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Picard Operators

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Abstract. A linear differential expression $Ly = y^{(n)} + q_{n-2}y^{(n-2)} + \cdots + q_0 y$ is called a Picard expression if its coefficients are elliptic functions (with common fundamental periods) and if the general solution of $Ly = Ey$ is an everywhere meromorphic function (with respect to the independent variable) for all $E \in \mathbb{C}$.

If L is a Picard expression we show that the differential equation $Ly = Ey$ has n linearly independent Floquet solutions except when E is any of a finite number of exceptional values. Also the conditional stability set of a Picard expression (and hence the spectrum of the associated operator in $L^2(\mathbb{R})$) consists of finitely many regular analytic arcs.

1. Introduction

Let L be the differential expression given by $Ly = y'' + qy$ where q is a continuous function of a real variable which is periodic with period a. Consider the following statements:

(A) There are precisely $2g+1$ real numbers $E_{2g} < \cdots < E_0$ such that the differential equation $y'' + qy = Ey$ fails to have two linearly independent Floquet solutions, i.e., solutions y satisfying $y(x+a) = \rho y(x)$, if and only if $E \in \{E_0, \ldots, E_{2g}\}$.

(B) There are precisely $2g+1$ real numbers $E_{2g} < \cdots < E_0$ such that the conditional stability set $\mathcal{S}$ of L, i.e., the set of all values of E for which at least one of the solutions of $y'' + qy = Ey$ is bounded, is given by

$$\mathcal{S} = (-\infty, E_{2g}] \cup [E_{2g-1}, E_{2g-2}] \cup \ldots \cup [E_1, E_0].$$

(C) There exists a monic ordinary differential operator P of order $2g+1$ such that $[P, L] = 0$.

(D) There exists a monic ordinary differential operator P of order $2g+1$ and $2g+1$ complex numbers $\lambda_0, \ldots, \lambda_{2g}$ such that $P^2 = (L-\lambda_0)\ldots(L-\lambda_{2g})$.

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First suppose that q is real-valued. Then endpoints of stability intervals coincide with points where only one periodic solution with period $2a$ exists. This was first shown by HAMEL [7] in 1913[1]. This implies the equivalence of (A) and (B), since a Floquet multiplier must be equal to ± 1 to be degenerate. Moreover it was proven in 1923 by BURCHNALL and CHAUNDY [1] that (C) and (D) are equivalent even if q is complex-valued. That there is also a relationship between (B) and (C) is a discovery which was made only about twenty years ago: NOVIKOV [14] proved in 1974 that a periodic stationary solution of an equation of the KdV hierarchy has only finitely many disjoint conditional stabilitity intervals (bands). Taking into account that every stationary equation of the KdV hierarchy admits a Lax representation $[P, L] = 0$ (see LAX [10]) yields then the conclusion that (C) implies (B). The converse, i.e., (B) implies (C) was first proven by DUBROVIN [3] in 1975. It also turns out that $\{\lambda_0, \ldots, \lambda_{2g}\} = \{E_0, \ldots, E_{2g}\}$.

When q is complex-valued the matter becomes more difficult. While the equivalence of (C) and (D) still holds the equivalence of suitable generalizations[2] of (A) and (B) is not longer given. In particular, it may happen that $y'' + qy = Ey$ does not have two linearly independent Floquet solutions and yet E is not the endpoint of a band. Conversely, it is conceivable that E is a band edge and yet $y'' + qy = Ey$ has two linearly independent Floquet solutions.

Now assume that q is an elliptic function and treat the independent variable as a complex variable. Under this condition the statement

(E) For every complex number E every solution of the differential equation $y''+qy = Ey$ is a meromorphic function of the independent variable.

was recently shown to be equivalent with (C) by F. GESZTESY and myself [6].

Naturally the question arises what happens when L is a periodic differential expression of n-th order and this is the topic the present paper is concerned with.

The requirement that all solutions of a linear homogeneous differential equation with elliptic coefficients are meromorphic is quite far-reaching. We therefore introduce the concept of a Picard differential expression.

Definition 1.1. A linear differential expression $Ly = y^{(n)} + q_{n-2}y^{(n-2)} + \cdots + q_0 y$ will be called a *Picard expression* if all of its coefficients are elliptic functions with common fundamental periods and if, for every complex number E, the general solution of $Ly = Ey$ is an everywhere meromorphic function (with respect to the independent variable).

The basic results of this paper are Theorems 4.1 and 5.3. The former states that the differential equation $Ly = Ey$ has n linearly independent Floquet solutions except when E is any of a finite number of exceptional values if L is a Picard expression. The latter tells us then that the conditional stability set of a Picard expression (and hence

[1] However, see also LIAPUNOV [11] who treated the case $y'' = \lambda p y$ with a periodic function p in 1899 and HAUPT [8] who corrects a mistake in HAMEL's paper. For a general reference on Floquet theory see, e.g., EASTHAM [4].

[2] The conditional stability set is not anymore a subset of the real line. It is, however, a set of regular analytic arcs which are called bands. Also, the points E where not two linearly independent Floquet solutions of $y'' + qy = Ey$ exist need not be real anymore.

the spectrum of the associated operator) consists of finitely many regular analytic arcs.

2. Preliminaries

In this section a number of well-known concepts are repeated in order to fix notation.

2.1. Elliptic functions

An elliptic function is a meromorphic function which is doubly periodic, i. e., has two periods 2ω and $2\omega'$ which are linearly independent over the reals. Without any loss of generality it is assumed henceforth that $2\omega \in \mathbb{R}$. Two periods 2ω and $2\omega'$ of an elliptic function are called fundamental if every period of the function is an integer linear combination of 2ω and $2\omega'$. If 2ω and $2\omega'$ are fundamental periods then so are $2m\omega + 2m'\omega'$ and $2k\omega + 2k'\omega'$ provided m, m', k, and k' are integer and $mk' - m'k = \pm 1$. In particular, then, 2ω and $2k\omega + 2\omega'$ are fundamental. Hence the fundamental periods 2ω and $2\omega'$ may be chosen in such a way that the angle θ between them is less than π/n.

A meromorphic function y which satisfies $y(x+2\omega) = \rho y(x)$ and $y(x+2\omega') = \rho' y(x)$ for two complex numbers ρ and ρ' is called elliptic of the second kind.

2.2. Floquet theory and Picard's theorem

Let L be a differential expression of the form

$$Ly = y^{(n)} + q_{n-2}y^{(n-2)} + \cdots + q_0 y$$

where the coefficients $q_0, \ldots, q_{n-2}$ are continuous complex-valued functions of a real variable periodic with period a. Denote the n-dimensional vector space of solutions of the differential equation $Ly = Ey$ by $W(E)$. Let $S(E)$ be the operator of translation by a acting on $W(E)$. Since $S(E)$ and L commute it follows that $S(E)$ is a linear operator which maps $W(E)$ to itself. Its eigenvalues are called Floquet multipliers and its eigenfunctions Floquet solutions of $Ly = Ey$. The fact that one may choose a basis of $W(E)$ whose elements are entire functions of E $\big($namely solutions $\phi_j(E,x)$ satisfying the initial conditions $\phi_j^{k-1}(E,0) = \delta_{j,k}$, $j,k = 1, \ldots, n\big)$ shows that Floquet multipliers are given as zeros of the polynomial

$$\mathcal{F}(E,\rho) = (-1)^n\rho^n + (-1)^{n-1}a_1(E)\rho^{n-1} + \cdots - a_{n-1}(E)\rho + 1 = 0 \tag{2.1}$$

where the functions $a_1, \ldots, a_{n-1}$ are entire.

Note that $\mathcal{F}(E,\cdot)$ has n distinct zeros unless the discriminant $D(E)$ of $\mathcal{F}(E,\cdot)$ which is an entire function of E is equal to zero. This, therefore happens at only countably many points. Denote by $m_g(E,\rho)$ and $m_f(E,\rho)$ the geometric and algebraic multiplicity, respectively, of the eigenvalue ρ of $S(E)$. The number $m_f(E,\rho) - m_g(E,\rho) \in \{0, 1, \ldots, n-1\}$ counts the "missing" Floquet solutions of $Ly = Ey$ with multiplier ρ. If Floquet solutions are missing only for finitely many complex numbers E we will say that L has finite Floquet deficiency.

For $\rho \neq 0$, consider the operator $T(\rho)$ defined to be the restriction of L to the domain

$$D(T(\rho)) \;=\; \left\{y \in H^{2,n}([0,a]) : y^{(k)}(a) = \rho y^{(k)}(0),\, k = 0, \dots , n-1\right\}.$$

$T(\rho)$ has discrete spectrum. In fact, its eigenvalues, which will be called Floquet eigenvalues, are given as the zeros of $\mathcal{F}(\,\cdot\,, \rho)$. Moreover, the algebraic multiplicities $m_a(E, \rho)$ of E as an eigenvalue of $T(\rho)$ is given as the order of E as a zero of $\mathcal{F}(\,\cdot\,, \rho)$ (see, e. g., [5]).

We now turn to the special case that $q_0, \dots , q_{n-2}$ are elliptic functions with common fundamental periods. The following theorem is due to PICARD [15]. HERMITE [9], however, had proven this theorem earlier in the special case of Lamé's equation.

Theorem 2.1. *If the differential equation $Ly = y^{(n)} + q_{n-2}y^{(n-2)} + \cdots + q_0 y = 0$ has elliptic coefficients with a common period lattice and a general solution which is everywhere meromorphic, then it has at least one solution which is elliptic of the second kind.*

Let S be the operator of translation by a fundamental period a of the coefficients acting on the space of solutions of $Ly = 0$. If all eigenvalues of S are distinct, then there exist n linearly independent solutions of $Ly = 0$ which are elliptic of the second kind.

Proof. Let 2ω be a fundamental period of the coefficients of the differential equation, W the space of solutions of the equation, and S the operator of translation by 2ω acting on W. As before, S is a linear operator from W to W and has an eigenvalue ρ and an associated eigenfunction u_1, i. e., $Ly = 0$ has a solution u_1 satisfying $u_1(z + 2\omega) = \rho u_1(z)$.

Now let $2\omega'$ be the other fundamental period of the coefficients. Consider the functions

$$(2.2) \qquad u_1(z)\,, \quad u_2(z) \;=\; u_1(z + 2\omega')\,, \quad \dots\,, \quad u_m(z) \;=\; u_1(z + 2(m-1)\omega')\,,$$

where $1 \leq m \leq n$ is chosen such that the functions in (2.2) are linearly independent but including $u_1(z + 2m\omega')$ would render a linearly dependent set of functions. Then,

$$(2.3) \qquad u_m(z + 2\omega') \;=\; b_1 u_1(z) + \cdots + b_m u_m(z)\,.$$

Next denote the linear operator of translation by $2\omega'$ acting on the span V of $\{u_1, \dots , u_m\}$ by S'. It follows from 2.3 that the range of S' is again V. Let ρ' be an eigenvalue of S' and v the associated eigenvector, i. e., v is a meromorphic solution of the differential equation satisfying $v(z + 2\omega') = \rho' v(z)$. But v satisfies also $v(z + 2\omega) = \rho v(z)$ since every element of V has this property. Hence v is elliptic of the second kind.

The numbers ρ and ρ' are of course Floquet multipliers corresponding to the periods 2ω and $2\omega'$, respectively. The process described in the above proof can be performed for each multiplier corresponding with the period 2ω. Also, of course, the roles of 2ω and $2\omega'$ may be interchanged. The last statement of the theorem follows then from the observation that solutions associated with different multipliers are linearly independent. □

2.3. Spectral theory

With any differential expression L given by

$$Ly = y^{(n)} + q_{n-2}y^{(n-2)} + \cdots + q_0 y$$

with continuous complex-valued periodic coefficients $q_0, \ldots, q_{n-2}$ of a real variable we associate the operator $T : H^{2,n} \to L^2(\mathbb{R})$, $Ty = Ly$.

The spectrum $\sigma(T)$ of T is the complement of the set of all complex numbers E for which $(T - E)^{-1} : L^2(\mathbb{R}) \to H^{2,n}$ exists as a bounded operator. The conditional stability set $\mathcal{S}(L)$ of L is the set of all complex numbers E for which the differential equation $Ly = Ey$ has a bounded solution.

It was shown by ROFE-BEKETOV [16] that $\sigma(T) = \mathcal{S}(L)$. For E to be in $\mathcal{S}(L)$ it is necessary and sufficient that $Ly = Ey$ has a Floquet multiplier of modulus one. Hence

$$\mathcal{S}(L) = \{E \in \mathbb{C} : \mathcal{F}(E, e^{it}) = 0 \text{ for some } t \in \mathbb{R}\}$$

where $\mathcal{F}$ is given by 2.1. Since $\mathcal{F}$ is entire in both its variables it follows that $\sigma(T) = \mathcal{S}(L)$ consists of (generally) infinitely many regular analytic arcs[3)]. These arcs are called spectral bands. They end at a point where the arc fails to be regular analytic or extend to infinity. The endpoints of the spectral bands are called band edges.

Definition 2.2. T (or L since no confusion can arise) is called a *finite-band operator* if $\sigma(T)$ consists of a finite number of regular analytic arcs.

3. Algebraic multiplicities of Floquet multipliers and Floquet eigenvalues

Throughout this section let $q_0, \ldots, q_{n-2}$ be complex-valued, continuous, periodic functions (with period 1) of a real variable and L the differential expression

$$L = \frac{d^n}{dx^n} + q_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \cdots + q_0 .$$

Let $\Phi(E, x)$ be a fundamental matrix of $Ly = Ey$ satisfying the initial condition $\Phi(E, 0) = I$ where I is the $n \times n$ identity matrix. The Floquet multipliers of the differential equation $Ly = Ey$ are then the eigenvalues of $\Phi(E, 1)$, the so called monodromy matrix.

Our aim is to determine multiplicities of Floquet eigenvalues and multipliers for large values of the spectral parameter E. For large values of E the equation $Ly = Ey$ can be treated as a perturbation of $y^{(n)} = Ey$. In this case there exist n linearly independent Floquet solutions $\exp(\lambda\sigma_k x)$ with associated Floquet multiplier $\exp(\lambda\sigma_k)$ where λ is

[3)] Let (a, b) be an open real interval. Then $z : (a, b) \to \mathbb{C}$ is called an analytic arc if z has a power series expansion near every $t \in (a, b)$. The arc is called regular analytic if, in addition, $z'(t) \neq 0$ for all $t \in (a, b)$. It is called regular analytic at $z(t_0)$ if $z'(t_0) \neq 0$. An arc $z = x + iy$ is regular analytic at $z(t_0) = x_0 + iy_0$ if and only if at least one of the functions $y = y(x)$ or $x = x(y)$ exists near x_0 respectively y_0 and admits a power series representation there.

such that $\lambda^n = -E$ and the σ_k are the different n-th roots of -1. The characteristic polynomial of the associated monodromy matrix is therefore given by

$$\mathcal{F}_0(E,\rho) = (-1)^n\rho^n + (-1)^{n-1}a_{F,1}(E)\rho^{n-1} + \cdots - a_{F,n-1}(E)\rho + 1$$

where the $a_{F,j}$ are the elementary symmetric polynomials in the variables $\exp(\lambda\sigma_1), \dots ,$ $\exp(\lambda\sigma_n)$.

In order to treat the general case we use the following theorem which, together with its proof, can be found in NAIMARK [13].

Theorem 3.1. *For any $c \in \mathbb{C}$ and any $\kappa \in \{0,1,\dots,2n-1\}$ let $T_{c,\kappa}$ be the region*

$$T_{c,\kappa} = \left\{\lambda : \frac{\kappa\pi}{n} < \arg(\lambda + c) < \frac{(\kappa+1)\pi}{n}\right\}.$$

Arrange $\sigma_1,\dots,\sigma_n$, the different n-th roots of -1, in an order such that for all $\lambda \in T_{c,\kappa}$ the inequalities $\Re((\lambda+c)\sigma_1) \le \Re((\lambda+c)\sigma_2) \le \cdots \le \Re((\lambda+c)\sigma_n)$ are satisfied. The equation $Ly + \lambda^n y = 0$ has linearly independent solutions $y_1(\lambda,x),\dots,y_n(\lambda,x)$, which, for fixed x, are analytic for sufficiently large $\lambda \in T_{c,\kappa}$. Moreover,

$$y_k^{(l-1)}(\lambda,x) = (\lambda\sigma_k)^{l-1}\exp(\lambda\sigma_k x)(1 + f_{l,k}(x)), \quad l,\, k = 1,\dots,n, \tag{3.1}$$

where $f_{l,k}(x) = O(\lambda^{-1})$ uniformly for $x \in [0,1]$.

There exists a constant $C \ge 1$ such that $|\exp(\pm 2c\sigma_j)| \le C$ for all $j = 1,\dots,n$. This implies that $|\exp(\lambda\sigma_j)| \le C\,|\exp(\lambda\sigma_k)|$ whenever $j < k$. The way in which the roots $\sigma_1,\dots,\sigma_n$ are ordered implies that $\Re((\lambda+c)\sigma_k) - \Re((\lambda+c)\sigma_j) \ge b\,|\lambda|$ for some constant $b > 0$ provided $k - j \ge 2$ and $|\lambda|$ is suitably large. Therefore, for large λ and if $k - j \ge 2$, one finds

$$|\exp(\lambda\sigma_j)| \le \frac{1}{4}\,|\exp(\lambda\sigma_k)|. \tag{3.2}$$

In particular, at most two of the numbers $\exp(\lambda\sigma_j)$ can coincide.

Now, for sufficiently large $E_0 = -\lambda_0^n$ choose numbers c and κ such that $|c| \le 2$ and $\{\lambda : |\lambda - \lambda_0| < 1\} \in T_{c,\kappa}$. Then apply Theorem 3.1 to establish the existence of solutions y_k of $Ly = Ey = -\lambda^n y$ satisfying 3.1. We thus have another fundamental matrix $Y(E,x)$ of $Ly = Ey$ associated with the solutions $y_1,\dots,y_n$. The two fundamental matrices obey, of course, a linear relationship $\Phi(E,x) = Y(E,x)A$ where A is an x-independent $n \times n$ matrix. In fact $A = Y(E,0)^{-1}$ in view of the initial condition satisfied by Φ. Thus the monodromy matrix equals $\Phi(E,1) = Y(E,1)Y(E,0)^{-1}$. It follows from Theorem 3.1 that

$$\left(Y(E,0)^{-1}\right)_{j,k} = \frac{(\lambda\sigma_j)^{1-k}}{n}\left(1 + O(\lambda^{-1})\right)$$

and hence that the entries of the monodromy matrix are

$$\Phi(E,1)_{l,k} = \frac{1}{n}\sum_{j=1}^{n}\left(\lambda\sigma_j\right)^{l-k}\exp(\lambda\sigma_j)\left(1 + O(\lambda^{-1})\right). \tag{3.3}$$

The characteristic polynomial of $\Phi(E,1)$ can be written as

$$\begin{aligned}\mathcal{F}(E,\rho) &= (-1)^n\rho^n + (-1)^{n-1}(a_{F,1}(E)+b_1(E))\rho^{n-1} + \cdots \\ &- (a_{F,n-1}(E)+b_{n-1}(E))\rho + 1\,.\end{aligned}$$

Since the coefficients of $\mathcal{F}(E,\,\cdot\,)$ are given as sums of products of terms as in 3.3 and since $\big|\exp(\lambda\sigma_j)\big| \le C\,|\exp(\lambda\sigma_k)|$ if $j<k$ one finds that

$$|b_j| \;\le\; \frac{M}{|\lambda|}\,\big|\exp(\lambda\sigma_{n+1-j})\,\ldots\,\exp(\lambda\sigma_n)\big|\,,\quad j \;=\; 1\,,\ldots,\,n-1\,,$$

for some suitable positive constant M.

The fact that the coefficients of the polynomials $\mathcal{F}_0(E,\,\cdot\,)$ and $\mathcal{F}(E,\,\cdot\,)$ are close to each other for large $|\lambda|$ forces the roots to be close also. This is made precise in the following proposition. Define

$$B_k(\gamma) \;=\; \{\rho : |\rho - \exp(\lambda\sigma_k)| < \gamma\,|\exp(\lambda\sigma_k)|\,\}\,.$$

Proposition 3.2. *For all $\gamma \in (0,1/4)$ there exists $R>0$ such that for $|E|>R$ and any fixed $k \in \{1,\ldots,n\}$ one of the following two statements on the Floquet multipliers of $Ly = Ey$ holds depending on whether*

$$\tag{3.4} \big|\exp(\lambda\sigma_k) - \exp(\lambda\sigma_j)\big| \;\ge\; \gamma\max\{\,|\exp(\lambda\sigma_k)|,\,\big|\exp(\lambda\sigma_j)\big|\,\} \quad \textit{for all}\quad j \ne k$$

is true or not.

1. *If 3.4 is satisfied then, counting multiplicities, there is precisely one Floquet multiplier in the disk $B_k(\gamma/(2C))$.*

2. *If 3.4 is violated for some $j = l \ne k$ then $|k-l| = 1$ and, again counting multiplicities, there are precisely two Floquet multipliers in the disk $B_k(3\gamma/2)$.*

Proof. Let $Q(E,\rho) = \sum_{j=1}^{n-1}(-1)^{n-j}b_j(E)\rho^{n-j}$. For all $\rho \in B_k(1)$ one obtains the estimate

$$\begin{aligned}|Q(E,\rho)| &\le \sum_{j=1}^{n-1}|b_j|\,|\rho|^{n-j} \\ &= \frac{2^nM}{|\lambda|}\sum_{j=1}^{n-1}\big|\exp(\lambda\sigma_{n+1-j})\ldots\exp(\lambda\sigma_n)\big|\,|\exp(\lambda\sigma_k)|^{n-j} \\ &\le \frac{(2C)^nMn}{|\lambda|}\,|\exp(\lambda\sigma_k)|^k|\exp(\lambda\sigma_{k+1})\ldots\exp(\lambda\sigma_n)|\,.\end{aligned}$$

Now assume that 3.4 holds and consider any ρ on the circle C_1 which bounds the disk $B_k(\gamma/(2C))$. Then, for all $j \ne k$

$$\big|\rho - \exp(\lambda\sigma_j)\big| \;\ge\; \frac{\gamma}{2}\max\{\,|\exp(\lambda\sigma_k)|,\,|\exp(\lambda\sigma_j)|\,\}\,.$$

Hence

$$
\begin{aligned}
|\mathcal{F}_0(E,\rho)| &= \prod_{m=1}^{n} |\rho - \exp(\lambda\sigma_m)| \\
&\geq \frac{1}{C}\left(\frac{\gamma}{2}\right)^n |\exp(\lambda\sigma_k)|^k\, |\exp(\lambda\sigma_{k+1})\dots\exp(\lambda\sigma_n)|\,.
\end{aligned}
$$

Thus, if we assume that $|\lambda| > (4C)^n MCn/\gamma^n$ we have established that $|\mathcal{F}_0(E,\rho)| > |Q(E,\rho)|$ for all $\rho \in C_1$. Rouché's theorem applies now and shows that $\mathcal{F}_0(E,\,\cdot\,)$ and $\mathcal{F}(E,\,\cdot\,) = \mathcal{F}_0(E,\,\cdot\,) + Q(E,\,\cdot\,)$ have the same number of zeros in $B_k(\gamma/(2C))$, namely one.

Next assume that for some $l \in \{1,\dots,n\}$

$$
|\exp(\lambda\sigma_k) - \exp(\lambda\sigma_l)| \leq \gamma \max\{\,|\exp(\lambda\sigma_k)|, |\exp(\lambda\sigma_l)|\,\}\,. \tag{3.5}
$$

This implies that

$$
\frac{3}{4}|\exp(\lambda\sigma_l)| \leq |\exp(\lambda\sigma_k)| \leq \frac{4}{3}|\exp(\lambda\sigma_l)|
$$

and, using also 3.2,

$$
\left|\exp(\lambda\sigma_k) - \exp\left(\lambda\sigma_j\right)\right| \geq \frac{2}{3}\max\left\{\,|\exp(\lambda\sigma_k)|,\ \left|\exp\left(\lambda\sigma_j\right)\right|\,\right\} \tag{3.6}
$$

for all j different from l and k. Now let ρ be on the circle C_2 which bounds $B_k(3\gamma/2)$. Then

$$
|\mathcal{F}_0(E,\rho)| \geq \left(\frac{1}{4}\right)^{n-2} \frac{3\gamma^2}{16}\,|\exp(\lambda\sigma_k)|^k\, |\exp(\lambda\sigma_{k+1})\dots\exp(\lambda\sigma_n)|\,.
$$

Apply once more Rouché's theorem and note that $B_k(3\gamma/2)$ contains two zeros of $\mathcal{F}_0(E,\,\cdot\,)$ to finish the proof. □

Next we want to estimate those (large) values of the spectral parameter E for which Floquet multipliers are degenerate, i.e., for which at least two multipliers coincide. Then condition 3.5 is satisfied for some pair k, l with $|k-l| = 1$ and this implies that $\exp(\lambda(\sigma_l - \sigma_k)) = 1 + z$ for some z which satisfies $|z| \leq 1/4$ when $\gamma = 3/16$ is chosen. Taking logarithms and solving for λ yields

$$
\lambda = \frac{2m\pi i}{\sigma_j - \sigma_k}\left(1 + \frac{\log(1+z)}{2m\pi i}\right) \tag{3.7}
$$

where m is a suitable integer and $\arg(\log(1+z)) \in (-\pi,\pi]$. Since $|\log(1+z)| \leq 1/2$ this implies that

$$
\frac{1}{2}\left|\frac{2m\pi i}{\sigma_j - \sigma_k}\right| \leq |\lambda| \leq \frac{3}{2}\left|\frac{2m\pi i}{\sigma_j - \sigma_k}\right|\,.
$$

Raising both sides of 3.7 to the n-th power and using the inequality $|(1+x)^n - 1| \leq 2n\,|x|$ which holds for $n\,|x| \leq 1$ yields

$$\begin{aligned} \left|E + \left(\frac{2m\pi i}{\sigma_j - \sigma_k}\right)^n\right| &\leq 2n\left|\frac{\log(1+z)}{2m\pi i}\right|\left|\frac{2m\pi i}{\sigma_j - \sigma_k}\right|^n \\ &\leq \frac{2^{n-1}n}{|\sigma_j - \sigma_k|}\,|\lambda|^{n-1} \\ &\leq \frac{2^{n-1}n}{|\sigma_j - \sigma_k|}\,|E|^{(n-1)/n} \end{aligned} \tag{3.8}$$

assuming that $|\log(1+z)/(2m\pi i)| \leq 1/n$ which is satisfied for large λ.

Note that $((\sigma_j - \sigma_k)/i)^n$ is always real since

$$\overline{\left(\frac{\sigma_j - \sigma_k}{i}\right)^n} = \left(\frac{\overline{\sigma_j} - \overline{\sigma_k}}{-i}\right)^n = \left(\frac{\sigma_k - \sigma_j}{-i\sigma_k\sigma_j}\right)^n = \left(\frac{\sigma_j - \sigma_k}{i}\right)^n.$$

Therefore 3.8 implies that

$$\left|\frac{\Im(E)}{\Re(E)}\right| = O(E^{-1/n})$$

and we have proven the following

Theorem 3.3. *Let L be defined as in the beginning of the section. For every $\varepsilon > 0$ there exists a disk $B(\varepsilon) \subset \mathbb{C}$ with the following two properties.*

1. *All values of E where at least two Floquet multipliers of the differential equation $Ly = Ey$ coincide lie in $B(\varepsilon)$ or in the cone $\{E : |\Im(E)|/|\Re(E)| \leq \varepsilon\}$.*
2. *Every degenerate Floquet multiplier outside $B(\varepsilon)$ has multiplicity two.*

This result has been obtained earlier by McKean [12] for $n = 3$ and by da Silva Menezes [2] for general n. Its proof, which is somewhat different from those earlier ones, is repeated here since several of the details are needed in the following.

We now turn to algebraic multiplicities of Floquet eigenvalues.

Theorem 3.4. *Let ρ_0 be a nonzero complex number. Then there exists a $R > 0$ such that every eigenvalue E of the Floquet operator $T(\rho_0)$ which satisfies $|E| > R$ has algebraic multiplicity two, at most.*

Proof. Assume that $E_0 = -\lambda_0^n$ is a suitably large eigenvalue of $T(\rho_0)$, i. e., a zero of $\mathcal{F}(\cdot, \rho_0)$. Let k be such that $\rho_k(\lambda)$ is a branch of a root of $\mathcal{F}(-\lambda^n, \cdot)$ passing through ρ_0.

Suppose first that 3.4 holds for $\lambda = \lambda_0$ and $\gamma = 1/8$. Then one can show that the algebraic multiplicity of E_0 is one. The proof is similar to, in fact somewhat simpler than the following for the case when 3.4 for $\lambda = \lambda_0$ and $\gamma = 1/8$ is violated. It will therefore be omitted.

Now assume that 3.5 holds for $\lambda = \lambda_0$ and $\gamma = 1/8$. Choose c and κ such that the disk $D = \{\lambda : |\lambda - \lambda_0| < r\}$ (where $r \in (0, 1/2)$ will be determined later) is in $T_{c,\kappa}$.

Then, for $\lambda \in D$ and $j \in \{1, \dots, n\}$

$$|\exp(\lambda\sigma_j) - \exp(\lambda_0\sigma_j)| \leq 2r\,|\exp(\lambda_0\sigma_j)|. \tag{3.9}$$

Let K denote the circle $\{\rho : |\rho - \exp(\lambda_0\sigma_k)| = 1/2\,|\exp(\lambda_0\sigma_k)|\}$ and B the open disk bounded by this circle.

The Floquet multipliers of $Ly + \lambda^n y = 0$ are denoted by $\rho_j(\lambda)$. For any $\gamma \in (0, 1/4)$ Proposition 3.2 shows that, for suitably large λ,

$$|\rho_j(\lambda) - \exp(\lambda\sigma_j)| \leq 3\gamma/2\,|\exp(\lambda\sigma_j)|.$$

This, (3.9), (3.6), and the triangle inequality imply

$$|\rho_j(\lambda) - \exp(\lambda_0\sigma_k)| \geq \left(\frac{2}{3} - 2r - 2\gamma\right) \max\{\,|\exp(\lambda_0\sigma_k)|, |\exp(\lambda_0\sigma_j)|\,\}$$

for all $j \neq k$, l and all $\lambda \in D$. Hence, if $2r + 2\gamma < 1/6$, which will henceforth be assumed, then $\rho_j(\lambda)$ is neither in B nor in K.

Next note that

$$\begin{aligned}
|\rho_k(\lambda) - \exp(\lambda_0\sigma_k)| &\leq (2r + 2\gamma)|\exp(\lambda_0\sigma_k)|, \\
|\rho_l(\lambda) - \exp(\lambda_0\sigma_k)| &\leq \frac{4}{3}\left(\frac{1}{8} + 2r + 2\gamma\right)|\exp(\lambda_0\sigma_k)|.
\end{aligned}$$

This shows that, for all $\lambda \in D$, the multipliers $\rho_k(\lambda)$ and $\rho_l(\lambda)$ are in B but not in K.

We have now shown that $\mathcal{F}(-\lambda^n, \rho) \neq 0$ for all $\lambda \in D$ and $\rho \in K$ and the only zeros of $\mathcal{F}(-\lambda^n, \cdot)$ in B are $\rho_k(\lambda)$ and $\rho_l(\lambda)$. One obtains from the residue theorem that

$$\rho_k(\lambda)^m + \rho_l(\lambda)^m = \frac{1}{2\pi i} \int_K \frac{\rho^m \mathcal{F}_\rho(-\lambda^n, \rho)}{\mathcal{F}(-\lambda^n, \rho)}\, d\rho$$

for every positive integer m and hence that $\rho_k + \rho_l$ and $\rho_k\rho_l$ are analytic in D.

Next define analytic functions ν, μ, and f by

$$\begin{aligned}
\exp(\lambda\sigma_k)\nu(\lambda) &= \rho_k(\lambda) + \rho_l(\lambda) - \exp(\lambda\sigma_k) - \exp(\lambda\sigma_l), \\
\exp(2\lambda\sigma_k)\mu(\lambda) &= \rho_k(\lambda)\rho_l(\lambda) - \exp(\lambda\sigma_k)\exp(\lambda\sigma_l)
\end{aligned}$$

and

$$f(\lambda, \rho) = \rho^2 - (\rho_k(\lambda) + \rho_l(\lambda))\rho + \rho_k(\lambda)\rho_l(\lambda). \tag{3.10}$$

Then

$$\begin{aligned}
f(\lambda, \rho) &= \rho^2 - \rho\,[\exp(\lambda\sigma_k) + \exp(\lambda\sigma_l) + \nu(\lambda)\exp(\lambda\sigma_k)] \\
&\quad + \exp(\lambda\sigma_k)\exp(\lambda\sigma_l) + \mu(\lambda)\exp(2\lambda\sigma_k).
\end{aligned}$$

The order of the zero λ_0 of $f(\cdot, \rho_0)$ equals the order of the zero E_0 of $\mathcal{F}(\cdot, \rho_0)$ and hence equals the algebraic multiplicity $m_a(E_0, \rho_0)$ of E_0 as an eigenvalue of $T(\rho_0)$.

From Proposition 3.2 we get estimates on the absolute values of $\nu(\lambda)$ and $\mu(\lambda)$ in D. From Cauchy's estimate we also get bounds on the absolute values of their derivatives in D. Specifically,

$$\begin{aligned} |\nu(\lambda)| &\leq 4\gamma, \quad |\nu'(\lambda)| \leq 4\gamma/r, \quad |\nu''(\lambda)| \leq 8\gamma/r^2, \\ |\mu(\lambda)| &\leq 5\gamma, \quad |\mu'(\lambda)| \leq 5\gamma/r, \quad |\mu''(\lambda)| \leq 10\gamma/r^2. \end{aligned}$$

Next compute

$$\begin{aligned} f_{\lambda,\lambda}(\lambda,\rho) = &- \left[\left(\sigma_k^2 + \nu(\lambda)\sigma_k^2 + 2\nu'(\lambda)\sigma_k + \nu''(\lambda)\right)\exp(\lambda\sigma_k) + \sigma_l^2\exp(\lambda\sigma_l)\right]\rho \\ &+ (\sigma_k+\sigma_l)^2\exp(\lambda(\sigma_k+\sigma_l)) \\ &+ \left(4\mu(\lambda)\sigma_k^2 + 4\mu'(\lambda)\sigma_k + \mu''(\lambda)\right)\exp(2\lambda\sigma_k), \end{aligned}$$

and note that $\rho_0 = (1+\eta)\exp(\lambda_0\sigma_k)$ and $\exp(\lambda_0\sigma_l) = (1+\xi)\exp(\lambda_0\sigma_k)$ where $|\eta| \leq 3\gamma/2$ and $|\xi| \leq 4\gamma/3$. Choosing $r = 1/20$ and $\gamma = 10^{-5}$ yields then that

$$|f_{\lambda,\lambda}(\lambda_0,\rho_0)\exp(-2\lambda_0\sigma_k) - 2\sigma_k\sigma_l| \leq \frac{1}{5}. \tag{3.11}$$

Hence $f_{\lambda,\lambda}(\lambda_0,\rho_0) \neq 0$ and $m_a(E_0,\rho_0) \leq 2$.

For later purposes we remark that we also obtain the following inequality:

$$|f_{\lambda,\rho}(\lambda_0,\rho_0) + \sigma_k + \sigma_l| \leq 10^{-3}|\exp(\lambda_0\sigma_k)|. \tag{3.12}$$

□

4. Geometric multiplicities of Floquet multipliers of Picard expressions

If L is a Picard expression, algebraic and geometric multiplicities of Floquet multipliers of $Ly = Ey$ can be different only when E is one of finitely many numbers, i. e., L has finite Floquet deficiency. This is stated more precisely in

Theorem 4.1. *If the differential expression given by*

$$Ly = y^{(n)} + q_{n-2}(z)y^{(n-2)} + \cdots + q_0(z)y$$

is Picard, then there exist n linearly independent solutions of $Ly = Ey$ which are elliptic of the second kind for all but finitely many values of the parameter E.

Proof. Since inside a compact set there can be only a finite number of values of E where Floquet multipliers associated with a fundamental period of the coefficients of L are degenerate and hence linearly independent solutions which are elliptic of the second kind number less than n we assume henceforth that $|E|$ is sufficiently large. According to Picard's theorem we have to prove that for one of the fundamental periods of the coefficients of L no Floquet multiplier of $Ly = Ey$ associated with this period is degenerate.

Choose the fundamental periods 2ω and $2\omega'$ such that the angle between them is less than π/n. Next fix a number z_0 in such a way that no singularity of $q_0, \ldots, q_{n-2}$

lies on the line through z_0 and $z_0 + 2\omega$ or on the line through z_0 and $z_0 + 2\omega'$. Since the q_k have only finitely many singularities in the fundamental period parallelogram the points which are not admissible choices for z_0 are on lines through the poles of $q_0, \dots, q_{n-2}$ parallel to the directions given by 2ω and $2\omega'$ and hence form a set of Lebesgue measure zero.

Multiply equation $Ly = Ey$ by $(2\omega)^n$, substitute $w(x) = y(2\omega x + z_0)$ and define $p_k(x) = (2\omega)^{n-k} q_k(2\omega x + z_0)$. This gives

$$w^{(n)} + p_{n-2}(x)w^{(n-2)} + \dots + p_0(x)y = (2\omega)^n Ew .$$

The coefficients are then continuous as functions of the real variable x. They are periodic with period one. Theorem 3.3 implies therefore that all Floquet multipliers associated with the period 2ω are pairwise distinct provided the spectral parameter $(2\omega)^n E$ lies outside the set

$$S = \left\{ z : \left| \frac{\Im(z)}{\Re(z)} \right| \le \frac{\theta}{3} \right\} \cup \{ z : |z| \le R \}$$

where is R a suitable positive constant. If $E \notin S$ Picard's theorem asserts the existence of a fundamental system of solutions of $Ly = Ey$ whose elements are elliptic function of the second kind.

Similarly one obtains that all Floquet multipliers associated with the period $2\omega'$ are pairwise distinct provided the spectral parameter $(2\omega')^n E$ lies outside the set

$$S' = \left\{ z : \left| \frac{\Im(z)}{\Re(z)} \right| \le \frac{\theta}{3} \right\} \cup \{ z : |z| \le R' \}$$

for a suitable positive constant R' and hence there exists a fundamental system of solutions which are elliptic of the second kind if $E \notin S'$.

The two sets S and S' do not intersect outside a big circle C. Hence for each value of E outside C we have proven the existence of n linearly independent solutions of $Ly = Ey$ which are elliptic functions of the second kind. □

5. Finite-band operators

In this section we will apply Theorem 4.1 to investigate the spectrum $\sigma(T)$ of the operator $T : H^{2,n}(\mathbb{R}) \to L^2(\mathbb{R})$ given by $Ty = Ly$ where L is a Picard expression. In the following two preparatory theorems, however, it is only required that L is periodic.

Theorem 5.1. *Let L be a differential expression of the form*

$$Ly = y^{(n)} + q_{n-2}y^{(n-2)} + \dots + q_0 y$$

where $q_0, \dots, q_{n-2}$ are continuous, complex-valued functions of a real variable which are periodic with period a. Let T be the associated operator on $H^{2,n}(\mathbb{R})$. Then, for every $\varepsilon > 0$ there exists a disk $B(\varepsilon) > 0$ such that for every E in $\sigma(T)$ but outside of $B(\varepsilon)$ the following holds:

1. *If n is odd, then E is in the set* $\{z : |\Re(z)|/|\Im(z)| < \varepsilon|z|^{-1/n}\}$.
2. *If $n = 4m-2$, $m \in \mathbb{N}$ then E is in the set*

$$\left\{z : |\Im(z)|/|\Re(z)| < \varepsilon|z|^{-1/n}, \Re(z) < 0\right\}.$$

3. *If $n = 4m$, $m \in \mathbb{N}$, then E is in the set* $\{z : |\Im(z)|/|\Re(z)| < \varepsilon|z|^{-1/n}, \Re(z) > 0\}$.

Moreover, in any case there are at most two spectral bands extending to infinity.

Proof. According to ROFE-BEKETOV [16] $\sigma(T) = \mathcal{S}(L)$, the conditional stability set. Let E be a suitably large element of $\sigma(T)$. Then at least one of the Floquet multipliers $\rho_k(E)$ has absolute value one. From Proposition 3.2 one obtains

$$|\exp(\lambda\sigma_k)| - |\rho_k(E)| \;\le\; |\rho_k(E) - \exp(\lambda\sigma_k)| \;\le\; \frac{3\gamma}{2}\,|\exp(\lambda\sigma_k)|$$

and hence that $|\Re(\lambda\sigma_k)|$ can be bounded by any constant α for a suitable choice of γ. Recall that choosing smaller γ forces us to exclude a larger disk in the E-plane from consideration. When n is odd $|\Re(\lambda\sigma_k)| \le \alpha$ and

$$E \;=\; (\lambda\sigma_k)^n \;=\; \sum_{l=0}^{n} i^l \binom{n}{l} (\Re(\lambda\sigma_k))^{n-l}(\Im(\lambda\sigma_k))^l$$

imply that $|\Re(E)| \le \alpha c(n)|E|^{1-1/n}$ and $|\Im(E)| \ge |E|/2$ where $c(n)$ is a constant depending only on n. The first claim (for odd n) follows now by choosing $\alpha = \varepsilon/c(n)$. The other cases are treated similarly.

Now we prove that at most two spectral arcs extend to infinity. For brevity we consider only the case where n is even but not a multiple of 4. If $E_0 = -(R\exp(it_0))^n \in \sigma(T)$ (where $R > 0$) is very large, then the previous result shows that $|t_0| < \varepsilon/(nR)$. Choose c and κ such that the set $D = \{\lambda : |\lambda - R| < r = 2\varepsilon/n\}$ is contained in $T_{c,\kappa}$ (cf. Theorem 3.1). D was chosen such that the arc $R\exp(it)$, $|t| < \varepsilon/(nR)$, is contained in D. Denote the Floquet multipliers of $Ly + \lambda^n y = 0$ by $\rho_j(\lambda)$, $j = 1, \ldots, n$. If $\lambda_0 = R\exp(it_0)$ is sufficiently large and if the roots $\sigma_1, \ldots, \sigma_n$ of -1 are suitably labeled then Proposition 3.2 shows that

$$\left|\rho_j(\lambda) - \exp(\lambda\sigma_j)\right| \;\le\; 3\gamma/2\left|\exp(\lambda\sigma_j)\right|$$

for all $\lambda \in D$ and $j = 1, \ldots, n$. Now let k be such that $|\rho_k(R\exp(it_0))| = 1$. Note that, since i and $-i$ are the only roots of -1 such that, for $\lambda \in D$, $\exp(\lambda\sigma_k)$ is close to the unit circle, we have $\sigma_k = \pm i$. In the following suppose $\sigma_k = i$. If $|\exp(iR) - \exp(-iR)| > 1/8$ it follows for all $\lambda \in D$ that $\rho_k(\lambda) \in B = \{\rho : |\rho - \exp(iR)| < 1/16\}$ and that $\rho_j(\lambda)$, $j \ne k$, is not in the closure of B provided that $C(r+\gamma)$ and $r + \gamma/(3C)$ are both smaller than $1/32$. This implies (as in the proof of Theorem 3.4) that $\rho_k(\lambda)$ and $\nu(\lambda) = \exp(-i\lambda)\rho_k(\lambda) - 1$ are analytic in D and that $|\nu(\lambda)| \le \gamma/(2C)$ and $|\nu'(\lambda)| \le \gamma/(2Cr)$.

Now consider the function $\rho_k(R\exp(it))$, $-\varepsilon/(nR) < t < \varepsilon/(nR)$. It crosses the unit circle in t_0 and its tangent is almost radial and points inside the circle if, say, $10\gamma < r = 2\varepsilon/n$. Therefore, as E revolves counterclockwise once around the origin on a circle of radius R^n, where R is a suitably large positive number satisfying

$|\exp(iR)-\exp(-iR)|>1/8$, then the multiplier $\rho_k(E)$ can cross the unit circle at most once since it always moves from the outside to the inside. There is a second multiplier which could cross the unit circle (associated with the root $-i$ of -1) and hence there are at most two values of E on this circle which belong to the spectrum of T.

Now assume there are more than two spectral arcs extending to infinity. Then there exists an arbitrarily large $R>0$ with $|\exp(iR)-\exp(-iR)|>1/8$ such that the circle with radius R centered at the origin intersects $\sigma(T)$ at least three times. Since this is impossible the theorem is proved. □

Recall that the spectral arc (or arcs) passing through E_0 are given by $\mathcal{F}(E,\rho_0\exp(it)) = 0$ where ρ_0 has modulus one and is such that $\mathcal{F}(E_0,\rho_0)=0$. It was proven in Section 3 that for large E the numbers $m_a(E,\rho)$ and $m_f(E,\rho)$ are both no larger than two. We will now discuss the various combinations.

Let us start with $m_a(E_0,\rho_0)=1$. Then, by the Weierstrass preparation theorem

$$\mathcal{F}(E,\rho_0\exp(it)) = (E-E_0-g(t))h(E,t)$$

where g is analytic for small t, $g(0)=0$, and h is analytic and nonzero in a neighborhood of $(E_0,0)$. Write $g(t)=\sum_{l=1}^{\infty}a_l t^l$. If $m_f(E_0,\rho_0)=1$ then $a_1\neq 0$ and the spectral arc $E=E_0+g(t)$ is regular analytic near $t=0$. If $m_f(E_0,\rho_0)=2$, then $a_1=0$ and $a_2\neq 0$ and two (possibly coinciding, if g is even) spectral arcs end in E_0.

If $m_a(E_0,\rho_0)=2$ the Weierstrass preparation theorem gives

$$\mathcal{F}(E,\rho_0\exp(it)) = \big((E-E_0)^2-2g_1(t)(E-E_0)+g_2(t)\big)\hat{h}(E,t)$$

where $g_1(t)=\sum_{l=1}^{\infty}b_l t^l$, $g_2(t)=\sum_{l=1}^{\infty}a_l t^l$ for sufficiently small t, and $\hat{h}$ is analytic and nonzero in a neighborhood of $(E_0,0)$. Then

$$E = E_0+g_1(t)\pm\sqrt{g_1(t)^2-g_2(t)}\,.$$

If $m_f(E_0,\rho_0)=1$, then $a_1\neq 0$. Let $t=s^2$ and

$$\varphi_\pm(s) = \pm\sqrt{g_1(s^2)^2-g_2(s^2)} = \pm\sqrt{a_1}\,is(1+\eta_1(s))$$

where η_1 is an even analytic function. Therefore $E_{\pm i}(s)=E_0+g_1(s^2)+\varphi_\pm(s)$ are regular analytic at E_0. In fact, these two arcs coincide since $E_{+i}(s)=E_{-i}(-s)$. Now let $t=-s^2$. Then $E_{\pm 1}(s)=E_0+g_1(s^2)\pm\sqrt{a_1}s(1+\eta_2(s))$ where η_2 is an even analytic function. This describes another arc which is regular analytic at E_0 and which intersects $E_{\pm i}(s)$ in a right angle.

Finally, consider $m_a(E_0,\rho_0)=m_f(E_0,\rho_0)=2$. In this case $a_1=0$, $a_2\neq 0$ and $b_1^2-a_2\neq 0$ as will be shown below. We get $E_\pm(t)=E_0+g_1(t)\pm\sqrt{b_1^2-a_2}\,t(1+\eta_3(t))$ where η_3 is analytic near zero. Since the linear term is given by $\big(b_1\pm\sqrt{b_1^2-a_2}\big)t$ which does not vanish identically $E_+(t)$ and $E_-(t)$ represent two, possibly coinciding, regular analytic arcs.

We still have to show that $b_1^2-a_2\neq 0$ if $m_a(E_0,\rho_0)=m_f(E_0,\rho_0)=2$. Using once more the Weierstrass preparation theorem we obtain

$$g(E,t) = (E-E_0)^2-2g_1(t)(E-E_0)+g_2(t) = f\big((-E)^{1/n},\rho_0\exp(it)\big)H(E,t$$

where f is the function defined in 3.10. Since $f(\lambda_0,\rho_0)=f_\lambda(\lambda_0,\rho_0)=f_\rho(\lambda_0,\rho_0)=0$,

$$\begin{aligned} g_{E,E}(E_0,0) &= \quad 2 \quad = f_{\lambda,\lambda}(\lambda_0,\rho_0)\lambda'(E_0)^2 H(E_0,0)\,,\\ g_{E,t}(E_0,0) &= -2b_1 = f_{\lambda,\rho}(\lambda_0,\rho_0)\lambda'(E_0)i\rho_0 H(E_0,0)\,,\\ g_{t,t}(E_0,0) &= \quad a_2 \quad = -\rho_0^2 f_{\rho,\rho}(\lambda_0,\rho_0)H(E_0,0)\,. \end{aligned}$$

These relations and (3.11) and (3.12) show that $|b_1^2/a_2|\le 7/12$.

In summary, we have shown

Theorem 5.2. *Let L and T be as in Theorem 5.1. Assume $E\in\sigma(T)$ has a sufficiently large absolute value. Then E is a band edge if and only if $m_a(E,\rho)=1$ and $m_f(E,\rho)=2$ for some Floquet multiplier ρ of $Ly=Ey$. In all other cases E is an interior point of one or more regular analytic arc(s).*

If n is odd $\sigma(T)$ is ultimately in a cone with the imaginary axis as axis while the potential band edges (where $m_f(E,\rho)=2$) are in a cone whose axis is the real axis. Therefore T is a finite-band operator whenever n, the order of L, is odd. The following result which states that Picard expressions are finite-band has therefore significance only when n is even.

Theorem 5.3. *Let L and T be as in Theorem 5.1. If $\sigma(T)$ does not contain closed regular analytic arcs and if L is a Picard expression, then $\sigma(T)$ consists of finitely many analytic arcs which are regular in their interior.*

Proof. Since there are no closed regular analytic spectral bands any band must have an endpoint or extend to infinity. By Theorem 5.1 at most two bands extend to infinity. Because of the algebraic structure of the singular points of the curve $\mathcal{F}(E,\rho)=0$ only finitely many spectral bands can end in a band edge. Hence, if there are only finitely many band edges the theorem is proven.

Inside any disk there can be at most finitely many band edges. Hence we have to prove that there exists a disk outside of which there are no band edges. A necessary condition for E to be a band edge is that there exists a $\rho\neq 0$ such that $m_a(E,\rho)=1$ and $m_f(E,\rho)=2$. In this case $m_g(E,\rho)=1$ and hence there does not exist a fundamental system of Floquet solutions of $Ly=Ey$. Hence, by Theorem 4.1, E can not be a band edge if its modulus is sufficiently large. □

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