# On Gelfand-Dickey Systems and Inelastic Solitons

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## 1 Introduction

The Boussinesq equation

$$u_{tt} = (u^2)_{xx} + u_{xxxx}$$

is known to have so called $N$-soliton solutions, i.e., solutions that exhibit asymptotically (as $t \to \pm\infty$) $N$ solitary waves of the typical $\mathrm{sech}^2$-form (see Hirota [5]). Here I am mainly interested in (a scaled version of) the Boussinesq equation in imaginary time, specifically,

$$u_{tt} = -\frac{2}{3}(u^2)_{xx} - \frac{1}{3}u_{xxxx}. \tag{1}$$

This equation renders "inelastic solitons", i.e., solitary waves of the $\mathrm{sech}^2$-form which may stick together after interaction thus forming a new $\mathrm{sech}^2$-wave (see Figure 1).

These inelastic solitons can be obtained via an auto-Bäcklund transformation for the Gelfand-Dickey system associated with the Boussinesq-type equation.[1] In the following I will define what Gelfand-Dickey systems and their "modified" counterparts, the Drinfeld-Sokolov systems, are. Section 2 then reviews the above mentioned

[1]As I realized only after finishing this work these solutions were obtained earlier by Tajiri and Nishitani (*J. Phys. Soc. J.*, 51:3720–3723, 1982) and by Lambert, Musette and Kesteloot (*Inv. Prob.*, 3:275–288, 1987) using different methods. However, the construction of these solutions in the present context should be viewed as an illustration of how the auto-Bäcklund transformation of Section 2 works.

auto-Bäcklund transformation (see [3] and [4]). Section 3 describes briefly (details will appear elsewhere) how the inelastic solitons are constructed.

Gelfand-Dickey systems are most easily defined in terms of Lax pairs. By a Lax pair is meant a pair of two ordinary differential expressions

$$\begin{aligned} L &= \partial_x^n + q_{n-2}\partial_x^{n-2} + ... + q_0, \\ P &= \partial_x^r + p_{r-2}\partial_x^{r-2} + ... + p_0, \end{aligned}$$

which are almost commuting, i.e., their commutator $[P, L]$ is a differential expression of order $n-2$ only. Under an additional homogeneity condition it is always possible to find uniquely coefficients $p_j$, $j = 0, ..., r-2$ such that this holds (Wilson [7]). This distinguishes between $n$ and $r$ and causes the two operators to play very different roles. The Lax equation

$$\frac{dL}{dt} = [P, L]$$

is then equivalent to a system of nonlinear evolution equations which is called a Gelfand-Dickey system. In particular the well-known KdV equation is recovered in the case $n = 2$ and $r = 3$, while the case $n = 3$ and $r = 2$ yields the Boussinesq-type equation (1).

An important ingredient in the construction of the auto-Bäcklund transformation is another system of evolution equations, the Drinfeld-Sokolov system which is defined as follows: Given functions $\phi_i(x, t)$, $i = 1, ..., n$ such that their sum is identically equal to zero, construct the matrix

$$M = \begin{pmatrix} 0 & ........ & 0 & \partial_x + \phi_n \\ \partial_x + \phi_1 & ........ & 0 & 0 \\ 0 & ........ & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & ........ & \partial_x + \phi_{n-1} & 0 \end{pmatrix}.$$

Then $M^n = \mathrm{diag}(L_1, ..., L_n)$ where each $L_j$ has the form of the above $L$:

$$\begin{aligned} L_j &= (\partial_x + \phi_{n+j-1})...(\partial_x + \phi_j) \\ &= \partial_x^n + q_{j,n-2}\partial_x^{n-2} + ... + q_{j,1}\partial_x + q_{j,0}. \end{aligned} \tag{2}$$

Note that $L_j$ is obtained from $L_{j-1}$ by commuting the first $n-1$ factors with the last one. This basic idea of commutation goes back to Darboux and was used by Deift [1] to construct the $N$-soliton solution of the KdV equation.

Now let $Q = \mathrm{diag}(P_1, P_2, ..., P_n)$ where $P_j,\ j = 1, ..., n$ is the uniquely defined differential expression of order $r$ that almost commutes with $L_j$. Then

$$\frac{dM}{dt} = [Q, M]$$

is equivalent to a system of $n-1$ nonlinear evolution equations, called the Drinfeld-Sokolov system or modified Gelfand-Dickey system.

## 2 An Auto-Bäcklund Transformation

Given a solution of the Drinfeld-Sokolov system, i.e., a set of $\phi_j$, $j = 1, ..., n$ such that $dM/dt = [Q, M]$ then it is easy to see that this implies $d(M^n)/dt = [Q, M^n]$, which is equivalent to

$$\frac{dL_j}{dt} = [P_j, L_j], \quad j = 1, ..., n.$$

This means one has found $n$ solutions of the associated Gelfand-Dickey system. This observation is due to Sokolov and Shabat [6]. Now the following question arises: Is it possible to reverse this process and to construct a solution $\phi_j,\ j = 1, ..., n$ of the Drinfeld-Sokolov system given a solution of the Gelfand-Dickey system? If so then one has immediately $n-1$ new solutions of the Gelfand-Dickey system. It is precisely this question which was answered affirmatively by Gesztesy and Simon in [2] in the case of the KdV equation and by Gesztesy, Race and myself in [3] in the case of a Boussinesq-type equation.

The answer in the general case was given in [4]. The method there allows the coefficients of $L$ to be matrices with entries in some commutative algebra with two independent derivations. For simplicity, however, I give in the following the scalar version using just functions of $x$ and $t$ as coefficients of $L$.

**Theorem 1 (Gesztesy, Race, Unterkofler, W.)** *Suppose that* $(q_{n-2}, ..., q_0)$ *is a real-valued solution of the Gelfand-Dickey system. Also assume that the* $q_i$ *and their* $x$*-derivatives up to order* $r+i$ *are continuous functions in* $\mathbb{R}^2$*. Let* $\psi_1, ..., \psi_n$ *be a fundamental system of solutions of* $L\psi = 0$ *and* $\psi_t = P\psi$ *and define* $\phi_1, ..., \phi_n$ *according to*

$$\phi_k = -\frac{\partial}{\partial x} \log \left| \frac{W_k}{W_{k-1}} \right|$$

*where* $W_0 = 1$ *and* $W_k = W(\psi_1, ..., \psi_k)$*, the Wronskian of* $\psi_1, ..., \psi_k$ *for* $k = 1, ..., n$*. (Note that this implies that* $\sum_{k=1}^n \phi_k = 0$*.) Then* $(\phi_1, ..., \phi_n)$ *satisfies the Drinfeld-Sokolov system. Furthermore define* $q_{k,i}$*,* $k = 1, ..., n$*,* $i = 0, ..., n-2$ *through (2). Then each tuple* $(q_{k,n-2}, ..., q_{k,0})$ *satisfies the Gelfand-Dickey system. In particular* $(q_{1,n-2}, ..., q_{1,0}) = (q_{n-2}, ..., q_0)$*.*

One can allow for an "energy" parameter $\lambda$ and consider $L\psi = \lambda\psi$ instead of $L\psi = 0$. The method can now be applied repeatedly to construct new solutions in each step, i.e., new operators $L_{j,1} = L_{j-1,2}$ starting from a given $L_{0,1}$. This way one may derive the following formula

$$q_{j,1,n-2} = q_{0,1,n-2} + n(\log W(\psi_{1,1}, ..., \psi_{j,1}))_{xx}, \tag{3}$$

where $q_{i,1,n-2}$ is the leading non-trivial coefficient in $L_{i,1}$ and $\psi_{i,1}$ is a solution of $L_{0,1}\psi = \lambda_i\psi$ and $P_{0,1}\psi = \psi_t$.

In general the solutions constructed by the method described above may have singularities since the Wronskians used may have zeros. In the KdV case as well as in the Boussinesq-type case it is possible to show that under certain conditions the new solutions inherit some properties from the original solution.

**Theorem 2 (Gesztesy, Race, W.)** *Let* $(q_1, q_0)$ *be such that the Gelfand-Dickey system for* $n = 3$*,* $r = 2$ *is satisfied. Furthermore assume that* $q_i, ..., q_i^{(3+i)}$ *are in* $C^0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *and that* $L\psi = 0$ *is disconjugate at time* $t_0$*.*

*Then* $L\psi = 0$ *is disconjugate at all times. Moreover, for a suitable choice of a solution system* $(\psi_1, ..., \psi_n)$*, the solutions constructed in*

*Theorem 1 satisfy the same smoothness and boundedness conditions as the original one, in particular there are no local singularities.*

A similar result was proven by Gesztesy and Simon [2] for the KdV case.

## 3 Inelastic Solitons

A solution of the Gelfand-Dickey system for $n = 3$ and $r = 2$

$$q_{1t} = 2q_{0x} - q_{1xx}, \quad q_{0t} = q_{0xx} - \frac{2}{3}(q_{1xxx} + q_1 q_{1x})$$

yields at once a solution of the Boussinesq-type equation

$$u_{tt} = -\frac{1}{6}a(u^2)_{xx} + bu_{xx} - \frac{1}{3}u_{xxxx}$$

upon letting $u = (4q_1 + 3b)/a$ $(a \neq 0)$. The Lax pair associated to this Gelfand-Dickey system is

$$L = \partial_x^3 + q_1\partial_x + q_0, \quad P = \partial_x^2 + \frac{2}{3}q_1.$$

Starting now from the trivial solution where both coefficients $q_1 = q_{1,1}$ and $q_0 = q_{1,0}$ of $L = L_1$ are constant, new nontrivial solutions of the Boussinesq-type equation are constructed. The coefficient $q_{2,1}$ of $L_2$ is given in terms of one solution $\psi_1$ of $L\psi = 0$ and $P\psi = \psi_t$ as

$$q_{2,1} = q_{1,1} + 3(\log \psi_1)_{xx}.$$

A fundamental system of solutions of $L\psi = 0$ and $P\psi = \psi_t$ is of course given by a set of exponential functions. If $\psi_1$ is now chosen to be one of these exponential functions then $q_{2,1} = q_{1,1}$, i.e., no new solution is constructed. If $\psi_1$ is chosen to be a linear combination of two of these exponentials then one obtains a one-soliton solution, i.e., a $\mathrm{sech}^2$-wave. This solution, however, involves two parameters instead of one in the Boussinesq case.

However if one linearly combines all three of the exponentials then something unexpected happens: initially there are two solitons well separated moving with constant velocity towards each other. When they eventually get into the same region they collide inelastically, i.e., one soliton only emerges after the interaction. This situation is shown in Figure 1, where $q_{2,1} - q_{1,1}$ is plotted as a function of $x$ for five different $t$. Defining the mass of a soliton to be the product of height and width then mass as well as momentum are conserved during this collision but (kinetic) energy gets destroyed.

Considering $q_{3,1}$ instead of $q_{2,1}$ or performing the transformation $t \to -t$ shows that one can also have the reverse situation, namely a single soliton moving along that all of a sudden decays into two different solitons under conservation of mass and momentum but producing kinetic energy while it decays.

Finally using the method of repeated commutation, i.e., formula (3) one can construct other interesting solutions. In the case $j = 2$ one gets according to the different possibilities of linearly combining $\psi_{1,1}$ and $\psi_{2,1}$ out of appropriate exponential functions besides the already known two further phenomena:

- Two elastically interacting solitons moving towards each other or following each other. In contrast to the Boussinesq case the smaller one is here the faster one. This situation is shown in Figure 2.

- Three solitons two of which collide inelastically forming one soliton after the collision while the third interacts elastically with both of the other two. This situation is shown in Figure 3.

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Figure 1: Two inelastically colliding solitons.

Figure 2: Two solitons interacting elastically the smaller one being faster than the bigger one.

Figure 3: Three solitons, two of which collide inelastically while the third one is interacting elastically with both of the others.

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