$ is an elliptic function of the type (4) (for different sets $\{x_j\}_{j=1}^M$) with M constant throughout $I_{\mathbb{R}}(q_0)$ and $n := \dim I_{\mathbb{R}}(q_0) \leq M$. The next breakthrough occurred in 1988 when Verdier [28] published new explicit examples of elliptic finite-gap potentials. Verdier's examples inspired Belokolos and Enol'skii [4] and Smirnov [23] and subsequently Taimanov [24] and Kostov and Enol'skii [18] to find further such examples by employing the reduction process of Abelian integrals to elliptic integrals (see, e.g., [5]). Finally, this development culminated in the recent result of Treibich and Verdier [25], [26], [27] that a general complex-valued potential of the form

$$q(z) = -\sum_{j=1}^{4} d_j \mathcal{P}(z - \omega_j), \quad z \in \mathbb{C} \tag{5}$$

($\omega_2 = \omega_1 + \omega_3$, $\omega_4 = 0$) is a finite-gap potential if and only if $d_j/2$ are triangular numbers, i.e., if and only if

$$d_j = s_j(s_j + 1) \ \text{ for some } \ s_j \in \mathbb{Z}, \ 1 \leq j \leq 4. \tag{6}$$

The methods of Treibich and Verdier are based on the notion of hyperelliptic tangent covers of the torus $\mathbb{C}/\Lambda$ (Λ the period lattice generated by $2\omega_1$, $2\omega_3$).

Motivated by the results above and by the fact that a complete characterization of all elliptic finite-gap solutions of the stationary KdV hierarchy is still open, we started to develop our own approach toward a solution of this problem. In contrast to all current approaches in this area, our methods to characterize elliptic finite-gap solutions of the KdV hierarchy rely on entirely different ideas based on a systematic use of a powerful theorem of Picard (see Theorem 3) concerning ordinary differential equations with elliptic coefficients. This approach immediately recovers and extends the results of [4], [23], [25], [26], [27], [28]. In the particular cases of Lamé -Ince potentials (1) and Treibich-Verdier potentials (5), (6) we refer to [12] and [13].

Picard's theorem naturally leads to the definition of a new class of elliptic potentials, the so called Picard class. More precisely, let q be an elliptic function. Then q is called a **Picard potential** if and only if

$$\psi'' + q\psi = E\psi$$

has a meromorphic fundamental system of solutions for each $E \in \mathbb{C}$ (see Definition 4).

The connection between Picard potentials and elliptic finite-gap potentials is now the following: By the Its-Matveev formula [17] for q and the corresponding Baker-Akhiezer function expressed in terms of the associated Riemann theta function one proves

Theorem 1. *Every elliptic finite-gap potential q is Picard.*

Naturally, one is led to conjecture that the converse of this theorem is also true. Hence it seems appropriate to formulate the following

Conjecture 2. *An elliptic potential q is finite-gap if and only if it is a Picard potential.*

A proof of this conjecture has recently been provided in [14].

Our main goal in this paper, however, is to investigate Picard potentials which are even about some fixed point z_0 by a systematic use of Picard's Theorem 3. In particular, we shall devise an algorithm to find the location of the points E where there exists only one solution (up to constant multiples) of $\psi'' + q\psi = E\psi$ which is elliptic of the second kind. More precisely, the computation of these points is reduced to the study of certain constrained linear algebraic eigenvalue problems. (Since these points are precisely the branch points and singular points of the hyperelliptic curve associated with q (see [14]) this enables one to compute the (arithmetic) genus of the underlying hyperelliptic curve.)

Our strategy is as follows: The requirement that the differential equation $\psi''+q\psi = E\psi$ has only meromorphic solutions is very restrictive and we show in Theorem 7 that q must be a sum of $\mathcal{P}$-functions with very special coefficients. If, in addition, q is even then the poles are either half-periods or else appear in pairs $(b, -b)$ (see Theorem 9). Next, one derives the existence of two solutions which are elliptic of the second kind (Theorem 11). Finally, one shows that these are linearly independent except in a finite number of cases. This is done in the following way: one extracts certain necessary conditions on the parameters on which these two solutions depend in order to guarantee the vanishing of their Wronskian. This information is then used to show that the values of E for which this may happen are given as eigenvalues of certain matrices of finite size. In some cases there might be additional constraints which require that some of these eigenvalues are to be discarded. In any case we are able to prove that there are two linearly independent solutions which are elliptic of the second kind for all but a finite number of values of E. Moreover, by counting the dimensions of the matrices involved, one can find an upper bound on the number of these values of E; i.e., one can find an upper bound for the genus n of the hyperelliptic curve K_n associated with q even without solving any of the eigenvalue problems.

While Section 2 briefly recalls Picard's theorem and introduces the definition of Picard and finite-gap potentials, Section 3 is devoted to basic results of Picard potentials. Our principal new results on even Picard potentials appear in Section 4 and are summarized in Theorem 13. Section 5 features as an example a particular two-gap potential, Appendix A explicitly gives the matrices which determine the number and location of the band edges, and Appendix B collects basic results on elliptic functions needed throughout this manuscript.

In the meantime we proved the analogue of the aforementioned conjecture for general second-order equations with elliptic coefficients without any symmetry assumption on the coefficients [14]. At present, however, the approach used to prove this general result does not provide a constructive method to obtain the genus n and the branch points respectively singular points of the underlying curve. Nevertheless it should be stressed at this point that this characterization of elliptic finite-gap potentials as Picard potentials yields the most effective criterion to date for determining whether or not a given elliptic potential is actually finite-gap.

2. Picard's theorem and finite-gap potentials.

Theorem 3 (see, e.g., [2, pp. 182–187], [16, pp. 375–376])**.** *Suppose the coefficients of a homogeneous linear differential equation of order n are elliptic functions corresponding to the same period lattice. Furthermore, assume that the equation has a meromorphic fundamental system of solutions. Then there exists at least one solution which is elliptic of the second kind.*

This theorem is the analogue of Floquet's theorem for periodic functions in the context of doubly periodic meromorphic (i.e., elliptic) functions.

In the following we want to restrict our attention to the special case of a second order differential equation of the form

$$\psi'' + q\psi = E\psi, \tag{7}$$

where q is an elliptic function and E is a spectral parameter. We then define the following subclass of elliptic functions.

Definition 4. Let q be an elliptic function. If the general solution of equation (7) is meromorphic for each complex number E then q is called a Picard potential.

It can be shown (see Corollary 8) that q is a Picard potential whenever (7) has a meromorphic fundamental system of solutions for a sufficiently large but finite number of distinct values of E.

The function $q(z) = -2\mathcal{P}(z)$, for instance, is a Picard potential. In fact, the meromorphic function $\psi(z) = \exp(\zeta(a)z)\frac{\sigma(z-a)}{\sigma(z)}$ solves (7) whenever a satisfies $\mathcal{P}(a) = E$. Except for the three values $E = e_1, e_2, e_3$ this equation has two solutions a (in Δ, the f.p.p., see Appendix B) which yield linearly independent solutions ψ of (7). One can also show, however, that for the three exceptional values of E the second solution is meromorphic.

In this paper we use the following definition of the term finite-gap potential.

Definition 5. Let q be a meromorphic, complex-valued, periodic function and E a complex number. For some $\varepsilon > 0$ let $\psi_1(\lambda, \cdot)$, $\psi_2(\lambda, \cdot)$ be nontrivial Floquet solutions of

$$\psi'' + q\psi = \lambda\psi, \quad |\lambda - E| < \varepsilon$$

in the sense that ψ_1 and ψ_2 are not identically equal to zero for any $|\lambda - E| < \varepsilon$. Assume that $\psi_1(\lambda, \cdot)$ and $\psi_2(\lambda, \cdot)$ are linearly independent for $0 < |\lambda - E| < \varepsilon$ and that they, together with their x-derivatives, depend continuously on λ. If the Wronskian $W(\psi_1, \psi_2)$ converges to zero as λ approaches E then E is called a band edge. The potential q is called a finite-gap potential if the number of its band edges is finite.

We emphasize here once more that this is not the definition usually employed. However, as we will show in [14], it is equivalent to the usual one. In fact, the proof of this equivalence is implied by the following characterization of general periodic finite-gap potentials (not necessarily elliptic).

Theorem 6 ([14]). *Assume that $q(x)$ is a periodic meromorphic function of period $\Omega > 0$ on $\mathbb{R}$ (without loss of generality we may consider translates $q(x+c)$, $c \in \mathbb{C}$ such that $q(x)$ has no poles on the real axis) and that $L = d^2/dx^2 + q(x)$ has two linearly independent Floquet solutions for all $E \in \mathbb{C}\backslash\{\hat{E}_j\}_{j=0}^{\hat{M}}$ for some $\hat{M} \in \mathbb{N} \cup \{0\}$ and precisely one Floquet solution for each $E = \hat{E}_j$ (assuming $\hat{E}_j \neq \hat{E}_{j'}$ for $j \neq j'$). Denote by $\hat{d}(E)$ the algebraic multiplicity of E as an (anti)periodic eigenvalue and by $\hat{p}(E)$ the minimal algebraic multiplicity of E as a Dirichlet eigenvalue on $[x_0, x_0+\Omega]$ as x_0 varies in $\mathbb{R}$. Let $\hat{q}(E) = \hat{d}(E) - 2\hat{p}(E)$. Then*

(i) *$\hat{q}(E)$ is positive on a finite set $\{\hat{E}_0, \dots, \hat{E}_M\}$, $M \geq \hat{M}$, and zero elsewhere. Let $\hat{q}_j = \hat{q}(\hat{E}_j)$, $j = 1, \dots, M$. Then $\sum_{j=0}^{M} \hat{q}_j = 2n+1$ for some nonnegative integer n; i.e., $\sum_{j=0}^{M} \hat{q}_j$ is an odd positive integer.*

(ii) *There exists a monic ordinary differential expression*

$$P_{2n+1} = \sum_{\ell=0}^{2n+1} p_\ell(x) \frac{d^\ell}{dx^\ell}, \quad p_{2n+1} = 1,$$

whose coefficients $p_0, \dots, p_{2n}$ are polynomials in q and its derivatives which commutes with L; i.e., q satisfies the stationary KdV equation $[P_{2n+1}, L] = 0$.

(iii) *The hyperelliptic curve associated with q is of (arithmetic) genus n and given by*

$$y^2 = \prod_{j=0}^{M} (E - \hat{E}_j)^{\hat{q}_j}.$$

It is nonsingular if all the multiplicities $\hat{q}_0, \dots, \hat{q}_M$ are equal to one.

The proof of Theorem 6 in [14] is based on well-known identities for the diagonal Green's function $G(E,x,x)$ in terms of the Floquet discriminant $\Delta(E)$ and a fundamental system of solutions of $L\psi(E,y) = E\psi(E,y)$ with respect to a reference point $x \in \mathbb{R}$, Hadamard-type factorizations of such solutions with respect to E, the nonlinear second-order differential equation satisfied by $G(E,x,x)$, and the recursion formalism for the KdV hierarchy.

3. Basic results on Picard potentials.

3.1. The general structure of Picard potentials.

Theorem 7. *If q is a non-constant Picard potential, then it may be represented as*

$$q(z) = C - \sum_{j=1}^{m} s_j(s_j+1)\mathcal{P}(z - b_j)$$

for suitable positive integers m, $s_1, \dots, s_m$ and complex numbers C, b_1, …, b_m, where the b_j are pairwise distinct modΔ.

Proof. Since q is a Picard potential the differential equation $\psi'' + q\psi = E\psi$ possesses two linearly independent meromorphic solutions. For this it is necessary that any singular point of this differential equation; i.e., any pole of q, be a regular singular point and that the exponents of the singularity relative to this point are unequal integers. This implies that any pole of q has order no greater than 2, since otherwise, it would not be a regular singular point. Theorem 14 then shows that for some integer m the function q may be expressed as

$$C + \sum_{j=1}^{m} A_j \mathcal{P}(z - b_j) + \sum_{j=1}^{m} B_j \zeta(z - b_j),$$

where $\sum_{j=1}^{m} B_j = 0$. This implies that the indicial equation for the singular point b_j is

$$f(\ell) = \ell(\ell - 1) + A_j = 0, \tag{8}$$

where ℓ is an exponent of the singularity and hence supposed to be an integer. This shows that A_j must be of the form $A_j = -s_j(s_j + 1)$, $s_j \in \mathbb{N}_0$ and that the solutions of (8) are then $\ell = s_j + 1$ and $\ell = -s_j$. The Frobenius method shows that for this choice of A_j there is always one solution of $\psi'' + q\psi = E\psi$ which is meromorphic near b_j. This solution is in fact of the form

$$\psi(z) = (z - b_j)^{s_j+1} \sum_{k=0}^{\infty} \alpha_k (z - b_j)^k, \quad \alpha_0 = 1. \tag{9}$$

It remains to be shown that $B_j = 0$, $j = 1, \dots, m$. One solution of $\psi'' + q\psi = E\psi$ is always given by (9). If all solutions are to be meromorphic near b_j, then there must be one of the form

$$\psi(z) = (z - b_j)^{-s_j} \sum_{k=0}^{\infty} \beta_k (z - b_j)^k, \quad \beta_0 = 1.$$

Inserting this into $\psi'' + q\psi = E\psi$ gives

$$(z - b_j)^2 \psi'' + Q_E(z)\psi = 0,$$

where Q_E is analytic near b_j, specifically

$$Q_E(z) = A_j + B_j(z - b_j) + (C_{j,2} - E)(z - b_j)^2 + \cdots + C_{j,k}(z - b_j)^k + \cdots$$

for suitable constants $C_{j,k}$. Hence, we obtain

$$\begin{aligned} 0 = f(-s_j)(z - b_j)^{-s_j} &+ \{f(1 - s_j)\beta_1 + G_1\}(z - b_j)^{1-s_j} \\ &+ \cdots + \{f(k - s_j)\beta_k + G_k\}(z - b_j)^{k-s_j} + \cdots, \end{aligned}$$

where

$$G_k = B_j\beta_{k-1} + (C_{j,2} - E)\beta_{k-2} + C_{j,3}\beta_{k-3} + \cdots + C_{j,k}\beta_0.$$

Next we note that $f(-s_j) = 0$ and that we may determine the β_k successively from the requirement that the coefficient of $(z - b_j)^{k-s_j}$ must be zero whenever $f(j - s_j) \neq 0$ for $j = 1, \dots, k$. Since the only zero of f besides $-s_j$ is $s_j + 1$ we may determine $\beta_1, \dots, \beta_{2s_j}$. The coefficient of $(z-b_j)^{s_j+1}$, however, is G_{2s_j+1} and there is no choice to make it vanish. In fact, if G_{2s_j+1} is not equal to zero then there will not be a second meromorphic solution. Hence, the equation $\psi'' + q\psi = E\psi$ has two meromorphic solutions if and only if $G_{2s_j+1} = 0$. Therefore, we now study the structure of this coefficient.

Note that

$$\beta_1 = \frac{-1}{f(1-s_j)} B_j(-E)^0 = \gamma_1 B_j(-E)^0,$$
$$\beta_2 = \frac{-1}{f(2-s_j)}(-E)^1 - \frac{1}{f(2-s_j)}\big(C_{j,2} - \frac{B_j^2}{f(1-s_j)}\big) = \gamma_2(-E)^1 + O(E^0),$$

defining positive constants γ_1 and γ_2. Assume that β_{2k-1} and β_{2k} are polynomials in E such that

$$\beta_{2k-1} = \gamma_{2k-1} B_j(-E)^{k-1} + O(E^{k-2}), \quad \beta_{2k} = \gamma_{2k}(-E)^k + O(E^{k-1})$$

and that γ_{2k-1} and γ_{2k} are positive. Then G_{2k+1} and G_{2k+2} are polynomials in E and

$$G_{2k+1} = (\gamma_{2k-1} + \gamma_{2k}) B_j(-E)^k + O(E^{k-1}), \quad G_{2k+2} = \gamma_{2k}(-E)^{k+1} + O(E^k).$$

Therefore, β_{2k+1} and β_{2k+2} are also polynomials in E if $k \leq s_j - 1$ and

$$\beta_{2k+1} = -\frac{\gamma_{2k-1} + \gamma_{2k}}{f(2k+1-s_j)} B_j(-E)^k + O(E^{k-1}),$$
$$\beta_{2k+2} = -\frac{\gamma_{2k}}{f(2k+2-s_j)}(-E)^{k+1} + O(E^k).$$

Defining γ_{2k+1} and γ_{2k+2} to be the coefficients of the leading terms in β_{2k+1} and β_{2k+2} we find that they are positive whenever $f(2k+1-s_j)$ and $f(2k+2-s_j)$ are negative; i.e., as long as $k \leq s_j - 1$. Hence, G_{2s_j+1} is a polynomial in E of order k and $G_{2s_j+1} = (\gamma_{2s_j-1} + \gamma_{2s_j}) B_j(-E)^{s_j} + O(E^{s_j-1})$. But $\gamma_{2s_j-1} + \gamma_{2s_j}$ is positive and therefore a necessary condition for G_{2s_j+1} to vanish for all complex E is that B_j be equal to zero. This proves the theorem. □

The above proof immediately implies the following

Corollary 8. *Let*

$$q(z) = C - \sum_{j=1}^{m} s_j(s_j+1)\mathcal{P}(z - b_j).$$

If $\psi'' + q\psi = E\psi$ has a meromorphic fundamental system of solutions for a number of distinct values of E which exceeds $\max\{s_1, \dots, s_m\}$, then q is a Picard potential.

It is not sufficient, however, to require the existence of two linearly independent meromorphic solutions of $\psi'' + q\psi = E\psi$ at just one point E in order to assure that q is Picard. This is shown by the example $q(z) = -2\mathcal{P}(z) - 2\mathcal{P}(z - \omega_2) - 4\eta_2(\zeta(z) - \zeta(z - \omega_2))$. Here we have to consider G_3 for both singular points. It turns out that G_3 for $z = \omega_2$ is just the negative of G_3 for $z = 0$ which in turn is given by $4\eta_2(e_2 + E)$. Hence, we have two linearly independent meromorphic solutions of $\psi'' + q\psi = -e_2\psi$ but only one for any other value of $E \in \mathbb{C}$.

3.2. Elliptic solutions of the second kind. If

$$q(z) = C - \sum_{j=1}^{m} s_j(s_j + 1)\mathcal{P}(z - b_j) \tag{10}$$

is a Picard potential then, by Picard's theorem (Theorem 3), $\psi'' + q\psi = E\psi$ has at least one solution which is elliptic of the second kind. In the following we want to study these solutions more closely. By Theorem 16, they may be represented as

$$\psi(z) = \exp(\lambda z) \prod_{j=1}^{N} \sigma(z - c_j)^{\ell_j} \tag{11}$$

for suitable constants $N \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $c_1, \dots, c_N \in \Delta$ with $c_m \neq c_n$ for $m \neq n$, and $\ell_1, \dots, \ell_N \in \mathbb{Z}$ with $\sum_{j=1}^{N} \ell_j = 0$. Note that the c_n are the roots and poles of ψ in the fundamental period parallelogram (f.p.p.) Δ. Since ψ may have zeros of order higher than one or poles only at singularities of the potential, the numbers ℓ_n may be different from zero or one only if c_n is a singularity of q. Since a factor with $\ell_n = 0$ can be dropped from the product on the right hand side of (11), we agree from this point on that $\ell_n = 0$ may occur only if c_n is a half-period (this greatly facilitates our notation later on). Now,

$$\frac{\psi''}{\psi} = \Big(\frac{\psi'}{\psi}\Big)' + \Big(\frac{\psi'}{\psi}\Big)^2$$

and, using fundamental properties of elliptic functions,

$$\frac{\psi'}{\psi} = \lambda + \sum_{j=1}^{N} \ell_j \zeta(z - c_j), \qquad \Big(\frac{\psi'}{\psi}\Big)' = -\sum_{j=1}^{N} \ell_j \mathcal{P}(z - c_j),$$

$$\Big(\frac{\psi'}{\psi}\Big)^2 = \sum_{j=1}^{N} \Big\{ \ell_j^2 \mathcal{P}(z - c_j) + 2\ell_j \Big(\lambda + \sum_{k=1, k\neq j}^{N} \ell_k \zeta(c_j - c_k)\Big)\zeta(z - c_j) \Big\}$$

$$- \sum_{j=1, j\neq r}^{N} \ell_j(\ell_j + 2\ell_r)\mathcal{P}(c_r - c_j).$$

for any $r \in \{1, \dots, N\}$ for which $\ell_r \neq 0$. Hence, still assuming $\ell_r \neq 0$,

$$\begin{aligned} q(z) &= E - \frac{\psi''}{\psi} \\ &= \sum_{j=1}^{N} \Big\{ (\ell_j - \ell_j^2)\mathcal{P}(z - c_j) - 2\ell_j \Big(\lambda + \sum_{k=1, k\neq j}^{N} \ell_k \zeta(c_j - c_k) \Big) \zeta(z - c_j) \Big\} \\ &\quad + \sum_{j=1, j\neq r}^{N} \ell_j(\ell_j + 2\ell_r)\mathcal{P}(c_r - c_j) + E = C - \sum_{j=1}^{m} s_j(s_j+1)\mathcal{P}(z - b_j). \end{aligned}$$

This implies that for each $r \in \{1, \dots, N\}$ with $\ell_r \neq 0$

$$\begin{aligned} &\lambda + \sum_{j=1, j\neq r}^{N} \ell_j \zeta(c_r - c_j) = 0, \\ &E = C - \sum_{j=1, j\neq r}^{N} \ell_j(\ell_j + 2\ell_r)\mathcal{P}(c_r - c_j). \end{aligned} \tag{12}$$

Moreover, m of the numbers c_j coincide with the b_j. Hence we may identify c_{N-m+j} with b_j and ℓ_{N-m+j} with either $-s_j$ or else $s_j + 1$ for $j = 1, \dots, m$. For every j such that $c_j \notin \{b_1, \dots, b_m\} \cup \{\omega_1, \dots, \omega_4\}$ we must have $\ell_j = 1$. We introduce

$$\begin{aligned} M_1 &= \{j \in \{1, \dots, m\} : \ell_{N-m+j} = -s_j\}, \\ M_2 &= \{j \in \{1, \dots, m\} : \ell_{N-m+j} = s_j + 1\}, \\ s &= \sum_{j=1}^{m} s_j, \qquad \hat{s} = N - m = \sum_{j\in M_1} s_j - \sum_{j\in M_2} (s_j + 1), \end{aligned}$$

and note that $M_1 \cup M_2 = \{1, \dots, m\}$ and $M_1 \cap M_2 = \emptyset$. We also recall that s_j may only be zero if b_j is a half-period. Hence we can rewrite (11) as

$$\begin{aligned} \psi(z) &= \exp(\lambda z) \frac{\big(\prod_{j\in M_2} \sigma(z - b_j)^{s_j+1}\big)\big(\prod_{j=1}^{\hat{s}} \sigma(z - c_j)\big)}{\prod_{j\in M_1} \sigma(z - b_j)^{s_j}} \\ &= \exp(\lambda z) \frac{\big(\prod_{j\in M_2} \sigma(z - b_j)^{2s_j+1}\big)\big(\prod_{j=1}^{\hat{s}} \sigma(z - c_j)\big)}{(\prod_{j\in M_1} \sigma(z - b_j)^{s_j})(\prod_{j\in M_2} \sigma(z - b_j)^{s_j})} \\ &= \exp(\lambda z) \frac{\prod_{j=1}^{s} \sigma(z - a_j)}{\prod_{j=1}^{m} \sigma(z - b_j)^{s_j}}, \end{aligned} \tag{13}$$

where $a_k = c_k$ for $k = 1, \dots, \hat{s}$, $a_k = b_{j_1}$ for $k = \hat{s}+1, \dots, \hat{s}+2s_{j_1}+1, \dots a_k = b_{j_\mu}$ for $k = s - 2s_{j_\mu}, \dots, s$ with $M_2 = \{j_1, j_2, \dots, j_\mu\}$.

Emphasizing the dependence of λ on the parameters $a_1, \dots, a_s$, we write from now on λ_a instead of λ. One obtains from (12)

$$\lambda_a = \sum_{j=1}^{n} \zeta(a_j - b_r) - \sum_{j=1, j\neq r}^{m} s_j \zeta(b_j - b_r). \tag{14}$$

Here we assumed that $s_r \neq 0$ and that none of the a_i is equal to b_r. There is always at least one pole of q such that this is true. Note also that we may use any of the equations (12) to express λ_a explicitly. Finally, we remark that the parameters $a_1, \dots, a_s$, and hence λ_a, depend on the spectral parameter E.

4. Even Picard potentials. We now focus our attention on even Picard potentials.

Theorem 9. *Let q be a Picard potential which is even about some point $z_0 \in \mathbb{C}$, i.e., $q(z + z_0) = q(-z + z_0)$. Then*

$$\begin{aligned} q(z) = C - \sum_{k=1}^{4} r_k(r_k+1)\mathcal{P}(z - z_0 - \omega_k) \\ - \sum_{j=1}^{\tilde{m}} s_j(s_j+1)[\mathcal{P}(z - z_0 - b_j) + \mathcal{P}(z - z_0 + b_j)] \end{aligned}$$

for appropriate non-negative integers $\tilde{m}$, $r_1, \dots, r_4$ and positive integers $s_1, \dots, s_{\tilde{m}}$ and complex numbers C and $b_1, \dots, b_{\tilde{m}}$. Here none of the b_i is a half-period and they are all pairwise distinct.

Proof. One only needs to compare the principal parts of the Laurent series of $q(z+z_0)$ and $q(-z + z_0)$ near all their poles which determine both functions up to a constant by Theorem 14. □

For simplicity we will assume from now on that $z_0 = 0$. We note that the number of poles of q in Δ is between $2\tilde{m}$ and $2\tilde{m} + 4$ depending on how many of the r_i are nonzero. We will adopt the following notation: $b_{k+\tilde{m}} = -b_k$ for $k = 1, \dots, \tilde{m}$, and $b_{k+2\tilde{m}} = \omega_k$ for $k = 1, \dots, 4$. This then enforces $s_{k+\tilde{m}} = s_k$ for $k = 1, \dots, \tilde{m}$. In addition, we let $s_{k+2\tilde{m}} = r_k$ for $k = 1, \dots, 4$. Also let

$$\begin{aligned} M_1 &= \{j \in \{1, \dots, 2\tilde{m}+4\} : b_j \notin \{a_1, \dots, a_s\}\}, \\ M_2 &= \{j \in \{1, \dots, 2\tilde{m}+4\} : b_j \in \{a_1, \dots, a_s\}\}, \\ 2d &= s - \sum_{j \in M_2} (2s_j + 1). \end{aligned}$$

These definitions coincide with those made in the last section if all of $\omega_1, \dots, \omega_4$ occur among the b_j in (10) (perhaps with $s_j = 0$).

Proposition 10. *Let q be an even Picard potential. If ψ_a is a solution of $\psi'' + q\psi = E\psi$ of the form* (13)*, then there is a pole b_r of q such that neither b_r nor $-b_r$ (modΔ) appears in $\{a_1, \dots, a_s\}$. In addition*

$$\lambda_a - \lambda_{-a} = \sum_{j=1}^{s} (\zeta(a_j - b_r) + \zeta(a_j + b_r)), \tag{15}$$

$$\lambda_a + \lambda_{-a} = -\sum_{k=1}^{3} 2r_k\eta_k. \tag{16}$$

Proof. In contradiction to the first statement, assume that all half-periods which are poles of q as well as b_ℓ or $-b_\ell$ for every index $\ell \in \{1, \dots, \tilde{m}\}$ appear in $\{a_1, \dots, a_s\}$ $(\mathrm{mod}\Delta)$. Then necessarily

$$s \geq \sum_{j=1}^{\tilde{m}}(2s_j + 1) + \sum_{k=1, r_k \neq 0}^{4}(2r_k + 1).$$

On the other hand,

$$s = \sum_{j=1}^{\tilde{m}} 2s_j + \sum_{k=1, r_k \neq 0}^{4} r_k,$$

which is impossible. Hence there is at least one half-period or a pair $(b_r, -b_r)$ among the poles of q which does not occur among the a_i.

This shows that we may use (14) to express both λ_a and λ_{-a} using the same pole b_r of q. This gives

$$\lambda_a - \lambda_{-a} = \sum_{j=1}^{n}(\zeta(a_j - b_r) - \zeta(-a_j - b_r))$$

and hence (15).

In order to prove (16) suppose first that there is a half-period, say ω_k, among the poles of q which does not occur among the a_j. Letting $\eta_4 = 0$ we get from $\zeta(a - \omega_k) + \zeta(-a - \omega_k) = -2\eta_k$ and $s = \sum_{j=1}^{\tilde{m}} 2s_j + r_1 + \cdots + r_4$ that

$$\begin{aligned}\lambda_a + \lambda_{-a} &= -2s\eta_k + 2\eta_k \sum_{j=1}^{\tilde{m}} 2s_j - \sum_{\ell=1, \ell\neq k}^{4} 2r_\ell(\eta_\ell - \eta_k) \\ &= \sum_{\ell=1}^{4}(-2r_\ell\eta_k - 2r_\ell(\eta_\ell - \eta_k)) = -\sum_{\ell=1}^{4} 2r_\ell\eta_\ell.\end{aligned}$$

Otherwise there is an $r \in \{1, \dots, \tilde{m}\}$ such that both b_r and $-b_r$ do not occur among the a_j. Therefore, we may rewrite two of the equations (12) to express λ_a in the following way:

$$\begin{aligned}\lambda_a &= \sum_{j=1}^{s} \zeta(a_j - b_r) - \sum_{k=1}^{4} r_k\zeta(\omega_k - b_r) \\ &\quad - \sum_{j=1, j\neq r}^{\tilde{m}} s_j(\zeta(b_j - b_r) + \zeta(-b_j - b_r)) - 2s_r\zeta(-2b_r)\end{aligned}$$

and

$$\begin{aligned}\lambda_a &= \sum_{j=1}^{s} \zeta(a_j + b_r) - \sum_{k=1}^{4} r_k\zeta(\omega_k + b_r) \\ &\quad - \sum_{j=1, j\neq r}^{\tilde{m}} s_j(\zeta(b_j + b_r) + \zeta(-b_j + b_r)) - 2s_r\zeta(2b_r).\end{aligned}$$

Adding these two equations we obtain

$$\begin{aligned}2\lambda_a &= \sum_{j=1}^{s}(\zeta(a_j-b_r)+\zeta(a_j+b_r)) - \sum_{k=1}^{4} r_k(\zeta(\omega_k-b_r)+\zeta(\omega_k+b_r))\\ &= \sum_{j=1}^{s}(\zeta(a_j-b_r)+\zeta(a_j+b_r)) - \sum_{k=1}^{3} 2r_k\eta_k.\end{aligned}$$

Therefore, $\lambda_a+\lambda_{-a} = -\sum_{k=1}^{3} 2r_k\eta_k$, concluding the proof. □

Theorem 11. *Suppose that q is an even Picard potential. Then if the function ψ_a given in* (13) *is a solution of the differential equation $\psi''+q\psi=E\psi$, so is the function ψ_{-a} which is obtained by replacing every a_j with $-a_j$, $j=1,\dots,s$ in* (13) *and* (14).

Proof. Consider

$$\psi_a(z) = \mathrm{e}^{\lambda_a z}\frac{\prod_{j=1}^{s}\sigma(z-a_j)}{\left(\prod_{k=1}^{4}\sigma(z-\omega_k)^{r_k}\right)\left(\prod_{j=1}^{\tilde m}\sigma(z-b_j)^{s_j}\sigma(z+b_j)^{s_j}\right)}$$

and compute $\psi_{-a}(z)/\psi_a(-z)$ to obtain

$$\begin{aligned}&\frac{\psi_{-a}(z)}{\psi_a(-z)} = \mathrm{e}^{(\lambda_a+\lambda_{-a})z}\frac{\sigma(z+\omega_1)^{r_1}\sigma(z+\omega_2)^{r_2}\sigma(z+\omega_3)^{r_3}}{\sigma(z-\omega_1)^{r_1}\sigma(z-\omega_2)^{r_2}\sigma(z-\omega_3)^{r_3}}\\ &= (-1)^{r_1+r_2+r_3}\exp((\lambda_a+\lambda_{-a}+2r_1\eta_1+2r_2\eta_2+2r_3\eta_3)z) = (-1)^{r_1+r_2+r_3}\end{aligned}$$

using (16) in the last equality.

Hence, $\psi_{-a}(z)$ is just a multiple of $\psi_a(-z)$. $\psi_a(-z)$, however, solves the equation $\psi''+q\psi=E\psi$, if $\psi_a(z)$ does since q is even; i.e., the equation is invariant under the transformation $z\to -z$. □

Next we compute the Wronskian of the two solutions ψ_a and ψ_{-a} of $\psi''+q\psi=E\psi$. Since

$$W(\psi_a,\psi_{-a}) = \psi_a\psi_{-a}\left(\frac{\psi_{-a}'}{\psi_{-a}} - \frac{\psi_a'}{\psi_a}\right)$$

we compute $\psi_a\psi_{-a}$ and get

$$\begin{aligned}&\psi_a\psi_{-a} = \\ &\mathrm{e}^{(\lambda_a+\lambda_{-a})z}\frac{\left(\prod_{j\in M_2}\sigma(z-b_j)^{2s_j+1}\sigma(z+b_j)^{2s_j+1}\right)\left(\prod_{j=1}^{2d}\sigma(z-a_j)\sigma(z+a_j)\right)}{\left(\prod_{j=1}^{\tilde m}\sigma(z-b_j)^{2s_j}\sigma(z+b_j)^{2s_j}\right)\left(\prod_{k=1}^{4}\sigma(z-\omega_k)^{2r_k}\right)}.\end{aligned} \tag{17}$$

Furthermore, by (15),

$$\begin{aligned}\frac{\psi_{-a}'}{\psi_{-a}} - \frac{\psi_a'}{\psi_a} &= \lambda_{-a} + \sum_{j=1}^{s}\zeta(z+a_j) - \lambda_a - \sum_{j=1}^{s}\zeta(z-a_j)\\ &= \sum_{j=1}^{s}(\zeta(z+a_j)-\zeta(z-a_j)-\zeta(b_r+a_j)+\zeta(b_r-a_j))\end{aligned}$$

for a suitable choice of r. Note that each of these summands is an elliptic function of order 2 with poles at $z=\pm a_j$ and zeros at $z=\pm b_\ell$. Therefore it may be represented by a quotient of products of σ-functions multiplied by an appropriate constant. This constant may be fixed by considering the behavior of the functions near any of the poles. Performing this we find

$$\frac{\psi'_{-a}}{\psi_{-a}}-\frac{\psi'_a}{\psi_a}=-\sum_{j=1}^{s}\frac{\sigma(2a_j)}{\sigma(a_j-b_r)\sigma(a_j+b_r)}\frac{\sigma(z-b_r)\sigma(z+b_r)}{\sigma(z-a_j)\sigma(z+a_j)}. \tag{18}$$

Now multiplying (17) and (18) and splitting the sum over j at $2d$ we obtain

$$W(\psi_a,\psi_{-a})=f_1(z)\sum_{j=1}^{2d}g_j(z)+f_2(z)\sum_{j\in M_2}h_j(z),$$

where

$$f_1(z)=\mathrm{e}^{(\lambda_a+\lambda_{-a})z}\frac{\sigma(z-b_r)\sigma(z+b_r)\prod_{j\in M_2}\sigma(z-b_j)^{2s_j+1}\sigma(z+b_j)^{2s_j+1}}{(\prod_{j=1}^{\tilde m}\sigma(z-b_j)^{2s_j}\sigma(z+b_j)^{2s_j})(\prod_{k=1}^{4}\sigma(z-\omega_k)^{2r_k})},$$

$$g_j(z)=\frac{-\sigma(2a_j)}{\sigma(a_j-b_r)\sigma(a_j+b_r)}\prod_{\ell=1,\ell\neq j}^{2d}\sigma(z-a_\ell)\sigma(z+a_\ell),$$

$$f_2(z)=\mathrm{e}^{(\lambda_a+\lambda_{-a})z}\sigma(z-b_r)\sigma(z+b_r)\prod_{j=1}^{2d}\sigma(z-a_j)\sigma(z+a_j),$$

$$h_j(z)=-(2s_j+1)\frac{\sigma(2b_j)}{\sigma(b_j-b_r)\sigma(b_j+b_r)}\sigma(z-b_j)^{2s_j}\sigma(z+b_j)^{2s_j}$$
$$\times\frac{\prod_{\ell\in M_2,\ell\neq j}\sigma(z-b_\ell)^{2s_\ell+1}\sigma(z+b_\ell)^{2s_\ell+1}}{(\prod_{\ell=1}^{\tilde m}\sigma(z-b_\ell)^{2s_\ell}\sigma(z+b_\ell)^{2s_\ell})(\prod_{k=1}^{4}\sigma(z-\omega_k)^{2r_k})}.$$

Our goal is to determine the number of values of the spectral parameter E for which the Wronskian is zero. Therefore, we first extract information on the a_j from the above expression for the Wronskian under which conditions this is true.

Since the Wronskian is in fact independent of z we may evaluate it at any point. We choose all the points $z=a_\ell$, $\ell=1,\dots,s$. We start with the case $1\le\ell\le 2d$. Since $f_2(z)$ contains the factor $\sigma(z-a_\ell)$ we obtain $f_2(a_\ell)=0$. Also every $g_j(z)$, with the exception of $g_\ell(z)$, contains this factor and hence $g_j(a_\ell)=0$ for all $j\neq\ell$. Thus

$$W(\psi_a,\psi_{-a})=-f_1(a_\ell)\frac{\sigma(2a_\ell)}{\sigma(a_\ell-b_r)\sigma(a_\ell+b_r)}\prod_{j=1,j\neq\ell}^{2d}\sigma(a_\ell-a_j)\sigma(a_\ell+a_j).$$

Note that a_ℓ is different from all the b_q, $-b_q$ (since $\ell\le 2d$), in particular, it is different from the half-periods implying that $\sigma(2a_\ell)\neq 0$. Also $a_\ell\neq a_k$ if $k\neq\ell$. Therefore, we find that the Wronskian is zero if and only if $\sigma(a_\ell+a_j)=0$ for some $j\in\{1,\dots,\ell-$

$1, \ell+1, \ldots, 2d\}$ and hence $a_j = -a_\ell \pmod{\Delta}$. In particular, we find that the number $2d$ is even; i.e., d is an integer.

Next we evaluate the Wronskian at those points b_ℓ which are among the a_j; i.e., for $\ell \in M_2$. In this case $f_1(z)$ and $h_j(z)$ for $j \neq \ell$ all contain the factor $\sigma(z - b_\ell)$ and therefore the Wronskian consists of one summand only:

$$W(\psi_a, \psi_{-a}) = -f_2(b_\ell)(2s_\ell + 1)\frac{\sigma(2b_\ell)}{\sigma(b_\ell - b_r)\sigma(b_\ell + b_r)} \times \frac{\sigma(2b_\ell)^{2s_\ell} \prod_{j \in M_2, j \neq \ell} \sigma(b_\ell - b_j)^{2s_j+1}\sigma(b_\ell + b_j)^{2s_j+1}}{\prod_{j=1, j\neq \ell}^{2\tilde{m}+4} \sigma(b_\ell - b_j)^{2s_j}},$$

which is zero if and only if b_ℓ is a half-period or if there is a $j \in M_2$ such that $b_j = -b_\ell$ $(\mathrm{mod}\Delta)$.

In summary we have found the following: if ψ_a and ψ_{-a} are linearly dependent solutions of $\psi'' + q\psi = E\psi$, then some of the numbers $a_1, \ldots, a_s$ may be half-periods while all others appear in pairs (a_j, a_{ℓ_j}) with $a_{\ell_j} = -a_j$. Moreover, if a_j is equal to a half-period ω_k which is a pole of q of the form $-r_k(r_k+1)/(z-\omega_k)^2$ then there are exactly $2r_k + 1$ of the a_ℓ which are equal to this half-period. If a_j is equal to a pole b_ℓ of the form $-s_\ell(s_\ell+1)/(z-b_\ell)^2$ where b_ℓ is not a half-period, then there are exactly $2s_\ell + 1$ of the a_m which are equal to this pole and exactly $2s_\ell + 1$ other a_m's which are equal to the pole $-b_\ell$.

This information is now being used to rewrite the solution ψ_a of $\psi'' + q\psi = E\psi$ for those values of the spectral parameter E where $W(\psi_a, \psi_{-a}) = 0$ as a product of two functions. The first one is a fixed function depending only on the poles of the potential q, on the half-periods, and the exponents associated with these. The second one is a polynomial in $\mathcal{P}(z)$ whose coefficients depend on those of the a_j which are neither half-periods nor poles of q and which are yet undetermined. According to the above argument there must be an even number, $2d$, of those and half of them are just the negatives of the other half.

We therefore define for $\ell = 1, \ldots, 2\tilde{m}$

$$t_\ell = \begin{cases} s_\ell + 1 & \text{if } 2s_\ell + 1 \text{ of the } a_j \text{ are equal to } b_\ell, \\ -s_\ell & \text{if none of the } a_j \text{ are equal to } b_\ell \end{cases} \tag{19}$$

and for $k = 1, \ldots, 4$

$$h_k = \begin{cases} r_k + 1 & \text{if } 2r_k + 1 \text{ of the } a_j \text{ are equal to } \omega_k, \\ -r_k & \text{if none of the } a_j \text{ are equal to } \omega_k. \end{cases} \tag{20}$$

Then $\psi_a(z) = f(z)Q(\mathcal{P}(z))$, where

$$f(z) = \mathrm{e}^{\lambda_a z}\Big(\prod_{k=1}^{4} \sigma(z-\omega_k)^{h_k}\Big)\Big(\prod_{\ell=1}^{2\tilde{m}} \sigma(z-b_\ell)^{t_\ell}\Big)\sigma(z)^{2d},$$

$$Q(\mathcal{P}(z)) = \prod_{j=1}^{d}(\mathcal{P}(z) - \mathcal{P}(a_j)) = \sum_{j=0}^{d} c_j \mathcal{P}(z)^j.$$

Here we used the fact that $\sigma(z-a_j)\sigma(z+a_j) = -\sigma(z)^2\sigma(a_j)^2(\mathcal{P}(z)-\mathcal{P}(a_j))$. Also we dropped the non-zero constant factor $(-1)^d\prod_{j=1}^d \sigma(a_j)^2$.

Since

$$\psi_a'' + q\psi_a = f\big\{\big(\frac{f''}{f}+q\big)Q + \big(2\frac{f'}{f}\mathcal{P}' + \mathcal{P}''\big)Q' + (\mathcal{P}')^2 Q''\big\}, \tag{21}$$

where the primes on Q denote differentiation with respect to $\mathcal{P}$, we compute $2\frac{f'}{f}\mathcal{P}'$ and $\frac{f''}{f}+q$ and obtain

$$\begin{aligned}\frac{f''(z)}{f(z)} + q(z) = A_1\mathcal{P}(z) + A_2 + \sum_{\ell=1}^{\tilde m}\frac{A_{3,\ell}}{\mathcal{P}(z)-\mathcal{P}(b_\ell)} \\ + \sum_{\ell,j=1,\ell\neq j}^{\tilde m}\frac{A_{4,\ell,j}}{(\mathcal{P}(z)-\mathcal{P}(b_\ell))(\mathcal{P}(z)-\mathcal{P}(b_j))},\end{aligned}$$

$$2\frac{f'(z)}{f(z)}\mathcal{P}'(z) + \mathcal{P}''(z) = B_1\mathcal{P}(z)^2 + B_2\mathcal{P}(z) + B_3 + \sum_{\ell=1}^{\tilde m}\frac{B_{4,\ell}}{\mathcal{P}(z)-\mathcal{P}(b_\ell)},$$

where

$$\begin{aligned}A_1 &= 4d^2 + 4dh_4 - 2d,\\ A_2 &= C - \sum_{\ell=1}^{\tilde m} 2t_\ell\mathcal{P}(b_\ell)(4d+2h_4+t_\ell) - \sum_{k=1}^{3} h_k e_k(4d+2h_4+h_k),\\ A_{3,\ell} &= \sum_{j=1}^{\tilde m} t_\ell t_j(4\mathcal{P}(b_\ell)^2 + 4\mathcal{P}(b_\ell)\mathcal{P}(b_j) + 4\mathcal{P}(b_j)^2 - g_2)\\ &\quad + \sum_{j=1}^{3} t_\ell h_j(4\mathcal{P}(b_\ell)^2 + 4\mathcal{P}(b_\ell)e_j + 4e_j^2 - g_2) - t_\ell^2\mathcal{P}''(b_\ell),\\ A_{4,\ell,j} &= \frac{1}{2}t_\ell t_j(\mathcal{P}'(b_\ell)^2 + \mathcal{P}'(b_j)^2),\end{aligned}$$

and

$$\begin{aligned}&B_1 = 6 - 8d - 4h_4, \quad B_2 = \sum_{\ell=1}^{\tilde m} 8t_\ell\mathcal{P}(b_\ell) + \sum_{k=1}^{3} 4h_k e_k,\\ &B_3 = \sum_{\ell=1}^{\tilde m} 8t_\ell\mathcal{P}(b_\ell)^2 + \sum_{k=1}^{3} 4h_k e_k^2 + g_2(2d+h_4-\frac{1}{2}), \quad B_{4,\ell} = 2t_\ell\mathcal{P}'(b_\ell)^2.\end{aligned}$$

Inserting this into (21) we get

$$\begin{aligned}
\psi_a'' + q\psi_a &= f\Big\{(A_1\mathcal{P}(z)+A_2)Q(\mathcal{P}(z)) + (B_1\mathcal{P}(z)^2+B_2\mathcal{P}(z)+B_3)Q'(\mathcal{P}(z)) \\
&+ (4\mathcal{P}(z)^3 - g_2\mathcal{P}(z) - g_3)Q''(\mathcal{P}(z)) + \sum_{\ell=1}^{\tilde m}\frac{A_{3,\ell}Q(\mathcal{P}(z)) + B_{4,\ell}Q'(\mathcal{P}(z))}{\mathcal{P}(z)-\mathcal{P}(b_\ell)} \\
&+ \sum_{\ell,j=1,\ell\neq j}^{\tilde m}\frac{A_{4,\ell,j}Q(\mathcal{P}(z))}{(\mathcal{P}(z)-\mathcal{P}(b_\ell))(\mathcal{P}(z)-\mathcal{P}(b_j))}\Big\} \qquad (22) \\
&= f\Big\{R_1(\mathcal{P}(z);c_0,\dots,c_d) + \frac{R_2(\mathcal{P}(z);c_0,\dots,c_d)}{\prod_{j=1}^{\tilde m}(\mathcal{P}(z)-\mathcal{P}(b_j))}\Big\}
\end{aligned}$$

for suitable polynomials R_1 and R_2. In fact, R_1 is a polynomial of degree $d+1$ and R_2 is a polynomial of degree $\tilde m - 1$ when considered as a function of $\mathcal{P}(z)$. As functions of $(c_0,\dots,c_d)$, however, both R_1 and R_2 are homogeneous polynomials of degree 1, i.e.,

$$R_1 = \sum_{k=0}^{d+1}\sum_{j=0}^{d} S_{d-k+1,d-j+1}c_j\mathcal{P}(z)^k, \quad R_2 = \sum_{k=0}^{\tilde m-1}\sum_{j=0}^{d} T_{\tilde m-k,d-j+1}c_j\mathcal{P}(z)^k$$

for suitable numbers $S_{k,j}$ and $T_{k,j}$. Next we note that

$$\sum_{j=0}^{d} S_{0,j}c_j = c_d(A_1 + dB_1 + 4d(d-1)) = 0.$$

Hence we obtain a solution of the equation

$$\psi_a'' + q\psi_a = E\psi_a$$

which satisfies $W(\psi_a,\psi_{-a}) = 0$ if and only if

$$\sum_{j=0}^{d} S_{k,j}c_j = Ec_k \quad \text{for all} \quad k = 1,\dots,d+1$$

and

$$\sum_{j=0}^{d} T_{k,j}c_j = 0 \quad \text{for all} \quad k = 1,\dots,\tilde m. \qquad (23)$$

This is a linear homogeneous system of $d + \tilde m + 1$ equations for the $d + 1$ variables $c_0,\dots,c_d$. It has nontrivial solutions if and only if the rank of the associated matrix is less or equal to d. A necessary, but in general not sufficient, condition for this is that E is an eigenvalue of S.

For any given even Picard potential there are several (but finitely many) choices to distribute some (or all) of the parameters $a_1,\dots,a_s$ among the half-periods and/or poles of q. Accordingly there are several (but finitely many) of the above described constrained eigenvalue problems to solve in order to find all the values of the spectral parameter E where $W(\psi_a,\psi_{-a}) = 0$. In each case there are only finitely many eigenvalues of the associated matrix S, some (or perhaps all) of which may be in contradiction to the constraints (23). Therefore we conclude that the following theorem holds.

Theorem 12. *Let q be an even Picard potential. Then there exists a solution ψ_a of the form* (13)*; i.e., a solution which is elliptic of the second kind, of the differential equation $\psi'' + q\psi = E\psi$ for every complex number E. The function ψ_{-a} is likewise a solution of the same equation for the same value of E and also elliptic of the second kind. For all but a finite number of values of E, these two solutions are linearly independent.*

In view of Definition 5 this immediately implies our final and most important result:

Theorem 13. *Every even Picard potential is finite-gap. The band edges are determined from a certain number of matrices S and T. More specifically, the band edges are those eigenvalues of the matrices S whose eigenvectors are in the kernel of T. The matrices S and T are determined after making one of two possible choices for each of the t_ℓ, $\ell = 1, \dots, \tilde{m}$ in* (19) *and for each of the h_k, $k = 1, \dots, 4$ in* (20)*. S and T are explicitly listed in Appendix A.*

5. A two-gap example. In [12] and [13] we discussed Lamé and Treibich-Verdier potentials, respectively. In these cases there are no matrices T and therefore the number of band edges and hence the number of bands (or gaps) is precisely determined from the number of eigenvalues of the matrices S.

We now turn to an example where the matrices T are present. Let $q(z) = -2\mathcal{P}(z) - 2\mathcal{P}(z-b) - 2\mathcal{P}(z+b)$, where b is not a half-period. Hence, we have in the notation of Sections 3 and 4, $m = 3$, $\tilde{m} = 1$, and $s = 3$.

Performing the analysis used in the proof of Theorem 7, it turns out that q is a Picard potential if and only if $\mathcal{P}'(2b) + \mathcal{P}'(b) = 0$. This condition is satisfied if $\mathcal{P}''(b) = 6\mathcal{P}(b)^2 - g_2/2 = 0$. Therefore, we assume in the following that

$$\mathcal{P}(b)^2 = \frac{1}{12} g_2. \tag{24}$$

A solution of $\psi'' + q\psi = E\psi$ which is elliptic of the second kind is of the form $\psi_a(z) = \mathrm{e}^{\lambda_a z} \frac{\sigma(z-a_1)\sigma(z-a_2)\sigma(z-a_3)}{\sigma(z)\sigma(z-b)\sigma(z+b)}$, where λ_a is given by (14) and a_1, a_2 and a_3 depend on E. In order to have a band edge the a_j must be as in one of the following three cases.

(1) $a_1 = a_2 = a_3 = 0$. In this case we have $d = 0$, $t_1 = -1$, $h_1 = h_2 = h_3 = 0$, and $h_4 = 2$. The matrices S and T are one-by-one matrices, namely $S = A_2 = 6\mathcal{P}(b)$ and $T = A_{3,1} = 6\mathcal{P}(b)^2 - g_2/2$. Hence, $E = 6\mathcal{P}(b)$ is the eigenvalue of S and this is a band edge since $T = 0$ by (24).

(2) $a_j = \omega_j$, for $j = 1, 2, 3$. Again $d = 0$ and $t_1 = -1$. Also $h_1 = h_2 = h_3 = 1$ and $h_4 = -1$. Now $S = A_2 = -6\mathcal{P}(b)$ and $T = -6\mathcal{P}(b)^2 + g_2/2 = 0$; i.e., $E = -6\mathcal{P}(b)$ is a band edge.

(3) $a_1 = \omega_i$ for some $i \in \{1, 2, 3\}$ and $a_2 = -a_3 = a$. Now $d = 1$, $t_1 = -1$, and $h_4 = -1$. Also $h_k = \delta_{i,k}$ for $k = 1, 2, 3$. Then

$$S = \begin{pmatrix} A_2 + B_2 & A_1 \\ A_{3,1} + B_3 & A_2 \end{pmatrix} = \begin{pmatrix} -6\mathcal{P}(b) + e_i & -2 \\ -6\mathcal{P}(b)^2 - 4\mathcal{P}(b)e_i + g_2 & 2\mathcal{P}(b) - 3e_i \end{pmatrix}$$

and

$$
\begin{aligned}
T &= (A_{3,1}\mathcal{P}(b) + B_{4,1}, A_{3,1}) \\
&= \big(-6\mathcal{P}(b)^3 - 4\mathcal{P}(b)^2 e_i + (\tfrac{5}{2}g_2 - 4e_i^2)\mathcal{P}(b) + 2g_3, \\
&\qquad 2\mathcal{P}(b)^2 - 4\mathcal{P}(b)e_i - 4e_i^2 + \tfrac{1}{2}g_2\big).
\end{aligned}
$$

Using equation (24) S and T may be simplified and one obtains

$$
\begin{aligned}
S &= \begin{pmatrix} -6\mathcal{P}(b) + e_i & -2 \\ -4\mathcal{P}(b)e_i + g_2/2 & 2\mathcal{P}(b) - 3e_i \end{pmatrix}, \\
T &= (\mathcal{P}(b)(2g_2 - 4e_i^2) + 2g_3 - \tfrac{1}{3}g_2 e_i, -4\mathcal{P}(b)e_i + \tfrac{2}{3}g_2 - 4e_i^2).
\end{aligned}
$$

S has two eigenvalues, $E = -3e_i$ and $E = e_i - 4\mathcal{P}(b)$. The associated eigenvectors are $(1, 2e_i - 3\mathcal{P}(b))$ and $(1, -\mathcal{P}(b))$. However, an eigenvalue E can only be a band edge if T applied to the eigenvector gives zero. Applying T to $(1, 2e_i - 3\mathcal{P}(b))$ indeed gives zero. Hence $E = -3e_i$ is an admissible eigenvalue for $i = 1, 2, 3$. Applying T to $(1, -\mathcal{P}(b))$ gives $\frac{4}{3}g_2\mathcal{P}(b) + 2g_3$. If this is zero then also $g_2^3 - 27g_3^2 = 0$ which never happens when all half-periods of $\mathcal{P}$ are finite. Hence, $E = e_i - 4\mathcal{P}(b)$ is not an admissible eigenvalue.

Collecting all the results in the previous three cases we find that if $\mathcal{P}(b)^2 = g_2/12$, then q has 5 band edges at $-\sqrt{3g_2}, -3e_1, -3e_2, -3e_3$, and $\sqrt{3g_2}$ and hence is a two-gap potential.

Appendix A, the matrices S and T. In this appendix we compute the matrices S and T introduced in Section 4. First we note that

$$
Q(\mathcal{P}(z)) = (\mathcal{P}(z) - \mathcal{P}(b)) \sum_{\nu=0}^{d-1} \sum_{\mu=\nu}^{d-1} c_{\mu+1}\mathcal{P}(b)^{\mu-\nu}\mathcal{P}(z)^\nu + \sum_{\mu=0}^{d} c_\mu \mathcal{P}(b)^\mu,
$$

$$
Q'(\mathcal{P}(z)) = (\mathcal{P}(z) - \mathcal{P}(b)) \sum_{\nu=0}^{d-2} \sum_{\mu=\nu}^{d-2} (\mu+2)c_{\mu+2}\mathcal{P}(b)^{\mu-\nu}\mathcal{P}(z)^\nu + \sum_{\mu=0}^{d-1} (\mu+1)c_{\mu+1}\mathcal{P}(b)^\mu,
$$

$$
\begin{aligned}
Q(\mathcal{P}(z)) &= (\mathcal{P}(z) - \mathcal{P}(b))(\mathcal{P}(z) - \mathcal{P}(c)) \sum_{\nu=0}^{d-2} \sum_{\mu=\nu}^{d-2} \sum_{\rho=\mu}^{d-2} c_{\rho+2}\mathcal{P}(b)^{\rho-\mu}\mathcal{P}(c)^{\mu-\nu}\mathcal{P}(z)^\nu \\
&\quad + (\mathcal{P}(z) - \mathcal{P}(b)) \sum_{\nu=0}^{d-1} \sum_{\mu=\nu}^{d-1} c_{\mu+1}\mathcal{P}(b)^{\mu-\nu}\mathcal{P}(c)^\nu + \sum_{\mu=0}^{d} c_\mu \mathcal{P}(b)^\mu.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{A_{3,\ell}Q(\mathcal{P}(z)) + B_{4,\ell}Q'(\mathcal{P}(z))}{\mathcal{P}(z) - \mathcal{P}(b_\ell)} &= \sum_{\nu=0}^{d-1} \sum_{\mu=\nu+1}^{d} A_{3,\ell}\mathcal{P}(b_\ell)^{\mu-\nu-1}c_\mu \mathcal{P}(z)^\nu \\
&\quad + \sum_{\nu=0}^{d-2} \sum_{\mu=\nu+2}^{d} B_{4,\ell}\mathcal{P}(b_\ell)^{\mu-\nu-2}\mu c_\mu \mathcal{P}(z)^\nu + \sum_{\mu=0}^{d} \frac{(A_{3,\ell}\mathcal{P}(b_\ell) + \mu B_{4,\ell})\mathcal{P}(b_\ell)^{\mu-1}c_\mu}{\mathcal{P}(z) - \mathcal{P}(b_\ell)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{A_{4,\ell,j}Q(\mathcal{P}(z))}{(\mathcal{P}(z) - \mathcal{P}(b_\ell))(\mathcal{P}(z) - \mathcal{P}(b_j))} &= \sum_{\nu=0}^{d-2} \sum_{\mu=\nu+2}^{d} \sum_{\rho=\nu}^{\mu-2} A_{4,\ell,j}\mathcal{P}(b_\ell)^{\mu-\rho-2}\mathcal{P}(b_j)^{\rho-\nu}c_\mu \mathcal{P}(z)^\nu \\
&\quad + \sum_{\nu=0}^{d-2} \sum_{\mu=\nu+1}^{d} \frac{A_{4,\ell,j}\mathcal{P}(b_j)^{\mu-\nu-1}\mathcal{P}(b_\ell)^\nu c_\mu}{\mathcal{P}(z) - \mathcal{P}(b_\ell)} + \sum_{\mu=0}^{d} \frac{A_{4,\ell,j}\mathcal{P}(b_j)^\mu c_\mu}{(\mathcal{P}(z) - \mathcal{P}(b_\ell))(\mathcal{P}(z) - \mathcal{P}(b_j))}.
\end{aligned}
$$

Inserting this into (22) gives for the entries of the matrix S $S = S_1 + S_2$, where

$$S_{1,\nu,\nu+1} = A_1 + (d-\nu)B_1 + 4(d-\nu)(d-\nu-1), \quad S_{1,\nu,\nu} = A_2 + (d-\nu+1)B_2,$$
$$S_{1,\nu,\nu-1} = (d-\nu+2)B_3 - (d-\nu+2)(d-\nu+1)g_2, \quad S_{1,\nu,\nu-2} = -(d-\nu+3)(d-\nu+2)g_3,$$

and

$$S_{2,\nu,\nu-1} = \sum_{\ell=1}^{\tilde m} A_{3,\ell}, \quad S_{2,\nu,\nu-2} = \sum_{\ell=1}^{\tilde m}(A_{3,\ell}\mathcal{P}(b_\ell) + (d-\nu+3)B_{4,\ell}) + \sum_{\ell,j=1,\ell\neq j}^{\tilde m} A_{4,\ell,j},$$
$$S_{2,\nu,\mu} = \sum_{\ell=1}^{\tilde m}(A_{3,\ell}\mathcal{P}(b_\ell) + (d-\mu+1)B_{4,\ell})\mathcal{P}(b_\ell)^{\nu-\mu-2}$$
$$+ \sum_{\ell,j=1,\ell\neq j}^{\tilde m}\sum_{\rho=\mu}^{\nu-2} A_{4,\ell,j}\mathcal{P}(b_\ell)^{\rho-\mu}\mathcal{P}(b_j)^{\nu-\rho-2} \quad \text{for } \mu < \nu-2$$

and where all other entries of S_1 and S_2 are zero. In particular, if $\tilde m = 0$, then $S = S_1$.

In order to obtain T, we first define the functions $\sigma_m(\ell)$ and $\tau_m(\ell, j)$ by

$$\prod_{k=1,k\neq\ell}^{\tilde m}(\mathcal{P}(z)-\mathcal{P}(b_k)) = \sum_{k=0}^{\tilde m-1}\sigma_{\tilde m-k-1}(\ell)\mathcal{P}(z)^k, \quad \prod_{k=1,k\neq\ell,j}^{\tilde m}(\mathcal{P}(z)-\mathcal{P}(b_k)) = \sum_{k=0}^{\tilde m-2}\tau_{\tilde m-k-2}(\ell,j)\mathcal{P}(z)^k;$$

i.e., $(-1)^m\sigma_m$ and $(-)^m\tau_m$ represent certain elementary symmetric polynomials. Then one obtains for the entries of the matrix T

$$T_{\nu,\mu} = \sum_{\ell=1}^{\tilde m}(A_{3,\ell}\mathcal{P}(b_\ell) + (d-\mu+1)B_{4,\ell})\mathcal{P}(b_\ell)^{d-\mu}\sigma_{\nu-1}(\ell)$$
$$+ \sum_{\ell,j=1,\ell\neq j}^{\tilde m} A_{4,\ell,j}\Big\{\mathcal{P}(b_j)^{d-\mu+1}\tau_{\nu-2}(\ell,j) + \sum_{\rho=0}^{d-\mu}\mathcal{P}(b_j)^{d-\rho-\mu}\mathcal{P}(b_\ell)^\rho\sigma_{\nu-1}(\ell)\Big\},$$

where we assume that $\tau_{-1} = 0$ and, as usual, that a sum is zero if the upper limit is smaller than the lower limit.

Appendix B, basic results on elliptic functions. In this section we collect some of the most basic results on elliptic functions and on functions which are elliptic of the second kind. For general references see, e.g., Akhiezer [2], Chandrasekharan [8], Markushevich [19], and Whittaker and Watson [29].

A function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ with two periods a and b, the ratio of which is not real, is called doubly periodic. If all its periods are of the form $m_1a + m_2b$ where m_1 and m_2 are integers then a and b are called fundamental periods of f.

A doubly periodic meromorphic function is called elliptic.

It is customary to denote the fundamental periods of an elliptic function by $2\omega_1 = 2\omega$ and $2\omega_3 = 2\omega'$ with $\Im(\omega'/\omega) > 0$. We also introduce $\omega_2 = \omega_1 + \omega_3$ and $\omega_4 = 0$. The numbers ω, ω' and $\omega_1, \dots, \omega_4$ are called half-periods. The fundamental period parallelogram (f.p.p.) Δ denotes the half-open domain consisting of the line segments $[0, 2\omega_1)$, $[0, 2\omega_3)$ and the interior of the parallelogram with vertices 0, $2\omega_1$, $2\omega_2$ and $2\omega_3$.

The class of elliptic functions with fundamental periods $2\omega_1, 2\omega_3$ is closed under addition, subtraction, multiplication, division by non-zero divisors and differentiation. If f is an entire elliptic function, then $f(z) = \text{const}$. An elliptic function $f \neq \text{const}$ must have at least one pole in Δ and the total number of poles in Δ is finite. The total number of poles (counting multiplicity) of an elliptic function f in Δ is called the order of f. The sum of residues of an elliptic function f at all its poles in Δ equals zero. In particular, the order of a non-constant elliptic function f is at least 2. The total number of points in Δ where the non-constant elliptic function f assumes the value A (counting multiplicity), denoted by $n(A)$, is equal to the order of f. In particular, $n(A) \geq 2$. Furthermore, $s(A)$, the sum of all the points in Δ where the non-constant elliptic function f assumes the value A, is congruent to $s(\infty)$, the sum of all the points in Δ where f has a pole; i.e., $s(A) = s(\infty) + 2m_1\omega_1 + 2m_3\omega_3$, where m_1 and m_3 are certain integers.

The function

$$\mathcal{P}(z|\omega_1, \omega_3) = \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \Big(\frac{1}{(z - 2m\omega_1 - 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2}\Big),$$

or $\mathcal{P}(z)$ for short, was introduced by Weierstrass. It is an even elliptic function of order 2 with fundamental periods $2\omega_1$ and $2\omega_3$. Every elliptic function may be written as $R_1(\mathcal{P}(z)) + R_2(\mathcal{P}(z))\mathcal{P}'(z)$, where R_1 and R_2 are rational functions of $\mathcal{P}$ and where the derivative $\mathcal{P}'$ of $\mathcal{P}$ is an odd elliptic function of order 3 with fundamental periods $2\omega_1$ and $2\omega_3$.

The Laurent expansions of $\mathcal{P}(z)$ and $\mathcal{P}'(z)$ at $z = 0$ are given by

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} c_k z^{2k-2}, \quad \mathcal{P}'(z) = \frac{-2}{z^3} + \sum_{k=2}^{\infty} (2k-2)c_k z^{2k-3},$$

where

$$\begin{aligned} c_2 &= 3 \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, \quad c_3 = 5 \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(2m\omega_1 + 2n\omega_3)^6}, \\ c_k &= \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m}, \quad k \geq 4. \end{aligned} \tag{25}$$

The numbers $g_2 = 20c_2$ and $g_3 = 28c_3$ are called invariants of $\mathcal{P}(z)$. Since $\mathcal{P}(z|\omega_1, \omega_3)$ is also uniquely characterized by its invariants g_2 and g_3 one frequently also uses the notation $\mathcal{P}(z|g_2, g_3)$.

The function $\mathcal{P}(z)$ satisfies the first order differential equation

$$(\mathcal{P}'(z))^2 = 4\mathcal{P}(z)^3 - g_2\mathcal{P}(z) - g_3 \tag{26}$$

and the second order differential equation

$$\mathcal{P}''(z) = 6\mathcal{P}(z)^2 - g_2/2.$$

The function $\mathcal{P}'$ being of order 3 has three zeros in Δ. Since $\mathcal{P}'$ is odd and elliptic it is obvious that these zeros are the half-periods $\omega_1, \omega_2 = \omega_1 + \omega_3$ and ω_3. Denote $\mathcal{P}(\omega_i) = e_i$, $i = 1, 2, 3$. Then (26) implies that $4e_i^3 - g_2 e_i - g_3 = 0$ for $i = 1, 2, 3$. Therefore $e_1 + e_2 + e_3 = 0$,

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = 2(e_1^2 + e_2^2 + e_3^2), \quad g_3 = 4e_1e_2e_3 = \frac{4}{3}(e_2^3 + e_2^3 + e_3^3).$$

Weierstrass also introduced two other functions denoted by ζ and σ. The Weierstrass ζ-function is defined by

$$\frac{d}{dz}\zeta(z) = -\mathcal{P}(z), \quad \lim_{z\to 0}(\zeta(z) - \frac{1}{z}) = 0.$$

It is a meromorphic function with simple poles at $2m\omega_1 + 2n\omega_3, m, n \in \mathbb{Z}$ having residues 1. It is not periodic but quasi-periodic in the sense that $\zeta(z + 2\omega_j) = \zeta(z) + 2\eta_j$, $j = 1, 2, 3, 4$, where $\eta_j = \zeta(\omega_j)$ for $j = 1, 2, 3$ and $\eta_4 = 0$. The Laurent expansion of ζ at $z = 0$ is given by

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} \frac{c_k}{2k-1} z^{2k-1}$$

with the c_k given in (25).

The Weierstrass σ-function is defined by

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z), \quad \lim_{z\to 0} \frac{\sigma(z)}{z} = 1.$$

$\sigma(z)$ is an entire function with simple zeros at the points $2m\omega_1 + 2n\omega_3, m, n \in \mathbb{Z}$. $\sigma(z)$ is also quasi-periodic, since

$$\sigma(z + 2\omega_j) = -\sigma(z)e^{2\eta_j(z+\omega_j)}, \quad j = 1, 2, 3. \tag{27}$$

Next we recall the following fundamental theorems.

Theorem 14. *Given an elliptic function f with fundamental periods $2\omega_1$ and $2\omega_3$, let $b_1, \dots, b_r$ be the poles of f in Δ. Suppose the principal part of the Laurent expansion near b_k is given by*

$$\sum_{i=1}^{\beta_k} \frac{A_{i,k}}{(z - b_k)^i}, \quad k = 1, \dots, r.$$

Then

$$f(z) = C + \sum_{k=1}^{r} \sum_{i=1}^{\beta_k} (-1)^{i-1} \frac{A_{i,k}}{(i-1)!} \zeta^{(i-1)}(z - b_k),$$

where C is a suitable constant and $\zeta(z)$ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$. Conversely, every such function is an elliptic function if $\sum_{k=1}^{r} A_{1,k} = 0$.

Theorem 15. *Given an elliptic function f of order n with fundamental periods $2\omega_1$ and $2\omega_3$, let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be the zeros and poles of f in Δ each counted a number of times equal to its order. Then*

$$f(z) = C \frac{\sigma(z - a_1) \cdots \sigma(z - a_n)}{\sigma(z - b_1) \cdots \sigma(z - b_{n-1})\sigma(z - b'_n)},$$

where C is a suitable constant, $\sigma(z)$ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$ and where $b'_n - b_n = (a_1 + \cdots + a_n) - (b_1 + \cdots + b_n)$ is a period of f. Conversely, every such function is an elliptic function.

Finally, we turn to elliptic functions of the second kind, the central object in our analysis. A meromorphic function $\psi : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ for which there exist two complex constants ω_1 and ω_3 with non-real ratio and two complex constants ρ_1 and ρ_3 such that for $i = 1, 3$ $\psi(z + 2\omega_i) = \rho_i\psi(z)$ is called elliptic of the second kind. We call $2\omega_1$ and $2\omega_3$ the quasi-periods of ψ. Together with $2\omega_1$ and $2\omega_3$, $2m_1\omega_1 + 2m_3\omega_3$ are also quasi-periods of ψ if m_1 and m_3 are integers. If every quasi-period of ψ can be written as an integer linear combination of $2\omega_1$ and $2\omega_3$, then these are called fundamental quasi-periods.

Theorem 16. *A function ψ which is elliptic of the second kind and has fundamental quasi-periods $2\omega_1$ and $2\omega_3$ can always be put in the form*

$$\psi(z) = C \exp(\lambda z) \frac{\sigma(z - a_1) \cdots \sigma(z - a_n)}{\sigma(z - b_1) \cdots \sigma(z - b_n)}$$

for suitable constants C, λ, $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Here $\sigma(z)$ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$. Conversely, every such function is elliptic of the second kind.

Proof. Let ψ be an elliptic function of the second kind. Then the function ψ'/ψ is meromorphic and doubly periodic and hence elliptic. Moreover, ψ'/ψ has only simple poles. Let the (distinct) zeros of ψ be denoted by $a_1, \dots, a_r$ and let m_j be the multiplicity of a_j for $j = 1, \dots, r$. Similarly, let $b_1, \dots, b_s$ denote the (distinct) poles of ψ and $k_1, \dots, k_s$ the associated multiplicities. Then the principal part of ψ'/ψ near a_j is given by $\frac{m_j}{z-a_j}$ while the principal part of ψ'/ψ near b_j is given by $\frac{-k_j}{z-b_j}$. Note that the zeros and poles of ψ are the only poles of ψ'/ψ and that

$$\sum_{j=1}^{r} m_j - \sum_{j=1}^{s} k_j = 0 \tag{28}$$

since this is the sum of the residues of all poles of the elliptic function ψ'/ψ. By Theorem 14,

$$\frac{\psi'(z)}{\psi(z)} = \lambda + \sum_{j=1}^{r} m_j \zeta(z-a_j) - \sum_{j=1}^{s} k_j \zeta(z-b_j) = \lambda + \sum_{j=1}^{r} m_j \frac{\sigma'(z-a_j)}{\sigma(z-a_j)} - \sum_{j=1}^{s} k_j \frac{\sigma'(z-b_j)}{\sigma(z-b_j)},$$

where λ is a suitable constant. Integration yields

$$\psi(z) = C \exp(\lambda z) \frac{\sigma(z-a_1)^{m_1} \cdots \sigma(z-a_r)^{m_r}}{\sigma(z-b_1)^{k_1} \cdots \sigma(z-b_s)^{k_s}},$$

where C is an appropriate integration constant. By (28) the number of σ-factors in the numerator and the number of σ-factors in the denominator is equal. Listing each zero or pole the number of times equal to its multiplicity proves the first part of the theorem. The converse statement simply follows from (27). □

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