Spectral Deformations and Soliton Equations

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1 Introduction

The main purpose of this paper is to describe the construction of new solutions V of the Korteweg–deVries (KdV) hierarchy of equations by deformations of a given finite–gap solution V_0. In order to describe the nature of these deformations we assume for a moment that the given real–valued quasi–periodic finite–gap solution V_0 is described in terms of the Its–Matveev formula [34] (see, e.g., (3.43)). The basic ingredients underlying this formula are a compact hyperelliptic curve K_n of genus n,

$$K_n: \quad y^2 = \prod_{m=0}^{2n} (E_m - z), \quad E_0 < E_1 < \cdots < E_{2n} \tag{1.1}$$

and an associated Dirichlet divisor

$$\begin{aligned} &\mathcal{D}_{\hat{\mu}_1(x_0)+\cdots+\hat{\mu}_n(x_0)}, \\ &\quad \hat{\mu}_j(x_0) = \left(\mu_j(x_0), \left(\prod_{m=0}^{2n} (E_m - \mu_j(x_0))\right)^{1/2}\right), \\ &\quad \mu_j(x_0) \in [E_{2j-1}, E_{2j}], \quad 1 \le j \le n, \quad x_0 \in \mathbb{R} \text{ fixed} \end{aligned} \tag{1.2}$$

(see Section 3). Here the parameters $\{E_m\}_{m=0}^{2n}$ in (1.1) (characterizing the branch points of K_n) and the projections $\{\mu_j(x_0)\}_{j=1}^{n}$ in (1.2) are spectral parameters of the underlying one–dimensional Schrödinger differential expression

$$\tau_0 = -\frac{d^2}{dx^2} + V_0 \tag{1.3}$$

in the following sense: The spectrum $\sigma(H_0)$ of the self–adjoint operator

$$H_0 = -\frac{d^2}{dx^2} + V_0 \quad \text{on} \quad H^2(\mathbb{R}) \tag{1.4}$$

in $L^2(\mathbb{R})$ is given by

$$\sigma(H_0) = \bigcup_{j=1}^{n} [E_{2(j-1)}, E_{2j-1}] \cup [E_{2n}, \infty) \tag{1.5}$$

and the spectrum $\sigma(H_{0,x_0}^D)$ of the Dirichlet operator H_{0,x_0}^D associated with τ_0 and an additional Dirichlet boundary condition at $x_0 \in \mathbb{R}$

$$\begin{aligned} H_{0,x_0}^D &= -\frac{d^2}{dx^2} + V_0, \\ \mathcal{D}(H_{0,x_0}^D) &= \{g \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{x_0\}) \,|\, g(x_0) = 0\} \end{aligned} \tag{1.6}$$

is given by

$$\sigma(H_{0,x_0}^D) = \{\mu_j(x_0)\}_{j=1}^{n} \cup \sigma(H_0). \tag{1.7}$$

Deformations of the spectral parameters E_m, $m = 0, ..., 2n$ and $\mu_j(x_0)$, $j = 1, ..., n$ in the corresponding Its–Matveev formula then yield new solutions V of the KdV hierarchy. In particular, it follows from (1.5) that deformations of $\{E_m\}_{m=0}^{2n}$ produce non–isospectral deformations of solutions of the KdV hierarchy, whereas deformations of $\{\hat{\mu}_j(x_0)\}_{j=1}^{n}$ are isospectral with respect to H_0.

Perhaps the simplest and best known non–isospectral deformation is the one where one or several spectral bands are contracted into points, e.g.,

$$[E_{2(m_0-1)},\ E_{2m_0-1}] \longrightarrow \lambda_{m_0}. \tag{1.8}$$

In this case K_n degenerates into the singular curve $\hat{K}_n$

$$K_n \longrightarrow \hat{K}_n \,:\, y^2 = (\lambda_{m_0} - z)^2 \prod_{\substack{m=0 \\ m \neq 2m_0-1,\, 2m_0}}^{2n} (E_m - z), \tag{1.9}$$

$$V_0 \longrightarrow V_1(\lambda_{m_0}) \tag{1.10}$$

and the resulting solution $V_1(\lambda_{m_0})$ represents a one–soliton solution on the background of another finite–gap solution $\tilde{V}_0$ corresponding to the hyperelliptic curve

$$\tilde{K}_{n-1} : \;\; y^2 = \prod_{\substack{m=0 \\ m \neq 2m_0-1,\, 2m_0}}^{2n} (E_m - z) \tag{1.11}$$

of genus $n-1$. Applying this procedure n–times finally yields the celebrated n–soliton solutions $V_n(\lambda_1, \dots, \lambda_n)$ of the KdV hierarchy (see [48], [49]).

On the other hand, varying $\hat{\mu}_j(x_0)$, $1 \leq j \leq n$ independently from each other traces out the isospectral manifold of solutions associated with the base solution V_0.

In Section 2 we give a brief account of the KdV hierarchy using a recursive approach. Section 3 describes real–valued quasi–periodic finite–gap solutions and the underlying Its–Matveev formula in some detail. (It also describes the mathematical terminology in connection with hyperelliptic curves needed in our main Section 5.) Section 4 introduces isospectral and non–isospectral deformations in a systematic way by alluding to single and double commutation techniques. In Section 5 we present our main new result on the isospectral set $I_{\mathbb{R}}(V_0)$ of smooth real–valued quasi–periodic finite–gap solutions of a given base solution V_0. (To be precise, we only represent the stationary, i.e., time–independent case since the insertion of the proper time–dependence poses no difficulties.) Finally, in Section 6 we sketch some generalizations and open problems in connection with infinite–gap solutions and consider the limit of N–soliton solutions as $N \to \infty$ in some detail.

Throughout this paper we confine ourselves to the KdV hierarchy. However, our methods extend to other $1+1$–dimensional completely integrable nonlinear evolution equations and to higher–dimensional systems such as the KP hierarchy. Work on these extensions is in progress and will appear elsewhere.

2 The KdV Hierarchy

In order to describe the hierarchy of KdV equations we first recall the recursive approach to the underlying Lax pairs (see, e.g., [3], [44], [46] for details). Consider the differential expressions

$$\begin{aligned} L(t) &= -\frac{d^2}{dx^2} + V(x,t), \\ \hat{P}_{2n+1}(t) &= \sum_{j=0}^{n} [-\frac{1}{2}\hat{f}_{j,x}(x,t) + \hat{f}_j(x,t)\frac{d}{dx}]L(t)^{n-j}, \end{aligned} \tag{2.1}$$

where the $\{\hat{f}_j\}_{j=0}^n$ satisfy the recursion relation

$$\begin{aligned} \hat{f}_0 &= 1, \\ 2\hat{f}_{j,x} &= -\frac{1}{2}\hat{f}_{j-1,xxx} + 2V\hat{f}_{j-1,x} + V_x\hat{f}_{j-1}, \ \ 1 \le j \le n. \end{aligned} \tag{2.2}$$

Define also $\hat{f}_{n+1}$ by

$$2\hat{f}_{n+1,x} = -\frac{1}{2}\hat{f}_{n-1,xxx} + 2V\hat{f}_{n-1,x} + V_x\hat{f}_{n-1}. \tag{2.3}$$

Then one can show that

$$[\hat{P}_{2n+1}, L] = 2\hat{f}_{n+1,x}, \tag{2.4}$$

where $[.,.]$ denotes the commutator. Explicitly one computes from (2.2) for the first few $\hat{f}_n$

$$\hat{f}_0 = 1, \tag{2.5}$$

$$\hat{f}_1 = \frac{1}{2}V + c_1, \tag{2.6}$$

$$\hat{f}_2 = -\frac{1}{8}V_{xx} + \frac{3}{8}V^2 + \frac{c_1}{2}V + c_2, \tag{2.7}$$

$$\begin{aligned} \hat{f}_3 &= \frac{1}{32}V_{xxxx} - \frac{5}{16}VV_{xx} - \frac{5}{32}V_x^2 + \frac{5}{16}V^3 \\ &\quad + \frac{c_1}{2}[-\frac{1}{4}V_{xx} + \frac{3}{4}V^2] + \frac{c_2}{2}V + c_3, \end{aligned} \tag{2.8}$$

where $\{c_j\}_{j\in\mathbb{N}}$ are integration constants. We shall use the convention that all homogeneous quantities, defined by $c_l \equiv 0$, $l \in \mathbb{N}$, are denoted by $f_j := \hat{f}_j(c_l \equiv 0)$, $P_{2n+1} := \hat{P}_{2n+1}(c_l \equiv 0)$, $l \in \mathbb{N}$, i.e.,

$$f_0 = 1, \tag{2.9}$$

$$f_1 = \frac{1}{2}V, \tag{2.10}$$

$$f_2 = -\frac{1}{8}V_{xx} + \frac{3}{8}V^2, \tag{2.11}$$

$$f_3 = \frac{1}{32}V_{xxxx} - \frac{5}{16}VV_{xx} - \frac{5}{32}V_x^2 + \frac{5}{16}V^3. \tag{2.12}$$

The KdV hierarchy is then defined as the sequence of evolution equations

$$KdV_n(V) : = V_t - [P_{2n-1}, L] = V_t - 2f_{n+1,x}(V) = 0, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.13}$$

(Since the $\hat{f}_{n+1}$ are differential polynomials in V we somewhat abuse notation by writing $\hat{f}_{n+1}(V)$ for $\hat{f}_{n+1}(x,t)$.) The first few equations of the KdV hierarchy (2.13) then read

$$KdV_0(V) = V_t - V_x = 0, \tag{2.14}$$

$$KdV_1(V) = V_t + \frac{1}{4}V_{xxx} - \frac{3}{2}VV_x = 0, \tag{2.15}$$

$$\begin{aligned} KdV_2(V) &= V_t - \frac{1}{16}V_{xxxxx} + \frac{5}{8}VV_{xxx} \\ &\quad + \frac{5}{4}V_xV_{xx} - \frac{15}{8}V^2V_x = 0, \end{aligned} \tag{2.16}$$

with $KdV_1(.)$ the usual KdV equation. The inhomogeneous version associated with (2.13) is

$$\begin{aligned} V_t - [\hat{P}_{2n+1}, L] &= V_t - 2\hat{f}_{n+1,x}(V) \\ &= V_t - 2\sum_{j=0}^{n} c_{n-j}\, f_{j+1,x}(V) = 0, \quad c_0 = 1. \end{aligned} \tag{2.17}$$

The special case of the stationary KdV hierarchy characterized by $V_t = 0$ then reads

$$f_{n+1,x}(V) = 0, \quad \text{resp.} \quad \sum_{j=0}^{n} c_{n-j} f_{j+1,x}(V) = 0. \tag{2.18}$$

Particularly simple solutions of (2.18) for $n=1,2$ are

$$V(x)=2\mathcal{P}(x+w';g_2,g_3), \tag{2.19}$$

$$KdV_1(2\mathcal{P})=0, \tag{2.20}$$

$$V(x)=6\mathcal{P}(x+w';\ g_2,g_3), \tag{2.21}$$

$$KdV_2(6\mathcal{P})-\frac{21}{8}g_2KdV_0(6\mathcal{P})=0, \tag{2.22}$$

where $\mathcal{P}(z;\ g_2,g_3)$ denotes the Weierstrass elliptic function with invariants g_2, g_3 and half–periods ω, ω', $\omega>0$, $-i\omega'>0$ [2].

Next define the polynomial $\hat F_n$ in z

$$\hat F_n(z,x,t)=\sum_{j=0}^{n} z^j\,\hat f_{n-j}(V(x,t))=\prod_{j=1}^{n}[z-\mu_j(x,t)],\ n\in\mathbb{N}\cup\{0\}, \tag{2.23}$$

whose zeros we denote by $\{\mu_j(x,t)\}_{j=1}^n$. Then (2.17) becomes

$$V_t=-\frac{1}{2}\hat F_{n,xxx}+2(V-z)\hat F_{n,x}+V_x\hat F_n. \tag{2.24}$$

In the following we specialize to the stationary case $V_t=0$. However, as will become clear from the paragraph following (3.42) (see also the end of Sections 4 and 6), corresponding solutions for any time–dependent element of the KdV hierarchy can easily be obtained.

Assuming $V_t=0$ we get

$$-\frac{1}{2}\,\hat F_{n,xxx}+2(V-z)\hat F_{n,x}+V_x\,\hat F_n=0. \tag{2.25}$$

Integrating (2.25) once results in

$$\hat F_{n,xx}\,\hat F_n-\frac{1}{2}\hat F_{n,x}^2-2(V-z)\hat F_n^2=-2\hat R_{2n+1}(z), \tag{2.26}$$

where the integration constant $-2\hat R_{2n+1}(z)$ is easily seen to be a polynomial in z of degree $2n+1$. Thus we may write

$$\hat R_{2n+1}(z)=\prod_{m=0}^{2n}(E_m-z) \tag{2.27}$$

denoting by $\{E_m\}_{m=0}^{2n}$ the zeros of $\hat{R}_{2n+1}$. A comparison of powers of z in (2.26) then yields the trace relation

$$V(x) = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j(x) \tag{2.28}$$

and the first–order system of differential equations

$$\mu_j'(x) = 2\hat{R}_{2n+1}(\mu_j(x))^{1/2} \prod_{\substack{l=1 \\ l\neq j}}^{n} [\mu_l(x) - \mu_j(x)]^{-1}, \quad 1 \leq j \leq n. \tag{2.29}$$

Since $V_t = 0$ implies

$$[\hat{P}_{2n+1}, L] = 0 \tag{2.30}$$

the (inhomogeneous) stationary KdV hierarchy is defined in terms of commuting ordinary differential operators. By a result of Burchnall and Chaundy [7], [8], (2.30) implies that $\hat{P}_{2n+1}$ and L fulfill an algebraic equation. One readily verifies that the polynomial $\hat{R}_{2n+1}$ enters this algebraic equation in the form

$$\hat{P}_{2n+1}^2 = \hat{R}_{2n+1}(L) = \prod_{m=1}^{2n} (E_m - L). \tag{2.31}$$

Hence one is led to hyperelliptic curves

$$y^2 = \hat{R}_{2n+1}(z) = \prod_{m=0}^{2n} (E_m - z) \tag{2.32}$$

in a natural way. Returning to our simple examples (2.19)–(2.22), one computes for n=1:

$$V(x) = 2\mathcal{P}(x+\omega';\ g_2, g_3), \tag{2.33}$$

$$P_3^2 = -L^3 + \frac{g_2}{4}L - \frac{g_3}{4} \tag{2.34}$$

(an elliptic curve), and for n=2:

$$V(x) = 6\mathcal{P}(x+\omega';\ g_2, g_3), \tag{2.35}$$

$$\hat{P}_5 = P_5 - \frac{21}{8}\, g_2 P_1, \tag{2.36}$$

$$(P_5 - \frac{21}{8}\, g_2\, P_1)^2 = (L^2 - 3g_3)(-L^3 + \frac{9}{4}\, g_2 L + \frac{27}{4}\, g_3). \tag{2.37}$$

3 Finite–Gap Potentials, Its–Matveev Formula

Any V satisfying a stationary higher order KdV equation of the type

$$\hat{f}_{n+1,x}(V) = \sum_{j=0}^{n} c_{n-j}\, f_{j+1,x}\,(V) = 0 \tag{3.1}$$

will be called a (stationary) finite–gap potential. In order to explain this terminology we make the following two hypotheses:
(H.3.1) $V \in C^{\infty}(\mathbb{R})$ is real–valued.
(H.3.2) $E_0 < E_1 < \cdots < E_{2n}$.

In particular, (H.3.2) implies simple zeros of $\hat{R}_{2n+1}$ and hence yields a nonsingular hyperelliptic curve (2.32). In addition one can show that (3.1) together with (H.3.1) and (H.3.2) imply quasi–periodicity and hence boundedness of V (see (3.36)). Hypotheses (H.3.1) and (H.3.2) will be assumed throughout the end of Section 5. Moreover, the one–dimensional Schrödinger operator H in $L^2(\mathbb{R})$ defined by

$$H = -\frac{d^2}{dx^2} + V \quad \text{on} \quad H^2(\mathbb{R}) \tag{3.2}$$

($H^p(\Omega)$, $\Omega \subseteq \mathbb{R}$, $p \in \mathbb{N}$ the usual Sobolev spaces) is self–adjoint with spectrum $\sigma(H)$ given by

$$\sigma(H) = \bigcup_{j=1}^{n} [E_{2(j-1)},\, E_{2j-1}] \cup [E_{2n},\, \infty). \tag{3.3}$$

Thus H has finitely many spectral gaps ρ_n,

$$\rho_0 = (-\infty,\, E_0), \quad \rho_j = (E_{2j-1},\, E_{2j}), \quad 1 \leq j \leq n. \tag{3.4}$$

Moreover, $\mu_j(y)$ defined in (2.23) are the eigenvalues of the Dirichlet operator H_y^D in $L^2(\mathbb{R})$

$$\begin{aligned} H_y^D &= -\frac{d^2}{dx^2} + V, \\ \mathcal{D}(H_y^D) &= \{g \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{y\}) \mid g(y) = 0\} \end{aligned} \tag{3.5}$$

with a Dirichlet boundary condition at $y \in \mathbb{R}$. In addition,

$$\mu_j(y) \in \overline{\rho}_j, \quad y \in \mathbb{R}, \quad 1 \leq j \leq n. \tag{3.6}$$

(See, e.g., [57] for proofs of (3.3)–(3.6).)

In order to describe the Its–Matveev formula [34] for potentials satisfying (3.1) and Hypotheses (H.3.1) and (H.3.2) we need to discuss the hyperelliptic curve

$$y^2 = \hat{R}_{2n+1}(z) = \prod_{m=0}^{2n} (E_m - z), \quad E_0 < E_1 < \cdots < E_{2n} \tag{3.7}$$

in more detail. (See [15]–[17], [24], [26], [30], [44], [46], [48], [50], [57] for reviews on the remaining material of Section 3. Our terminology will follow the one in [24] and [26].)

We employ the usual topological model associated with (3.7) by considering two copies of the cut plane

$$\Pi_0 = \mathbb{C} \backslash \bigcup_{j=0}^{n} \rho_j \tag{3.8}$$

and joining the upper and lower rims of the cuts $\overline{\rho}_j$ crosswise. This leads to the compact hyperelliptic curve K_n consisting of points

$$P = (z,\ \hat{R}_{2n+1}(z)^{1/2}), \quad z \in \mathbb{C} \ \text{ and } \ P_\infty \tag{3.9}$$

(P_∞ the point at infinity obtained by one–point compactification) with branch points

$$(E_m, 0), \quad 0 \le m \le 2n, \quad P_\infty. \tag{3.10}$$

We also need the projection

$$\Pi : \begin{cases} K_n & \longrightarrow \mathbb{C} \cup \{\infty\} \\ P = (z, \hat{R}_{2n+1}(z)^{1/2}) & \longrightarrow z \\ P_\infty & \longrightarrow \infty \end{cases} \tag{3.11}$$

and the involution (sheet exchange map)

$$* : \begin{cases} K_n & \longrightarrow K_n \\ P = (z,\ \hat{R}_{2n+1}(z)^{1/2}) & \longrightarrow P^* = (z,\ -\hat{R}_{2n+1}(z)^{1/2}). \end{cases} \tag{3.12}$$

The upper sheet Π_+ of K_n is then declared as follows. Define

$$\lim_{\epsilon\downarrow 0} \hat{R}_{2n+1}(\lambda+i\epsilon)^{1/2} = -|\hat{R}_{2n+1}(\lambda+i0)^{1/2}|, \quad \lambda < E_0 \tag{3.13}$$

on Π_+ and analytically continue with respect to λ. Local coordinates ζ near $P_0 = (z_0, \hat{R}_{2n+1}(z_0)^{1/2})$, P^∞ then read

$$\zeta = \begin{cases} (z-z_0), & z_0 \in \mathbb{C}\backslash\{E_m\}_{m=0}^{2n} \\ (z-E_m)^{1/2}, & z_0 = E_m,\ 0 \le m \le 2n \\ z^{-1/2}, & z_0 = \infty. \end{cases} \tag{3.14}$$

A convenient homology basis $\{a_j, b_j\}_{j=1}^n$ on K_n, $n \in \mathbb{N}$ is then chosen as follows: the cycle a_j surrounds the cut $\overline{\rho}_j$ clockwise on Π_+ while b_j starts at the lower rim of $\overline{\rho}_j$ on Π_+, intersects a_j, then encircles E_0 clockwise thereby changing into the lower sheet Π_-, and returns on Π_- to its initial point. The cycles are chosen in such a way that their intersection matrix reads

$$a_j \circ b_l = \delta_{j,l}, \quad 1 \le j, l \le n. \tag{3.15}$$

A basis for the holomorphic differentials (Abelian differentials of the first kind, DFK) on K_n is given by

$$\eta_j = \hat{R}_{2n+1}(z)^{-1/2} z^{j-1}\, dz, \quad 1 \le j \le n. \tag{3.16}$$

We choose the standard normalization

$$\omega_j = \sum_{l=1}^n c_{j,l}\, \eta_l \quad \text{with} \quad \int_{a_j} \omega_l = \delta_{j,l}, \quad 1 \le j, l \le n \tag{3.17}$$

and define the b–periods of ω_l by

$$\tau_{j,l} = \int_{b_j} \omega_l, \quad 1 \le j, l \le n. \tag{3.18}$$

Riemann's period relations and (H.3.2) then imply

$$\tau_{j,l} = \tau_{l,j}, \quad \tau = i\,T, \quad T = (T_{j,l}) > 0. \tag{3.19}$$

Abelian differentials of the second kind (DSK) $\omega^{(2)}$ are characterized by vanishing residues and conveniently normalized by

$$\int_{a_j} \omega^{(2)} = 0, \quad 1 \leq j \leq n. \tag{3.20}$$

The Riemann theta–function θ and Jacobi variety $J(K_n)$ associated with K_n are then defined as

$$\theta(\underline{z}) = \sum_{\underline{m}\in\mathbb{Z}^n} e^{2\pi i(\underline{m},\underline{z})+\pi i(\underline{m},\tau\underline{m})}, \quad \underline{z} \in\mathbb{C}^n \tag{3.21}$$

and

$$J(K_n) =\mathbb{C}^n/L_n, \tag{3.22}$$

where L_n denotes the period lattice

$$L_n = \{\underline{z} = (\underline{N} + \tau\underline{M}) \in\mathbb{C}^n \mid \underline{M}, \underline{N} \in \mathbb{Z}^n\}. \tag{3.23}$$

Divisors $\mathcal{D}$ on K_n are defined as integer–valued maps

$$\mathcal{D} : K_n \longrightarrow \mathbb{Z} \tag{3.24}$$

where only finitely many $\mathcal{D}(P) \neq 0$. The degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is defined by

$$\deg(\mathcal{D}) = \sum_{P\in K_n} \mathcal{D}(P). \tag{3.25}$$

The set of all divisors on K_n is denoted by $\mathrm{Div}(K_n)$ and forms

an Abelian group under addition. The set of positive divisors will be denoted by $\mathrm{Div}_+(K_n)$,

$$\mathrm{Div}_+(K_n) = \{\mathcal{D} \in \mathrm{Div}(K_n) \mid \mathcal{D} : K_n \longrightarrow \mathbb{N} \cup \{0\}\} \tag{3.26}$$

(one writes $\mathcal{D} \geq 0$ for $\mathcal{D} \in \mathrm{Div}_+(K_n)$) and the set of positive divisors of degree $r \in \mathbb{N}$ is as usual identified with the r–th symmetric product $\sigma^r K_n$ of K_n. We also use the notation

$$\mathcal{D}_{P_1+\cdots+P_r} : \begin{cases} K_n & \longrightarrow \mathbb{N} \cup \{0\} \\ P & \longrightarrow \begin{cases} m & \text{if } P \text{ occurs } m\text{–times in } \{P_1, \ldots, P_r\} \\ 0 & \text{if } P \notin \{P_1, \ldots, P_r\} \end{cases} \end{cases} \tag{3.27}$$

for divisors in $\sigma^r K_n$. The Abel (Jacobi) map with base point $P_0 \in K_n$ is then defined by

$$\underline{A}_{P_0} : \begin{cases} K_n & \longrightarrow J(K_n) \\ P & \longrightarrow \left\{ \int_{P_0}^{P} \omega_j \right\}_{j=1}^{n} \pmod{L_n} \end{cases} \tag{3.28}$$

respectively by

$$\underline{\alpha}_{P_0} : \begin{cases} \operatorname{Div}(K_n) & \longrightarrow J(K_n) \\ \mathcal{D} & \longrightarrow \sum\limits_{P \in K_n} \mathcal{D}(P)\, \underline{A}_{P_0}(P). \end{cases} \tag{3.29}$$

If $f \not\equiv 0$ is a meromorphic function on K_n, the divisor (f) of f is defined by

$$(f) : \begin{cases} K_n & \longrightarrow \mathbb{Z} \\ P & \longrightarrow \nu_f(P), \end{cases} \tag{3.30}$$

where $\nu_f(P)$ denotes the order of f at P. Divisors of the type (3.30) are called principal. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(K_n)$ are called linearly equivalent, $\mathcal{D} \sim \mathcal{E}$ iff they differ by a principal divisor, i.e., iff

$$\mathcal{D} = \mathcal{E} + (f) \tag{3.31}$$

for some meromorphic $f \not\equiv 0$ on K_n. The equivalence class of $\mathcal{D}$ is denoted by $[\mathcal{D}]$ (if $\mathcal{D} \geq 0$, $|\mathcal{D}|$ usually denotes the set of positive divisors linearly equivalent to $\mathcal{D}$). By Abel's theorem,

$$\mathcal{D} \sim \mathcal{E} \text{ iff } \begin{cases} \deg(\mathcal{D}) = \deg(\mathcal{E}) \\ \underline{A}_{P_0}(\mathcal{D}) = \underline{A}_{P_0}(\mathcal{E}). \end{cases} \tag{3.32}$$

The Jacobi inversion theorem states

$$\underline{\alpha}_{P_0}(\sigma^n K_n) = J(K_n). \tag{3.33}$$

Finally, a positive divisor $\mathcal{D} \in \sigma^n K_n$ is called nonspecial iff the equivalence class $|\mathcal{D}|$ of positive divisors of $\mathcal{D}$ only consists of $\mathcal{D}$ itself, i.e., iff

$$|\mathcal{D}| = \{\mathcal{D}\}. \tag{3.34}$$

Otherwise $\mathcal{D} \geq 0$ is called special. One can show that $\mathcal{D}_{P_1+\cdots+P_n} \in \sigma^n K_n$ is special iff there exists at least one pair (P, P^*) such that

$$(P, P^*) \in \{P_1, \ldots, P_n\}. \tag{3.35}$$

After these preliminaries we can describe in detail the Its–Matveev formula [34] for real–valued finite–gap potentials V satisfying (3.1). It reads

$$\begin{aligned} V(x) = \sum_{m=0}^{2n} E_m - 2 \sum_{j=1}^{n} \lambda_j \qquad\qquad (3.36) \\ -2 \frac{d^2}{dx^2} \ln \theta \left(\underline{\zeta}_{P_\infty} + \underline{\alpha}_{P_\infty} (\mathcal{D}_{\hat{\mu}_1(x_0)+\cdots+\hat{\mu}_n(x_0)}) + \frac{(x-x_0)}{2\pi} \underline{U}_0 \right). \end{aligned}$$

Here

$$\underline{\zeta}_{P_\infty} = \frac{1}{2} \left\{ j + \sum_{l=1}^{n} \tau_{j,l} \right\}_{j=1}^{n} \in \mathbb{C}^n \tag{3.37}$$

denotes the vector of Riemann constants, $\underline{U}_0$ given by

$$U_{0,j} = \int_{b_j} \omega_0^{(2)}, \quad \int_{a_j} \omega_0^{(2)} = 0, \quad 1 \leq j \leq n \tag{3.38}$$

denotes the vector of b–periods of the normalized DSK

$$\begin{aligned} \omega_0^{(2)} &= -2^{-1}\, i\, \hat{R}_{2n+1}(z)^{-1/2} \prod_{j=1}^{n} (\lambda_j - z)\, dz \qquad (3.39) \\ &= [\zeta^{-2} + 0(1)]\, d\zeta \quad \text{near} \quad P_\infty \end{aligned}$$

with a single pole at P_∞. (3.39) also identifies the numbers $\{\lambda_j\}_{j=1}^n$ in (3.36). (One infers $\lambda_j \in \rho_j$, $1 \leq j \leq n$.) Moreover, the Dirichlet divisor $\mathcal{D}_{\hat{\mu}_1(x)+\cdots+\hat{\mu}_n(x)}$ is obtained as follows.

$$\hat{\mu}_j(x) = (\mu_j(x),\, \hat{R}_{2n+1}(\mu_j(x))^{1/2}), \quad 1 \leq j \leq n, \tag{3.40}$$

where $\{\mu_j(x)\}_{j=1}^n$ satisfy the system (2.29) with prescribed initial conditions

$$\hat{\mu}_j(x_0) = (\mu_j(x_0),\, \hat{R}_{2n+1}(\mu_j(x_0))^{1/2}), \quad 1 \leq j \leq n \tag{3.41}$$

at x_0. In particular, the Abel map linearizes the system (2.29) since (modulo L_n)

$$\begin{aligned}\underline{\alpha}_{P_\infty}\left(\mathcal{D}_{\hat{\mu}_1(x)+\cdots+\hat{\mu}_n(x)}\right) &= \underline{\alpha}_{P_\infty}\left(\mathcal{D}_{\hat{\mu}_1(x_0)+\cdots+\hat{\mu}_n(x_0)}\right)\\ &\quad+\frac{(x-x_0)}{2\pi}\,\underline{U}_0, \quad x\in\mathbb{R}. \qquad (3.42)\end{aligned}$$

So far we have only discussed the stationary case. However, (3.36) easily extends to the time–dependent situation [34]. E.g.,

$$\begin{aligned}V(x,t) &= \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n}\lambda_j \qquad (3.43)\\ &\quad -2\partial_x^2 \ln\theta\Big(\underline{\zeta}_{P_\infty} + \underline{\alpha}_{P_\infty}\left(\mathcal{D}_{\hat{\mu}_1(x_0,t_0)+\cdots+\hat{\mu}_n(x_0,t_0)}\right)\\ &\quad +\frac{(x-x_0)}{2\pi}\,\underline{U}_0 + \frac{3(t-t_0)}{2\pi}\,\underline{U}_2\Big)\end{aligned}$$

satisfies the KdV_1 equation (see (2.16)), i.e.,

$$KdV_1(V) = V_t + \frac{1}{4}V_{xxx} - \frac{3}{2}VV_x = 0, \qquad (3.44)$$

where $\underline{U}_2$ is the vector of b–periods of the normalized DSK $\omega_2^{(2)}$ with a single pole at P_∞ of the type

$$\omega_2^{(2)} = [\zeta^{-4} + 0(1)]\,d\zeta \quad \text{near} \quad P_\infty, \qquad (3.45)$$

$$U_{2,j} = \int_{b_j}\omega_2^{(2)}, \quad \int_{a_j}\omega_2^{(2)} = 0, \quad 1\le j\le n. \qquad (3.46)$$

In this case the Dirichlet divisor $\mathcal{D}_{\hat{\mu}_1(x,t)+\cdots+\hat{\mu}_n(x,t)}$ is obtained as follows.

$$\hat{\mu}_j(x,t) = (\mu_j(x,t),\ \hat{R}_{2n+1}(\mu_j(x,t))^{1/2}), \quad 1\le j\le n, \qquad (3.47)$$

where $\{\mu_j(x,t)\}_{j=1}^n$ satisfy the system

$$\partial_x\,\mu_j(x,t) = 2\hat{R}_{2n+1}(\mu_j(x,t))^{1/2}\prod_{\substack{l=1\\ l\neq j}}^{n}[\mu_l(x,t)-\mu_j(x,t)]^{-1},$$

$$\partial_t \mu_j(x,t) = 2\Big[\sum_{m=0}^{2n} E_m - 2\sum_{\substack{l=1\\ l\neq j}}^{n} \mu_l(x,t)\Big]\partial_x \mu_j(x,t),$$
$$1 \le j \le n \quad (3.48)$$

with prescribed initial conditions

$$\hat{\mu}_j(x_0,t_0) = (\mu_j(x_0,t_0),\ \hat{R}_{2n+1}(\mu_j(x_0,t_0))^{1/2}), \quad 1 \le j \le n \quad (3.49)$$

at (x_0,t_0). Again the Abel map linearizes the system (3.48) since

$$\underline{\alpha}_{P_\infty}\Big(\mathcal{D}_{\hat{\mu}_1(x,t)+\cdots+\hat{\mu}_n(x,t)}\Big) = \underline{\alpha}_{P_\infty}\Big(\mathcal{D}_{\hat{\mu}_1(x_0,t_0)+\cdots+\hat{\mu}_n(x_0,t_0)}\Big) + \frac{(x-x_0)}{2\pi}\underline{U}_0 + \frac{3(t-t_0)}{2\pi}\underline{U}_2, \quad (x,t) \in \mathbb{R}^2 \quad (3.50)$$

(modulo L_n).

4 Spectral Deformations, Commutation Techniques

Since virtually all explicitly known solutions of the KdV hierarchy, such as soliton solutions, rational solutions, and solitons on the background of quasi–periodic finite–gap solutions, can be obtained from the Its–Matveev formula upon suitable deformations (singularizations) of the underlying hyperelliptic curve K_n (see e.g. [17]–[20], [26], [48], [49], [64] and the references therein), we propose a systematic study of such deformations in this section. Our main strategy will be to exploit single and double commutation techniques to be explained below.

We illustrate the main idea by the following simple example. Consider again the potential (2.19)

$$V(x) = 2\mathcal{P}(x+\omega';\ g_2,g_3) + \mathcal{P}(\omega';\ g_2,g_3) \quad (4.1)$$

associated with the nonsingular elliptic curve (see (2.34))

$$y^2 = (-e_1+e_3-z)(-e_2+e_3-z)(-z), \quad (4.2)$$
$$e_1 = \mathcal{P}(\omega;\ g_2,g_3),\ e_2 = \mathcal{P}(\omega+\omega';\ g_2,g_3),\ e_3 = \mathcal{P}(\omega';\ g_2,g_3).$$

(For convenience we added $\mathcal{P}(\omega')$ in (4.1) in order to guarantee $E_2 = 0$.) Then $H = -\dfrac{d^2}{dx^2} + V$ has spectrum (see (3.3))

$$\sigma(H) = [-e_1 + e_3, \ -e_2 + e_3] \cup [0, \infty). \tag{4.3}$$

Fix $\kappa > 0$ and deform

$$[-e_1 + e_3, \ -e_2 + e_3] \longrightarrow -\kappa^2 \tag{4.4}$$

by taking $\omega \to \infty$, $\omega' = (i\pi/2\kappa)$. Then V in (4.1) converges to the one–soliton potential V_1

$$\begin{aligned} &V(x) = 2\mathcal{P}(x + \omega'; \ g_2, g_3) + \mathcal{P}(\omega'; \ g_2, g_3) \\ &\longrightarrow V_1(x) = -2\kappa^2[\cosh(\kappa x)]^{-2} \end{aligned} \tag{4.5}$$

and the associated elliptic curve (4.2) degenerates into a singular curve

$$y^2 = (-e_1 + e_3 - z)(-e_2 + e_3 - z)(-z) \longrightarrow y^2 = (-\kappa^2 - z)^2(-z). \tag{4.6}$$

The corresponding operator $H_1 = -\dfrac{d^2}{dx^2} + V_1$ then has the spectrum

$$\sigma(H_1) = \{-\kappa^2\} \cup [0, \infty). \tag{4.7}$$

A further degeneration $\kappa \to 0$ finally yields

$$V(x) = 0 \quad \text{and} \quad y^2 = (-z)^3. \tag{4.8}$$

This point of view has been adopted in [48] and [49] and the general n–soliton potentials have been derived from the Its–Matveev formula by a singularization of K_n where all compact spectral bands degenerate into a single point

$$[E_{2(j-1)}, \, E_{2j-1}] \longrightarrow -\kappa_j^2, \ \ 1 \le j \le n, \ \kappa_1 > \kappa_2 > \cdots > \kappa_n, \ E_{2n} = 0 \tag{4.9}$$

(see (3.3)).

Here we shall in a sense reverse the above point of view. Instead of starting with a finite–gap potential such as (4.1) and degenerating compact spectral bands into single points (such as in (4.4) with the result (4.5)–(4.7)), we shall start with a finite–gap potential V_0 and insert eigenvalues into its spectral gaps. In the context of the above example this amounts to starting with

$$V_0(x) = 0, \quad y^2 = -z \tag{4.10}$$

and inserting the eigenvalue $-\kappa^2$ into the spectral gap $\rho_0 = (-\infty, 0)$ of V_0 to arrive at

$$V_1(x) = -2\kappa^2[\cosh(\kappa x)]^{-2}, \;\; y^2 = (-\kappa^2 - z)^2(-z). \tag{4.11}$$

The spectral deformations described so far were clearly non–isospectral. In addition we will also discuss various isospectral deformations of potentials below. In short, these isospectral deformations either "insert eigenvalues" at points where there were already eigenvalues or they formally insert eigenvalues with certain "defects" such as zero or infinite norming constants. In either case no new eigenvalue is actually inserted and the deformation is isospectral. A systematic and detailed approach to these ideas can be found in [25]–[27].

We start with the single commutation method or Crum–Darboux method [11]–[14], [18], [19], [36], [61]. Assume that $V_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ is real–valued and that the differential expression

$$\tau_0 = -\frac{d^2}{dx^2} + V_0(x) \tag{4.12}$$

is nonoscillatory and in the limit point case at $\pm\infty$. Consider the self–adjoint realization H_0 of τ_0 in $L^2(\mathbb{R})$

$$\begin{aligned} H_0 &= -\frac{d^2}{dx^2} + V_0, \\ \mathcal{D}(H_0) &= \{g \in L^2(\mathbb{R}) \,\big|\, g,\, g' \in AC_{\mathrm{loc}}(\mathbb{R}),\ \tau_0\, g \in L^2(\mathbb{R})\} \end{aligned} \tag{4.13}$$

(here $AC_{\mathrm{loc}}(\cdot)$ denotes the set of locally absolutely continuous functions) with

$$E_0 = \inf[\sigma(H_0)] > -\infty. \tag{4.14}$$

The basic idea behind the single commutation method is the following: choose

$$\lambda_1 \in \rho_0 = (-\infty, E_0) \tag{4.15}$$

and factor

$$H_0 = AA^* + \lambda_1 = -\frac{d^2}{dx^2} + V_0 \tag{4.16}$$

with

$$A = \frac{d}{dx} + \phi,\ \phi(x) = {\psi'}_0(\lambda_1, x)/\psi_0(\lambda_1, x),\ H_0\, \psi_0(\lambda_1) = \lambda_1\, \psi_0\,(\lambda_1) \tag{4.17}$$

for some real–valued distributional solution $\psi_0(\lambda_1, x)$. Commuting A and A^* yields

$$H_1 = A^*A + \lambda_1 = -\frac{d^2}{dx^2} + V_1, \tag{4.18}$$

$$V_1(x) = V_0(x) - 2\frac{d^2}{dx^2}\ \ln\, \psi_0(\lambda_1, x). \tag{4.19}$$

We note that $\tau_1 = -\frac{d^2}{dx^2} + V_1(x)$ is in the limit point case at $\pm\infty$ and that

$$\sigma(H_1)\backslash\{\lambda_1\} = \sigma(H_0). \tag{4.20}$$

Depending on the choice of $\psi_0(\lambda_1, x)$, λ_1 either belongs to $\sigma(H_1)$ and one has inserted an eigenvalue λ_1 into $\rho_0 = (-\infty, E_0)$ which represents the non–isospectral case, or $\lambda_1 \not\in \sigma(H_1)$, i.e., $\sigma(H_1) = \sigma(H_0)$ which is the isospectral case. The above procedure can easily be iterated and we only summarize the final results.

Consider weak solutions $\psi_{0,\pm}\,(\lambda_1, x)$ such that

$$\begin{gathered} 0 < \psi_{0,\pm}\,(\lambda, .) \in L^2((R, \pm\infty)),\ R \in \mathbb{R},\ \lambda < E_0, \\ H_0\, \psi_{0,\pm}(\lambda) = \lambda\psi_{0,\pm}\,(\lambda),\ \lambda < E_0. \end{gathered} \tag{4.21}$$

Pick

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N < E_0 \tag{4.22}$$

and define in $L^2(\mathbb{R})$

$$H(\lambda_1, \epsilon_1, \ldots, \lambda_N, \epsilon_N) = -\frac{d^2}{dx^2} + V(\lambda_1, \epsilon_1, \ldots, \lambda_N, \epsilon_N), \tag{4.23}$$

$$V(\lambda_1,\epsilon_1,\dots,\lambda_N,\epsilon_N,x)=V_0(x)$$
$$-2\frac{d^2}{dx^2}\ln W(\psi_{0,\epsilon_1}(\lambda_1),\dots,\psi_{0,\epsilon_N}(\lambda_N))(x),$$
$$\epsilon_l\in\{+,-\},\quad 1\le l\le N. \tag{4.24}$$

Then $\tau_N=-\frac{d^2}{dx^2}+V(\lambda_1,\epsilon_1,\dots,\lambda_N,\epsilon_N,x)$ is in the limit point case at $\pm\infty$ and $H(\lambda_1,\epsilon_1,\dots,\lambda_N,\epsilon_N)$ and H_0 are isospectral, i.e.,

$$\sigma(H(\lambda_1,\epsilon_1,\dots,\lambda_N,\epsilon_N))=\sigma(H_0) \tag{4.25}$$

(in fact, one can show that they are unitarily equivalent [13]). If on the other hand one replaces $\psi_{0,\epsilon_l}(\lambda_l,x)$ in (4.24) by a genuine linear combination of $\psi_{0,+}(\lambda_l,x)$ and $\psi_{0,-}(\lambda_l,x)$

$$\psi_{0,\epsilon_l}(\lambda_l,x)\longrightarrow\alpha\psi_{0,+}(\lambda_l,x)+\beta\psi_{0,-}(\lambda_l,x),\ \alpha>0,\ \beta>0 \tag{4.26}$$

then $\lambda_l\in\rho_0=(-\infty,E_0)$ becomes actually an eigenvalue of the resulting operator. Since we are going to use the single commutation method only in the isospectral context in Section 5 we shall not give any further details on the non–isospectral case.

In the special case of V_0 in (4.13) being a finite–gap potential of the type (3.36),

$$V_0(x)=\sum_{m=0}^{2n}E_m-2\sum_{j=1}^{n}\lambda_j \tag{4.27}$$
$$-2\frac{d^2}{dx^2}\ln\theta\left(\underline{\zeta}_{P_\infty}+\underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat\mu_1^0(x_0)+\cdots+\hat\mu_n^0(x_0)})+\frac{(x-x_0)}{2\pi}\underline{U}_0\right),$$

(4.24) becomes

$$V(\lambda_1,\epsilon_1,\dots,\lambda_N,\epsilon_N,x)=\sum_{m=0}^{2n}E_m-2\sum_{j=1}^{n}\lambda_j \tag{4.28}$$
$$-2\frac{d^2}{dx^2}\ln\theta(\underline{\zeta}_{P_\infty}-\underline{\alpha}_{P_\infty}(\mathcal{D}_{Q_1+\cdots+Q_N})$$
$$+\underline{\alpha}_{P_\infty}\left(\mathcal{D}_{\hat\mu_1^0(x_0)+\cdots+\hat\mu_n^0(x_0)}\right)+\frac{(x-x_0)}{2\pi}\underline{U}_0),$$
$$Q_l=\left(\lambda_l,-\epsilon_l\left|\hat R_{2n+1}(\lambda_l+i0)^{1/2}\right|\right),\quad \epsilon_l\in\{+,-\},\quad 1\le l\le N.$$

In this particular context it can be shown that (4.23)–(4.25) extend to the case $\lambda_N \leq E_0$ (in addition to (4.22)).

The single commutation method has the obvious drawback that λ_1 in (4.15) is confined to being below $E_0 = \inf[\sigma(H_0)]$ since for $\lambda_1 > \inf[\sigma(H_0)]$, ψ_0 in (4.17), (4.19) would have at least one zero by Sturm's oscillation theory and hence V_1 in (4.19) would necessarily be singular. In order to overcome this drawback and insert an eigenvalue λ_1 into any spectral gap of H_0 one is led to the double commutation method (going back at least to [23] and described in detail in [13], [14], [22], [25–27], [38]), a refinement of two single commutations at the same spectral point λ_1.

Assuming

$$\lambda_1 \in \mathbb{R} \backslash \sigma(H_0) \tag{4.29}$$

one factors again

$$H_0 = A_\pm \, A_\pm^* + \lambda_1 = -\frac{d^2}{dx^2} + V_0, \tag{4.30}$$

$$H_{1,\pm} = A_\pm^* \, A_\pm + \lambda_1 = -\frac{d^2}{dx^2} + V_{1,\pm}, \tag{4.31}$$

$$V_{1,\pm}(x) = V_0(x) - 2\frac{d^2}{dx^2} \ln \psi_{0,\pm}(\lambda_1, x), \tag{4.32}$$

where

$$A_\pm = \frac{d}{dx} + \phi_\pm, \quad \phi_\pm(x) = {\psi'}_{0,\pm}(\lambda_{1,x})/\psi_{0,\pm}(\lambda_1, x), \tag{4.33}$$

$$\psi_{0,\pm}(\lambda_{1,.}) \in L^2((R, \pm\infty)),\ R \in \mathbb{R},\ H_0 \, \psi_{0,\pm,}(\lambda_1) = \lambda_1 \, \psi_{0,\pm}(\lambda_1)$$

and $V_{1,\pm}$ are now singular in general. Introducing

$$\Psi_{\gamma_1,\pm}(x) \quad = \quad \psi_{0,\pm}(\lambda_1, x)^{-1} \, [1 \mp \gamma_{1,\pm} \int_{\pm\infty}^x dx' \, \psi_{0,\pm}(\lambda_1, x')^2], \qquad \gamma_{1,\pm} \geq 0, \tag{4.34}$$

$$\Phi_\pm(x) = \Psi'_{\gamma_{1,\pm}}(x)/\Psi_{\gamma_{1,\pm}}(x), \tag{4.35}$$

$$B_\pm = \frac{d}{dx} + \Phi_\pm, \quad B_\pm^+ = -\frac{d}{dx} + \Phi_\pm, \tag{4.36}$$

one infers by inspection that

$$H_{1,\pm} = A_\pm^* A_\pm + \lambda_1 = B_\pm \, B_\pm^+ + \lambda_1. \tag{4.37}$$

A further commutation of $B_\pm$ and $B_\pm^+$ then leads to

$$H_{\gamma_{1,\pm}} = B_\pm^+ \, B_\pm + \lambda_1 = -\frac{d^2}{dx^2} + V_{\gamma_{1,\pm}}, \tag{4.38}$$

$$V_{\gamma_{1,\pm}}(x) = V_0(x) - 2\frac{d^2}{dx^2} \ln\,[1 \mp \gamma_{1,\pm} \int_{\pm\infty}^x dx' \, \psi_{0,\pm}(\lambda_1, x')^2]. \tag{4.39}$$

One can prove that $\tau_{\gamma_{1,\pm}} = -\dfrac{d^2}{dx^2} + V_{\gamma_{1,\pm}}$ is in the limit point case at $\pm\infty$ and that

$$\sigma(H_{\gamma_{1,\pm}}) = \sigma(H_0) \cup \{\lambda_1\} \quad \text{iff} \quad 0 < \gamma_{1,\pm} < \infty. \tag{4.40}$$

Hence $\gamma_{1,\pm} \in (0, \infty)$ represents the non–isospectral case. The two cases $\gamma_{1,\pm} = 0, \infty$ on the other hand represent the isospectral case, i.e.,

$$\sigma(H_{\infty,\pm}) = \sigma(H_0), \tag{4.41}$$

where

$$H_{\infty,\pm} = -\frac{d^2}{dx^2} + V_{\infty,\pm}, \tag{4.42}$$

$$V_{\infty,\pm}(x) = V_0(x) - 2\frac{d^2}{dx^2} \ln\,[\mp \int_{\pm\infty}^x dx' \, \psi_{0,\pm}(\lambda_1, x')^2]. \tag{4.43}$$

This procedure can easily be iterated and we summarize again the final results.

Consider weak solutions $\psi_{0,\pm}(\lambda, x)$ such that

$$\psi_{0,\pm}(\lambda, .) \in L^2((R, \pm\infty)) \text{ is real–valued, } R \in \mathbb{R},$$

$$H_0 \, \psi_{0,\pm}(\lambda) = \lambda \psi_{0,\pm}(\lambda), \quad \lambda \in \mathbb{R} \backslash \sigma(H_0). \tag{4.44}$$

Pick

$$\lambda_j \in \mathbb{R} \backslash \sigma(H_0), \quad 1 \le j \le N, \quad \lambda_j \ne \lambda_l \text{ for } j \ne l \tag{4.45}$$

and define in $L^2(\mathbb{R})$

$$H_{\gamma_1,\dots,\gamma_N,\pm}(\lambda_1,\dots,\lambda_N) = -\frac{d^2}{dx^2} + V_{\gamma_1,\dots,\gamma_N,\pm}(\lambda_1,\dots,\lambda_N), \quad (4.46)$$

$$\begin{aligned} &V_{\gamma_1,\dots,\gamma_N,\pm}(\lambda_1,\dots,\lambda_N,x) = V_0(x) \\ &-2\frac{d^2}{dx^2} \ln \det\left\{\left[\delta_{l,l'} \mp \gamma_{l,\pm} \int_{\pm\infty}^x dx'\, \psi_{0,\pm}(\lambda_l,x')\, \psi_{0,\pm}(\lambda_l',x')\right]_{l,l'=1}^N\right\}, \\ &\qquad \gamma_{l,\pm} \geq 0, \quad 1 \leq l \leq N. \end{aligned} \quad (4.47)$$

Then $\tau_{N,\pm} = -\frac{d^2}{dx^2} + V_{\gamma_1,\dots,\gamma_N,\pm}(\lambda_1,\dots,\lambda_N,x)$ is in the limit point case at $\pm\infty$ and

$$\sigma(H_{\gamma_1,\dots,\gamma_N,\pm}(\lambda_1,\dots,\lambda_N)) = \sigma(H_0) \cup \{\lambda_l\}_{l=1}^N \text{ iff } \gamma_{l,\pm} \in (0,\infty), \quad 1 \leq l \leq N, \quad (4.48)$$

illustrating the nonisospectral case. Similarly, defining

$$H_{\infty,\pm}(\lambda_1,\dots,\lambda_N) = -\frac{d^2}{dx^2} + V_{\infty,\pm}(\lambda_1,\dots,\lambda_N), \quad (4.49)$$

$$\begin{aligned} &V_{\infty,\pm}(\lambda_1,\dots,\lambda_N,x) = V_0(x) \\ &\quad -2\frac{d^2}{dx^2} \ln \det\left\{\left[\mp \int_{\pm\infty}^x dx' \psi_{0,\pm}(\lambda_l,x')\psi_{0,\pm}(\lambda_{l'},x')\right]_{1\leq l,\, l'\leq N}\right\} \end{aligned} \quad (4.50)$$

yields the isospectral counterpart, i.e.,

$$\sigma(H_{\infty,\pm}(\lambda_1,\dots,\lambda_N)) = \sigma(H_0) \quad (4.51)$$

(actually, one can show that $H_{\infty,\pm}(\lambda_1,\dots,\lambda_N)$ and H_0 are unitarily equivalent [25]).

In the particular case where V_0 is the finite–gap potential (4.27), equation (4.50) becomes

$$V_{\infty,\pm}(\lambda_1,\dots,\lambda_N,x) = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \lambda_j \quad (4.52)$$

$$-2\frac{d^2}{dx^2}\ \ln\,\theta\Big(\underline{\zeta}_{P_\infty} \mp 2\underline{\alpha}_{P_\infty}(\mathcal{D}_{Q_1+\cdots+Q_N})$$
$$+\underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1^0(x)+\cdots+\hat{\mu}_n^0(x_0)}) + \frac{(x-x_0)}{2\pi}\,\underline{U}_0\Big),$$
$$Q_l = \Big(\lambda_l, -\Big|\hat{R}_{2n+1}(\lambda_l+i0)^{1/2}\Big|\Big), \quad 1 \le l \le N.$$

A comparison of (4.52) and (4.28) reveals that in the finite–gap context one double commutation at λ_1 corresponds to two single commutations at λ_1 and λ_2 in the limit $\lambda_2 \to \lambda_1$. Actually this fact is independent of the finite–gap context and holds in general. Indeed, taking into account the identity

$$\begin{aligned}&\int_{\pm\infty}^{x} dx'\,\psi_{0,\pm}(\lambda_1,x')\psi_{0,\pm}(\lambda_2,x') \\ &\quad= (\lambda_1-\lambda_2)^{-1}\,W(\psi_{0,\pm}(\lambda_1),\ \psi_{0,\pm}(\lambda_2))(x), \\ &\quad \lambda_1,\ \lambda_2 \in \mathbb{R}\backslash\sigma(H_0), \quad \lambda_1 \ne \lambda_2\end{aligned} \tag{4.53}$$

and the fact that $W(\psi_{0,+}(\lambda_1),\psi_{0,-}(\lambda_1))$ is a nonzero constant, one infers, e.g.,

$$\begin{aligned} V(\lambda_1,\epsilon_1,\,\lambda_2,\epsilon_2,x) &= V_0(x) - 2\frac{d^2}{dx^2}\ \ln\,W(\psi_{0,\epsilon_1}(\lambda_1),\psi_{0,\epsilon_2}(\lambda_2))(x) \\ &\underset{\lambda_2\to\lambda_1}{\longrightarrow} \begin{cases} V_0(x), & \epsilon_1 = -\epsilon_2 \\ V_{\infty,\epsilon_1}(\lambda_1,x), & \epsilon_1 = \epsilon_2. \end{cases} \end{aligned} \tag{4.54}$$

Finally, with a slight adjustment only, one can also use directly formulas (4.39) resp. (4.47) to produce potentials isospectral to V_0. E.g., if λ_1 is already an eigenvalue of H_0,

$$\lambda_1 \in \sigma_p(H_0) \tag{4.55}$$

then $H_{\gamma_{1,\pm}}$ in (4.38) and (4.39), with $\psi_{0,+}(\lambda_1,x) = c\,\psi_{0,-}(\lambda_1,x)$ the corresponding eigenfunction of H_0, are well defined. In this case one only changes the corresponding norming constant of the eigenfunction of $H_{\gamma_{1,\pm}}$ associated with λ_1 and hence $H_{\gamma_{1,\pm}}$ and H_0 are isospectral

$$\sigma(H_{\gamma_{1,\pm}}) = \sigma(H_0). \tag{4.56}$$

(A further extension, allowing $\gamma_{1,\pm} = -\|\psi_{0,\pm}(\lambda_1)\|_2^2$, removes the eigenvalue λ_1 from H_0, i.e., $\sigma(H_{\gamma_{1,\pm}}) = \sigma(H_0)\backslash\{\lambda\}$ in this case.) These facts are illustrated, e.g., in [1], [58].

It should perhaps be pointed out again at this occasion that the substitution

$$\psi_{0,\pm}(\lambda_j, x) \longrightarrow \psi_{0,\pm}(\lambda_j, x, t) \tag{4.57}$$

in (4.24), (4.47), (4.50), where $\psi_{0,\pm}(\lambda_j, x, t)$ satisfies

$$H_0\,\psi_{0,\pm}(\lambda_j) = \lambda_j\,\psi_{0,\pm}(\lambda_j),\ \partial_t\,\psi_{0,\pm}(\lambda_j) = P_{2n+1}\,\psi_{0,\pm}(\lambda_j),\ 1 \le j \le N \tag{4.58}$$

and V_0 satisfies the n–th KdV equation

$$KdV_n(V_0) = 0, \tag{4.59}$$

produces again solutions $V(\lambda_1, \epsilon_1, \dots, \lambda_N, \epsilon_N, x, t)$ and $V_{\gamma_{1,\dots,},\gamma_N,\pm}\,(\lambda_{1,\dots}, \lambda_N, x, t)$, $V_{\infty\pm}\,(\lambda_{1,\dots}, \lambda_N, x, t)$ of the n–th KdV equation.

5 Isospectral Sets of Quasi–Periodic Finite–Gap Potentials

In this section we fix a real–valued quasi–periodic finite–gap potential $V_0(x)$ satisfying Hypotheses (H.3.1) and (H.3.2) and

$$\begin{aligned} \hat{f}_{n+1,x}(V_0) \quad &= \quad \sum_{j=0}^{n} c_{n-j}\, f_{j+1,x}\,(V_0) = 0 \\ &\qquad \text{for some fixed } \{c_j\}_{j=0}^{n} \subset \mathbb{R}, \quad c_0 = 1 \end{aligned} \tag{5.1}$$

with the associated nonsingular compact hyperelliptic curve $K_n = K_n(V_0)$

$$K_n : y^2 = \hat{R}_{2n+1}(z) = \prod_{m=0}^{2n} (E_m - z), \quad E_0 < E_1 < \cdots < E_{2n} \tag{5.2}$$

(cf. (2.23), (2.26), and (2.27)). Thus V_0 can be represented by the Its–Matveev formula (4.27)

$$V_0(x) = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \lambda_j \tag{5.3}$$

$$-2\frac{d^2}{dx^2}\ \ln\,\theta\left(\underline{\zeta}_{P_\infty}+\underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1^0(x_0)+\cdots+\hat{\mu}_n^0(x_0)})+\frac{(x-x_0)}{2\pi}\,\underline{U}_0\right).$$

The isospectral set $I_{\mathbb{R}}(V_0)$ of real–valued quasi–periodic finite–gap potentials of V_0 is then defined by

$$I_{\mathbb{R}}(V_0)=\{V\in C^\infty(\mathbb{R}),\ \text{real–valued}\ \Big|\ \hat{f}_{n+1,x}(V)=0\,, \qquad K_n(V)=K_n(V_0)\}, \tag{5.4}$$

where $\hat{f}_{n+1,x}$ is given in terms of the sequence $\{c_j\}_{j=0}^n$, $c_0=1$ in (5.1) and $K_n(V)=K_n(V_0)$ denotes the fixed hyperelliptic curve (5.2).

In order to give an explicit realization of $I_{\mathbb{R}}(V_0)$ we need to introduce the following sets $\mathcal{D}_{\mathbb{R}_\pm}\subset\sigma^n\,K_n$ of positive divisors in "real position" (see Section 3 for the terminology employed)

$$\mathcal{D}_{\mathbb{R}_-}=\{\mathcal{D}_{P_1+\cdots+P_n}\in\sigma^nK_n\mid\Pi(P_j)\in\overline{\rho}_0=[-\infty,E_0],\ 1\le j\le n\}, \tag{5.5}$$

$$\mathcal{D}_{\mathbb{R}_+}=\{\mathcal{D}_{P_1+\cdots+P_n}\in\sigma^nK_n\ \Big|\ \Pi(P_j)\in\overline{\rho}_{\pi(j)}=[E_{2\pi(j)-1},\,E_{2\pi(j)}], \qquad 1\le j\le n\}, \tag{5.6}$$

where π denotes some permutation of $\{1,\ldots,n\}$.

The Its–Matveev formula (3.36) and the fact that Dirichlet divisors $\mathcal{D}_{\hat{\mu}_1(x)+\cdots+\hat{\mu}_n(x)}$ are nonspecial then yields the following theorem (see, e.g., [4], [5], [17], [21], [35], [44], [48], [50], [57]).

Theorem 5.1 *The map*

$$i_+:\begin{cases} I_{\mathbb{R}}(V_0) & \longrightarrow \mathcal{D}_{\mathbb{R}_+}\\ V_{\hat{\mu}_1,\ldots,\hat{\mu}_n} & \longrightarrow \mathcal{D}_{\hat{\mu}_1(x_0)+\cdots+\hat{\mu}_n(x_0)}\end{cases} \tag{5.7}$$

is bijective, where

$$V_{\hat{\mu}_1,\ldots,\hat{\mu}_n}(x)=\sum_{m=0}^{2n}E_m-2\sum_{j=1}^n\lambda_j \qquad -2\frac{d^2}{dx^2}\ \ln\,\theta(\underline{\zeta}_{P_\infty}+\underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1(x_0)+\cdots+\hat{\mu}_n(x_0)})+\frac{(x-x_0)}{2\pi}\,\underline{U}_0) \tag{5.8}$$

and the associated Dirichlet divisor $\mathcal{D}_{\hat{\mu}_1(x)+\cdots+\hat{\mu}_n(x)}$ is obtained from (3.40) by solving the system (2.29) with initial conditions (3.41).

Next we state the following "real" version of the Jacobi inversion theorem (3.33).

Lemma 5.2 *Denote by* $[\underline{z}]$ *the equivalence class of* $\underline{z} \in \mathbb{C}^n$ *in* $J(K_n) = \mathbb{C}^n/L_n$. *Then*

$$\underline{\alpha}_{P_\infty}(\mathcal{D}_{\mathbb{R}_-}) = \{[\underline{x}] \in J(K_n) \mid \underline{x} \in \mathbb{R}^n\}. \tag{5.9}$$

Sketch of proof. Due to the fact that $\hat{R}_{2n+1}(z)^{1/2}$ is real–valued iff $z \in \bigcup_{j=0}^{n} \overline{\rho}_j$ and

$$\begin{aligned}
\underline{A}_{P_\infty}((E_{2j},0)) &= \frac{1}{2}\left[(0,\ldots,\underbrace{0}_{j},1,\ldots,1) + (\tau_{j,1},\ldots,\tau_{j,n})\right], \\
\underline{A}_{P_\infty}((E_{2j-1},0)) &= \frac{1}{2}\left[(0,\ldots,\underbrace{0}_{j-1},1,\ldots,1) + (\tau_{j,1},\ldots,\tau_{j,n})\right]
\end{aligned} \tag{5.10}$$

one can show that

$$\underline{\alpha}_{P_\infty}(\mathcal{D}_{Q_1+\cdots+Q_n}) \subseteq \{[\underline{x}] \in J(K_n) \mid \underline{x} \in \mathbb{R}^n\} \text{ iff } \mathcal{D}_{Q_1+\cdots+Q_n} \in \mathcal{D}_{\mathbb{R}_-}. \tag{5.11}$$

(5.9) then follows from (3.33) by restricting $\underline{\alpha}_{P_\infty}$ to $\mathcal{D}_{\mathbb{R}_-}$.

Next we introduce the notion of admissibility of divisors: a positive divisor $\mathcal{D}_{P_1} + \cdots + P_n \in \sigma^n K_n$ is called admissible iff there is no pair $(P, P^*) \in \{P_1, \ldots, P_n\}$ with $P \in K_n \backslash \{P_\infty\}$. The set of all admissible divisors is denoted by $\mathcal{A}$.

We note that admissible divisors $\mathcal{D}_{P_1+\ldots+P_n} \in \mathcal{A}$ are either nonspecial or their speciality stems from one or more points P_∞ contained in $\{P_1, \ldots, P_n\}$.

Lemma 5.3 *Given* $\mathcal{D}_{\hat{\mu}_1^0+\cdots+\hat{\mu}_n^0} \in \mathcal{D}_{\mathbb{R}_+}$ *and* $\mathcal{D}_{\hat{\mu}_1+\cdots+\hat{\mu}_n} \in \mathcal{D}_{\mathbb{R}_+}$ *there exists a unique divisor* $\mathcal{D}_{Q_1+\cdots+Q_n} \in \mathcal{D}_{\mathbb{R}_-} \cap \mathcal{A}$ *such that*

$$\underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1+\cdots+\hat{\mu}_n}) = \underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1^0+\cdots+\hat{\mu}_n^0}) - \underline{\alpha}_{P_\infty}(\mathcal{D}_{Q_1+\cdots+Q_n}). \tag{5.12}$$

Sketch of proof. Since $\hat{R}_{2n+1}(z)^{1/2}$ is real–valued if $z \in \bigcup_{j=1}^{n} \overline{\rho}_j$, (5.12) is equivalent to

$$\underline{\alpha}_{P_\infty}(\mathcal{D}_{Q_1+\cdots+Q_n}) = -\sum_{j=1}^{n} \underline{A}_{\hat{\mu}_j^0}(\hat{\mu}_{\pi(j)}) \in \{[\underline{x}] \in J(K_n) \mid \underline{x} \in \mathbb{R}^n\} \tag{5.13}$$

for some permutation π of $\{1,\ldots,n\}$. Thus the existence of some $\mathcal{D}_{Q_1+\cdots+Q_n} \in \mathcal{D}_{\mathbb{R}_-}$ satisfying (5.12) follows from Lemma 5.2. If $\mathcal{D}_{Q_1+\cdots+Q_n}$ is nonspecial then $\mathcal{D}_{Q_1+\cdots+Q_n} \in \mathcal{A}$ is clearly the unique solution of (5.12). If on the other hand $n \geq 2$ and $\{Q_1,\ldots,Q_n\}$ contains a pair (P,P^*) with $\Pi(P) \in (-\infty, E_0]$, say $Q_1 = P$, $Q_2 = P^*$, then simply replace Q_1 and Q_2 by P_∞ since

$$\mathcal{D}_{Q_1+Q_2+Q_3+\cdots+Q_n} \sim \mathcal{D}_{P_\infty+P_\infty+Q_3+\cdots+Q_n} \tag{5.14}$$

by Abel's theorem (3.32). By continuing this process of replacing pairs (P,P^*), $P \neq P_\infty$ by (P_∞,P_∞) one finally ends up with a unique admissible divisor linearly equivalent to the original $\mathcal{D}_{Q_1+\cdots+Q_n}$.

Our new main result on $I_{\mathbb{R}}(V_0)$ then reads

Theorem 5.4 [27] *The map*

$$i_- : \begin{cases} I_{\mathbb{R}}(V_0) & \longrightarrow \mathcal{D}_{\mathbb{R}_-} \cap \mathcal{A} \\ V_{\hat{\mu}_1,\ldots,\hat{\mu}_n} & \longrightarrow \mathcal{D}_{Q_1+\cdots+Q_n} \end{cases} \tag{5.15}$$

is bijective, where $\mathcal{D}_{Q_1+\cdots+Q_n} \in \mathcal{D}_{\mathbb{R}_-} \cap \mathcal{A}$ *is the unique solution of*

$$\begin{aligned} \underline{\alpha}_{P_\infty}(\mathcal{D}_{Q_1+\ldots+Q_n}) \quad &= \quad \underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1^0(x_0)+\cdots+\hat{\mu}_n^0(x_0)}) \\ &\quad -\underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1(x_0)+\cdots+\hat{\mu}_n(x_0)}). \end{aligned} \tag{5.16}$$

Moreover,

$$\begin{aligned} V_{\hat{\mu}_1,\ldots,\hat{\mu}_n}(x) = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \lambda_j \\ -2\frac{d^2}{dx^2} \ln\theta\left(\underline{\zeta}_{P_\infty} + \underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1(x_0)+\cdots+\hat{\mu}_n(x_0)}) + \frac{(x-x_0)}{2\pi}\,\underline{U}_0\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \lambda_j \\
&\quad -2\frac{d^2}{dx^2} \ln \theta \Big(\underline{\zeta}_{P_\infty} - \underline{\alpha}_{P_\infty}(\mathcal{D}_{Q_1+\cdots+Q_n}) \\
&\quad +\underline{\alpha}_{P_\infty}(\mathcal{D}_{\hat{\mu}_1^0(x_0)+\cdots+\hat{\mu}_n^0(x_0)}) + \frac{(x-x_0)}{2\pi}\underline{U}_0 \Big) \\
&= V(\lambda_{j_1}, \epsilon_{j_1}, \dots, \lambda_{j_m}, \epsilon_{j_m}, x) \\
&= V_0(x) - 2\frac{d^2}{dx^2} \ln W(\psi_{0,\epsilon_{j_1}}(\lambda_{j_1}), \dots, \psi_{0,\epsilon_{j_m}}(\lambda_{j_m}))(x), \qquad (5.17)
\end{aligned}
$$

where

$$
\begin{aligned}
&\{Q_1, \dots, Q_n\} = \{P_\infty, \dots, P_\infty, Q_{j_1}, \dots, Q_{j_m}\}, \qquad (5.18) \\
&\quad Q_{j_l} = \left(\lambda_{j_l}, -\epsilon_{j_l} \left| \hat{R}_{2n+1}(\lambda_{j_l} + i0)^{1/2} \right| \right), \\
&\quad \lambda_{j_l} \in (-\infty, E_0], \ 1 \le l \le m \le n.
\end{aligned}
$$

Sketch of proof. Existence and uniqueness of $\mathcal{D}_{Q_1+\cdots+Q_n} \in \mathcal{D}_{\mathbb{R}_-} \cap \mathcal{A}$ in (5.15) associated with $V_{\hat{\mu}_1,\dots,\hat{\mu}_n}$ by (5.16) follows from Lemma 5.3. (5.17) and (5.18) are a consequence of (4.24) and (4.28).

Remark 5.5 An explicit realization of $I_{\mathbb{R}}(V_0)$ in the case where V_0 is a real–valued periodic finite–gap potential has first been derived by Finkel, Isaacson, and Trubowitz [21]. We also refer to [9], [35], [37], [51]–[53], [59], and [62] for further investigations in this direction. Our realization (5.17) of $I_{\mathbb{R}}(V_0)$ differs from the one in [21] in two respects. First of all, for fixed genus n, (5.17) involves at most an $n \times n$ Wronskian as opposed to a $2n \times 2n$ Wronskian in [21] (involving n additional Dirichlet eigenfunctions) and secondly, (5.17) does not assume periodicity but applies to the quasi–periodic finite–gap case. The upshot of (5.17) is the following: the entire isospectral torus $I_{\mathbb{R}}(V_0)$ of the given base potential V_0 is generated by at most n–single commutations associated with $(\lambda_1, \epsilon_1, \dots, \lambda_n, \epsilon_n)$, where the points $Q_j = (\lambda_j, -\epsilon_j |\hat{R}_{2n+1}(\lambda_j + i0)^{1/2}|)$, $1 \le j \le n$ vary independently of each other on both rims of the cut $\overline{\rho}_0 = [-\infty, E_0]$ (avoiding pairs of the type (Q, Q^*), $Q \ne P_\infty$ in $\{Q_1, \dots, Q_n\}$).

One can prove an analogous representation for $I_{\mathbb{R}}(V_0)$ by using the isospectral double commutation approach (4.49)–(4.52) [27].

6 Some Generalizations

In our final section we comment on some natural generalizations of the approach in Sections 4 and 5 and mention some open problems.

a) Infinitely Many Spectral Gaps in $\sigma(H_0)$:

The case where $V_0 \in \mathbb{C}^\infty(\mathbb{R})$ is real–valued and periodic of period $a > 0$ with infinitely many spectral gaps in $\sigma(H_0)$ is well understood [21], [35], [37], [46], [47], [54], [55], [59], [62]. If

$$\sigma(H_0) = \bigcup_{j\in\mathbb{N}} [E_{2(j-1)}, E_{2j-1}], \tag{6.1}$$

then V_0 can be approximated uniformly on $\mathbb{R}$ by a sequence of real–valued finite–gap potentials $V_{0,n}$ (of the same period a) associated with K_n in (5.2) as $n \to \infty$. In this context determinants of the type (4.24) and (4.50) converge to Fredholm determinants as $n \to \infty$ (we shall illustrate this in some detail in a similar context at the end of this section).

These results have been extended to particular classes of real–valued almost periodic potentials $V_0 \in C^\infty(\mathbb{R})$ with suitable conditions on the asymptotic behavior of E_j as $j \to \infty$ in [10], [39]–[44].

It should perhaps be pointed out that with the exceptions of [4]–[6], [31], [32], [60], the corresponding complex–valued analog received much less attention in the literature. In particular, the Jacobi inversion problem on the noncompact Riemann surface K_∞ associated with V_0 in the complex–valued periodic or almost–periodic infinite–gap case (a crucial step in the corresponding generalization of the Its–Matveev formula) appears to be open.

b) Harmonic Oscillators etc:

The double commutation approach in connection with (4.55) and (4.56) can be used to produce families of isospectral unbounded potentials with purely discrete spectra. In order to see the connection with spectral deformations in Section 4 consider the harmonic oscillator example

$$V_0(x) = x^2 - 1 \tag{6.2}$$

and the (suitably scaled) Mathieu potential

$$V_\epsilon(x) = 2\epsilon^{-2}[1 - \cos(\epsilon x)] - 1, \quad \epsilon > 0. \tag{6.3}$$

As is well known [57], all periodic and anti–periodic eigenvalues of $-\frac{d^2}{dx^2} + V_\epsilon$ restricted to $[x_0, x_0 + (2\pi/\epsilon)]$, $\epsilon > 0$ are simple and hence

$$H_\epsilon = -\frac{d^2}{dx^2} + V_\epsilon \ \text{ on } \ H^2(\mathbb{R}), \quad \epsilon > 0 \tag{6.4}$$

has infinitely many spectral gaps for all $\epsilon > 0$

$$\sigma(H_\epsilon) = \bigcup_{j\in\mathbb{N}} [E_{2(j-1)}(\epsilon),\ E_{2j-1}(\epsilon)]. \tag{6.5}$$

As $\epsilon \downarrow 0$,

$$V_\epsilon(x) \underset{\epsilon\downarrow 0}{\longrightarrow} V_0(x) = x^2 - 1 \tag{6.6}$$

and, since

$$E_{2(j-1)}(\epsilon),\, E_{2j-1}(\epsilon) \underset{\epsilon\downarrow 0}{\longrightarrow} 2(j-1), \quad j \in \mathbb{N}, \tag{6.7}$$

one infers

$$\sigma(H_\epsilon) \underset{\epsilon\downarrow 0}{\longrightarrow} \sigma(H_0) = \{2(j-1)\}_{j\in\mathbb{N}} \tag{6.8}$$

(see, e.g., [33], [63]). In this scaling limit $\epsilon \downarrow 0$, the noncompact Riemann surface $K_\infty(\epsilon)$ associated with V_ϵ, $\epsilon > 0$ degenerates into a highly singular curve consisting of infinitely many double points $\{2(j-1)\}_{j\in\mathbb{N}}$. A careful study of this limit on the level of degenerating hyperelliptic curves and their θ–functions, to the best of our knowledge, has not been undertaken yet. Isospectral families of the limit potential $V_0(x) = x^2 - 1$ have been constructed in [45] and [56] but apart from the harmonic oscillator case we are not aware of any other detailed study of isospectral families for unbounded potentials with purely discrete spectra.

Finally, we mention another possible generalization in a bit more detail:

c) N–Soliton Solutions as $N \to \infty$:

Here we choose

$$H_0 = -\frac{d^2}{dx^2} \text{ on } H^2(\mathbb{R}), \quad V_0(x) = 0 \tag{6.9}$$

and choose double commutation to insert N eigenvalues

$$\{\lambda_j = -\kappa_j^2\}_{j=1}^N,\ \kappa_j > 0,\ 1 \leq j \leq N,\ \kappa_j \neq \kappa_{j'} \text{ for } j \neq j' \tag{6.10}$$

into the spectral gap $\rho_0 = (-\infty, 0)$ of H_0. The result is the N–soliton potential [22], [38]

$$V_N(x) = -2\frac{d^2}{dx^2} \ln \det[1_N + C_N(x)], \tag{6.11}$$

$$C_N(x) = \left[\frac{c_l\, c_{l'}}{\kappa_l + \kappa_{l'}}\, e^{-(\kappa_l+\kappa_{l'})x}\right]_{1\leq l,\, l'\leq N}, \tag{6.12}$$

where

$$c_l > 0, \quad 1 \leq l \leq N \tag{6.13}$$

are (norming) constants (related to $\gamma_{l,+}$ in (4.47) by $c_l^2 = \gamma_{l,+}$, $1 \leq l \leq N$, i.e., $V_N(x) = V_{c_1^2,\ldots,c_N^2,+}(\lambda_1, \ldots, \lambda_N, x)$). Introducing

$$H_N = -\frac{d^2}{dx^2} + V_N \text{ on } H^2(\mathbb{R}), \tag{6.14}$$

one verifies that

$$\sigma(H_N) = \{-\kappa_j^2\}_{j=1}^N \cup [0, \infty) \tag{6.15}$$

with purely absolutely continuous essential spectrum of multiplicity two

$$\sigma_{ess}(H_N) = \sigma_{ac}(H_N) = [0, \infty), \tag{6.16}$$

$$\sigma_p(H_N) \cap [0, \infty) = \sigma_{sc}(H_N) = \emptyset \tag{6.17}$$

and simple discrete eigenvalues $\{-\kappa_j^2\}_{j=1}^N$. (Here $\sigma_{ess}(.)$, $\sigma_{ac}(.)$, $\sigma_{sc}(.)$, and $\sigma_p(.)$ denote the essential, absolutely continuous, singularly continuous, and point spectrum (the set of eigenvalues) respectively.)

The unitary scattering matrix $S_N(k)$ in $\mathbb{C}^2$ associated with the pair (H_N, H_0) is reflectionless and reads

$$\begin{aligned} S_N(k) &= \begin{pmatrix} T_N(k) & 0 \\ 0 & T_N(k) \end{pmatrix}, \\ T_N(k) &= \prod_{j=1}^{N} \left(\frac{k + i\kappa_j}{k - i\kappa_j} \right), \ k \in \mathbb{C} \backslash \{i\kappa_j\}_{j=1}^N \end{aligned} \tag{6.18}$$

($\lambda = k^2$ the spectral parameter of H_0). As briefly mentioned in Section 4, the singular curve associated with H_N is of the type

$$K_{0,N}: \quad y^2 = \left[\prod_{j=1}^{N} (-\kappa_j^2 - z)^2 \right] (-z) \tag{6.19}$$

which can be obtained from the nonsingular curve

$$K_N: \quad y^2 = \prod_{m=0}^{2n} (E_m - z), \quad E_0 < E_1 < \cdots < E_{2N} = 0 \tag{6.20}$$

by degenerating the compact spectral bands $[E_{2(j-1)}, E_{2j-1}]$ into the eigenvalues $-\kappa_j^2$

$$[E_{2(j-1)}, E_{2j-1}] \longrightarrow -\kappa_j^2, \quad 1 \le j \le N. \tag{6.21}$$

At this point it seems natural to ask what happens if $N \to \infty$. This can be answered as follows.

Theorem 6.1 [28], [29] *Assume* $\{\kappa_j > 0\}_{j \in \mathbb{N}} \in l^\infty(\mathbb{N})$, $\kappa_j \neq \kappa_{j'}$ *for* $j \neq j'$ *and choose* $\{c_j > 0\}_{j \in \mathbb{N}}$ *such that* $\{c_j^2/\kappa_j\}_{j \in \mathbb{N}} \in l^1(\mathbb{N})$. *Then* V_N *converges pointwise to some* $V_\infty \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *as* $N \to \infty$ *and*

(i) $\lim_{x \to +\infty} V_\infty(x) = 0$ *and*

$$\lim_{n \to \infty} \sup_{x \in K} \left| V_N^{(m)}(x) - V_\infty^{(m)}(x) \right| = 0, \quad m \in \mathbb{N} \cup \{0\} \tag{6.22}$$

for any compact $K \subset \mathbb{R}$.

(ii) Denoting

$$H_\infty = -\frac{d^2}{dx^2} + V_\infty \quad on \quad H^2(\mathbb{R}) \tag{6.23}$$

we have

$$\sigma_{ess}(H_\infty) = \{-\kappa_j^2\}'_{j\in\mathbb{N}} \cup [0,\infty), \tag{6.24}$$

$$\sigma_{ac}(H_\infty) = [0,\infty), \tag{6.25}$$

$$[\sigma_p(H_\infty) \cup \sigma_{sc}(H_\infty)] \cap (0,\infty) = \emptyset, \tag{6.26}$$

$$\{-\kappa_j^2\}_{j\in\mathbb{N}} \subseteq \sigma_P(H_\infty) \subseteq \overline{\{-\kappa_j^2\}_{j\in\mathbb{N}}}. \tag{6.27}$$

The spectral multiplicity of H_∞ on $(0,\infty)$ equals two while $\sigma_p(H_\infty)$ is simple. In addition, if $\{\kappa_j\}_{j\in\mathbb{N}}$ is a discrete subset of $(0,\infty)$ (i.e., 0 is its only limit point) then

$$\sigma_{sc}(H_\infty) = \emptyset, \tag{6.28}$$

$$\sigma(H_\infty) \cap (-\infty, 0) = \sigma_d(H_\infty) = \{-\kappa_j^2\}_{j\in\mathbb{N}}. \tag{6.29}$$

More generally, if $\{\kappa_j\}'_{j\in\mathbb{N}}$ is countable then (6.28) holds.

Here A' denotes the derived set of $A \subset \mathbb{R}$ (i.e., the set of accumulation points of A) and $\sigma_d(.)$ denotes the discrete spectrum (cf. also the paragraph following (6.17)).

We refer to [29] for a complete proof of this result. Here we only mention that the condition $\{c_j^2/\kappa_j\}_{j\in\mathbb{N}} \in l^1(\mathbb{N})$ implies convergence in trace norm topology of the $N \times N$ matrix $C_N(x)$ (see (6.12)) embedded into $l^2(\mathbb{N})$ to the trace class operator $C_\infty(x)$ in $l^2(\mathbb{N})$ given by

$$C_\infty(x) = \left[\frac{c_l\, c_{l'}}{\kappa_l + \kappa_{l'}}\, e^{-(\kappa_l + \kappa_{l'})x}\right]_{l,l'\in\mathbb{N}}. \tag{6.30}$$

Moreover, one has in analogy to (6.11),

$$V_\infty(x) = -2\frac{d^2}{dx^2}\ \ln\ \det{}_1[1 + C_\infty(x)], \tag{6.31}$$

where $\det_1(.)$ denotes the Fredholm determinant associated with $l^2(\mathbb{N})$.

We emphasize that Theorem 6.1 solves the following inverse spectral problem: *Given any bounded and countable subset* $\{-\kappa_j^2\}_{j\in\mathbb{N}}$ *of* $(-\infty,0)$, *construct a (smooth and real–valued) potential* V *such that* $H=-\frac{d^2}{dx^2}+V$ *has a purely absolutely continuous spectrum equal to* $[0,\infty)$ *and the set of eigenvalues of* H *includes the prescribed set* $\{-\kappa_j^2\}_{j\in\mathbb{N}}$. (In particular, $\{-\kappa_j^2\}_{j\in\mathbb{N}}$ can be dense in a bounded subset of $(-\infty,0)$.)

Under the stronger hypothesis $\{\kappa_j\}_{j\in\mathbb{N}}\in l^1(\mathbb{N})$ one obtains

Theorem 6.2 [28], [29] *Assume* $\{\kappa_j>0\}_{j\in\mathbb{N}}\in l^1(\mathbb{N})$, $\kappa_j\neq\kappa_{j'}$ *for* $j\neq j'$ *and choose* $\{c_j>0\}_{j\in\mathbb{N}}$ *such that* $\{c_j^2/\kappa_j\}_{j\in\mathbb{N}}\in l^1(\mathbb{N})$. *Then in addition to the conclusions of Theorem 6.1 we have*

(i)

$$\lim_{n\to\infty}\|V_N^{(m)}-V_\infty^{(m)}\|_p=0,\quad 1\le p\le\infty,\quad m\in\mathbb{N}\cup\{0\}. \tag{6.32}$$

(ii)

$$\sigma_{ess}\,(H_\infty)=\sigma_{ac}(H_\infty)=[0,\infty), \tag{6.33}$$

$$\sigma_p\,(H_\infty)\cap(0,\infty)=\sigma_{sc}(H_\infty)=\emptyset, \tag{6.34}$$

$$\sigma_d\,(H_\infty)=\{-\kappa_j^2\}_{j\in\mathbb{N}}. \tag{6.35}$$

The unitary scattering matrix $S_\infty(k)$ in $\mathbb{C}^2$ associated with the pair (H_∞,H_0) is reflectionless and given by

$$\begin{aligned} S_\infty(k) &= \begin{pmatrix} T_\infty(k) & 0\\ 0 & T_\infty(k)\end{pmatrix}, \\ T_\infty(k) &= \prod_{j=1}^{\infty}\left(\frac{k+i\kappa_j}{k-i\kappa_j}\right),\ k\in\mathbb{C}\backslash\{\{i\kappa_j\}_{j\in\mathbb{N}}\cup\{0\}\}. \end{aligned} \tag{6.36}$$

Note that Theorem 6.2 constructs a new class of reflectionless potentials involving an infinite negative point spectrum of H_∞ accumulating at zero.

For a detailed proof of Theorem 6.2 see [29]. We remark that the condition $\{\kappa_j\}_{j\in\mathbb{N}}\in l^1(\mathbb{N})$ implies that $V_\infty\in L^1(\mathbb{R})$ (but $V_\infty\notin$

$L^1(\mathbb{R};\,(1+|x|)\,dx))$ and that the product $T_N(k)$ converges absolutely to $T_\infty(k)$ as $N\to\infty$.

We conclude with the observation that the simple substitution

$$c_j \longrightarrow c_j\, e^{\kappa_j^3\, t}, \quad j\in\mathbb{N} \tag{6.37}$$

in (6.30) and (6.31), denoting the result in (6.31) by $V_\infty(x,t)$, produces solutions of the KdV_1 equation (see (2.8))

$$KdV_1(V_\infty) = V_{\infty,t} + \frac{1}{4}\,V_{\infty,xxx} - \frac{3}{2}\,V_\infty\,V_{\infty,x} = 0. \tag{6.38}$$

In particular, substitutions of the type (6.37) together with Theorem 6.2 provide new soliton solutions of the KdV hierarchy [28],[29].

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