ERROR ESTIMATES FOR NON-SELFADJOINT INVERSE STURM-LIOUVILLE PROBLEMS WITH FINITE SPECTRAL DATA

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1. Introduction

This extended abstract is a summary of the main results in [4]. We consider a stability result for the inverse problem associated with the Sturm-Liouville equation

$$-y'' + q_0(x)y = \lambda y, \quad x \in (0,1),$$

in which the potential $q_0 \in L^2(0,1)$ is allowed to be complex-valued and the spectral data consists of the first N Dirichlet-Dirichlet eigenvalues and the first N Dirichlet-Neumann eigenvalues, determined to within an accuracy ε. As the spectral data is finite the problem may be expected to have infinitely many solutions (this appears to be unproven in the non-selfadjoint case). The usual philosophy in the numerical analysis literature is to construct recovery algorithms which select one of the infinitely many possible solutions. Numerical experiments are then carried out in which finite spectral data are generated from some known potential and the algorithm is declared to be good or bad according to how well it manages to recover the selected potential, in some norm. This process is meaningless unless one can prove that all of the infinitely many solutions to the finite data inverse problems are 'close', in some suitable sense. The point of this article is to establish such results. For reviews of reconstruction methods for inverse Sturm-Liouville problems see Rundell [10] and McLauglin [6].

There appears to be little published on stability for inverse Sturm-Liouville problems with finite data. For full data and a real potential, Ryabushko [12] proves the result

$$\|q_1 - q_2\|_2 \leq C(\|\lambda(q_1) - \lambda(q_2)\|_2 + \|\mu(q_1) - \mu(q_2)\|_2) \tag{1}$$

where λ is the sequence of Dirichlet-Dirichlet eigenvalues, μ is the sequence of Dirichlet-Neumann eigenvalues, and $\|\cdot\|_2$ denotes either the norm in $L^2(0,1)$ or in $\ell^2(\mathbb{N})$ as appropriate. The potentials q_1 and q_2 are assumed to have the same mean value. McLaughlin [5] proved that when the average value of the potential is zero, there is a local diffeomorphism between the potential in $L^2(0,1)$ and the sequences $\{\lambda_n - n^2\pi^2, \rho_n\}$ in $\ell^2 \times \ell^2$, where $\{\rho_n\}$ are the 'norming constants'. The closest result to ours, in spirit, is that of Hitrik [1], which concerns an inverse scattering problem in $L^2(\mathbb{R})$ when finitely many values of the reflection coefficient are known.

Notation We use the notation $f^{(n,m)}(x_0, y_0)$ to denote the value at (x_0, y_0) of the partial derivative $\frac{\partial^{n+m} f}{\partial x^n \partial y^m}$, since it will be particularly important to indicate the points at which partial derivatives are evaluated.

2. Statement of the main result

Assume q_0 and q are complex-valued functions in $L^2([0,1])$. Let $\lambda_j(q)$, $j \in \mathbb{N}$ denote the eigenvalues of the boundary value problem

$$-y'' + qy = \lambda y, \quad y(0) = 0, \quad y(1) = 0$$

and assume that they are repeated according to their algebraic multiplicities. Similarly, let $\mu_j(q)$, $j \in \mathbb{N}$ be the eigenvalues of the boundary value problem

$$-y'' + qy = \lambda y, \quad y(0) = 0, \quad y'(1) = 0$$

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also repeated according to their algebraic multiplicities. The quantities $\lambda_j(q_0)$ and $\mu_j(q_0)$ denote the eigenvalues of those problems where q is replaced by q_0. We will always assume that these eigenvalues are labeled in such a way that identical values are adjacent and that their moduli form nondecreasing sequences.

The main result of [4] is the following:

Theorem 2.1. *Assume q_0 and q are complex-valued functions in $L^2([0,1])$ with the same mean value. Define $a_j = |\lambda_j(q) - \lambda_j(q_0)|$ and $b_j = |\mu_j(q) - \mu_j(q_0)|$ and let $\varepsilon_0 > 0$ and $N_0 \in \mathbb{N}$ be fixed. Then there exists a constant C, depending only on q_0, ε_0, and N_0 such that the following is true:*

If $0 \le \varepsilon \le \varepsilon_0$, $N \ge N_0$, and $\max\{a_1, ..., a_N, b_1, ..., b_N\} \le \varepsilon$ then

$$\left| \int_0^x (q(t) - q_0(t))dt \right| \le C \exp(\|q\|_2) \left(\varepsilon \log N + \frac{\|a\|_2 + \|b\|_2}{N^{1/2}} \right)$$

for all $x \in [0,1]$.

In fact this result is obtained as a result of a proof which also establishes the following.

Theorem 2.2. *With the notation of Theorem 2.1, for all $x \in [0,1]$*

$$\left| \int_0^x (q(t) - q_0(t))dt \right| \le C \exp(\|q\|_2) \left(\left\| \{a_j/j\}_{j=1}^\infty \right\|_1 + \left\| \{b_j/j\}_{j=1}^\infty \right\|_1 \right).$$

3. The transformation operator

We introduce the transformation operator (see, e.g., Levitan [3]). Let $\mathcal{D}_0$ and $\mathcal{D}$ be the sets

$$\mathcal{D}_0 = \{y \in \mathrm{AC}([0,1]) : y' \in \mathrm{AC}([0,1]), -y'' + q_0 y \in L^2([0,1]), y(0) = 0\}$$

and

$$\mathcal{D} = \{Y \in \mathrm{AC}([0,1]) : Y' \in \mathrm{AC}([0,1]), -Y'' + qY \in L^2([0,1]), Y(0) = 0\}.$$

Then there exists an integral operator $\mathcal{K} : \mathcal{D}_0 \to \mathcal{D}$, the transformation operator, defined by

$$Y(x) = (\mathcal{K}y)(x) = y(x) + \int_0^x K(x,t)y(t)dt \tag{2}$$

such that $-(\mathcal{K}y)'' + q\mathcal{K}y = \mathcal{K}(-y'' + q_0 y)$ for all $y \in \mathcal{D}_0$. The kernel of this operator can be characterized in two ways. Firstly, it admits an expansion

$$K(x,t) = \sum_{n=0}^\infty K_n(x,t)$$

where

$$K_0(x,t) = \frac{1}{2} \int_{(x-t)/2}^{(x+t)/2} (q(s) - q_0(s))ds$$

and

$$K_n(x,t) = \int_{(x-t)/2}^{(x+t)/2} \int_0^{(x-t)/2} (q(\alpha+\beta) - q_0(\alpha-\beta))K_{n-1}(\alpha+\beta, \alpha-\beta)d\beta d\alpha.$$

Secondly, it can also be written as follows. Let $f = K(1,\cdot)$ and $g = K_x(1,\cdot) = K^{(1,0)}(1,\cdot)$ and extend f and g to odd functions on $[-1,1]$. f is absolutely continuous and g is integrable. Let

$$\tilde{K}_0(x,t) = \frac{1}{2} \int_{x-t-1}^{x+t-1} (f'(s) + g(s))ds \tag{3}$$

and

$$\tilde{K}_n(x,t) = \frac{1}{2} \int_x^1 \int_{t+x-u}^{t-x+u} (q(u) - q_0(v))\tilde{K}_{n-1}(u,v)dvdu \tag{4}$$

Then one has the second representation of $K(x,t)$ given by

$$K(x,t) = \sum_{n=0}^{\infty} \tilde{K}_n(x,t) \tag{5}$$

By induction, the functions $\tilde{K}_n$ may be shown to satisfy the inequality

$$|\tilde{K}_n(x,t)| \leq \|\tilde{K}_0\|_\infty \frac{Q^n(1-x)^{3n/2}}{n!} \tag{6}$$

where $Q = (\|q\|_2^2 + \|q_0\|_2^2)^{1/2} \leq \|q\|_2 + \|q_0\|_2$. As an immediate consequence of this result we obtain

Theorem 3.1. *Suppose that* $q, q_0 \in L^2([0,1])$. *Then*

$$\left|\int_0^x (q-q_0)(s)ds\right| = 2|K(x,x)| \leq 4\exp(\|q\|_2 + \|q_0\|_2)\|\tilde{K}_0\|_\infty.$$

□

In order to connect this estimate to the differences $\lambda_j(q) - \lambda_j(q_0)$ and $\mu_j(q) - \mu_j(q_0)$ we use the definition (3) of $\tilde{K}_0$ in terms of the functions $f = K(1,\cdot)$ and $g = K_x(1,\cdot)$. The problem of estimating $\|\tilde{K}_0\|_\infty$ is reduced to the problem of estimating $\|f\|_\infty$ and $\|G\|_\infty$, where $G(\xi) = \int_0^\xi g(s)ds$. To do this we introduce some notation.

We denote by $s(\lambda,\cdot)$ the solution of the initial value problem

$$-y'' + qy = \lambda y, \quad y(0) = 0, \quad y'(0) = 1$$

and by $s_0(\lambda,\cdot)$ the corresponding solution for the potential q_0. The $\lambda_j(q)$ are the zeros of $s(\cdot,1)$ while the $\mu_j(q)$ are the zeros of $s'(\cdot,1)$. The transformation equation (2) yields

$$s(\lambda,x) = s_0(\lambda,x) + \int_0^x K(x,t)s_0(\lambda,t)dt. \tag{7}$$

Algebraically simple eigenvalues. Suppose that all the $\lambda_k(q_0)$ are algebraically simple. Evaluating (7) at $x=1$, $\lambda = \lambda_k(q_0)$ yields, upon recalling that $K(1,t) = f(t)$,

$$s(\lambda_k(q_0),1) = \int_0^1 f(t)s_0(\lambda_k(q_0),t)dt. \tag{8}$$

Similarly, evaluating at $\lambda_k(q)$ yields

$$0 = s_0(\lambda_k(q),1) + \int_0^1 f(t)s_0(\lambda_k(q),t)dt. \tag{9}$$

Define

$$\alpha_k := \sqrt{2}\,k\pi \int_0^1 f(t)s_0(\lambda_k(q_0),t)dt. \tag{10}$$

Equations (8,9,10) together with the fact that $s_0(\lambda_k(q_0),1) = 0$ imply that

$$\frac{\alpha_k}{\sqrt{2}\,k\pi} = s_0(\lambda_k(q_0),1) - s_0(\lambda_k(q),1) + \int_0^1 f(t)\left(s_0(\lambda_k(q_0),t) - s_0(\lambda_k(q),t)\right)dt. \tag{11}$$

Elementary estimates now yield

$$\begin{aligned}\frac{|\alpha_k|}{\sqrt{2}\,k\pi} &\leq (1+\|f\|_1)\|s_0(\lambda_k(q_0),\cdot) - s_0(\lambda_k(q),\cdot)\|_\infty \\ &\leq (1+\|f\|_1)|\lambda_k(q_0) - \lambda_k(q)| \sup_{\lambda\in[\lambda_k(q_0),\lambda_k(q)]} \left\|s_0^{(1,0)}(\lambda,\cdot)\right\|_\infty.\end{aligned} \tag{12}$$

(Note that the interval $[\lambda_k(q_0),\lambda_k(q)]$ is, in general, a straight line-segment in $\mathbb{C}$.) Further progress now depends on asymptotic estimates of the solution s_0 and of its partial derivatives with respect to λ, as well as the corresponding estimates of the eigenvalues $\lambda_j(q)$ and $\lambda_j(q_0)$. The following two results may be proved by standard techniques. The proofs are given in [4]. Note that the results do not assume self-adjointness of the problem.

Lemma 3.2. *Let $z^2 = \lambda$, $\mathrm{Im}(z) \geq 0$, and suppose $q_0 \in L^1(0,1)$. Then there exist constants C and $\tilde{C}$ not depending on q_0 such that*

$$\left| s_0(\lambda, x) - \frac{\sin(zx)}{z} \right| \leq C|z|^{-1} \left\{ \exp\left(|z|^{-1} \int_0^x |q_0(t)|dt \right) - 1 \right\} \leq \frac{\tilde{C}\|q_0\|_2}{|z|^2}, \quad (|z| \to \infty); \tag{13}$$

moreover for every $k \in \mathbb{N}_0$ there exists a constant c_k not depending on q_0 such that

$$|s_0^{(k,0)}(\lambda, x)| \leq c_k \mathrm{e}^{\mathrm{Im}(z)x} |z|^{-k-1} \tag{14}$$

and

$$|s_0^{(k,1)}(\lambda, x)| \leq c_k \mathrm{e}^{\mathrm{Im}(z)x} |z|^{-k}. \tag{15}$$

Lemma 3.3. *Suppose that q_0 and q lie in $L^2(0,1)$. Then there exist sequences $(\beta_j(q_0))$ and $(\beta_j(q))$ in ℓ^2 such that*

$$\lambda_j(q_0) = j^2\pi^2 + \beta_j(q_0), \quad \lambda_j(q) = j^2\pi^2 + \beta_j(q).$$

In particular, for all sufficiently large j the eigenvalues $\lambda_j(q_0)$ and $\lambda_j(q)$ are simple.

Substituting these results back into (12) yields

$$|\alpha_k| \leq C(Q)k\pi(1 + \|f\|_1)\frac{|\lambda_k(q) - \lambda_k(q_0)|}{k^2\pi^2} \leq \pi^{-1}C(Q)(1 + \|f\|_2)\frac{a_k}{k} \tag{16}$$

where $C = C(Q)$ is a constant depending only on $Q = (\|q\|_2^2 + \|q_0\|_2^2)^{1/2}$. We now make some observations about the numbers α_k. Let $\phi_k = k\pi\sqrt{2}s_0(\lambda_k(q_0), \cdot)$. Then from Lemma 3.2, eqn. (13), the ϕ_k are quadratically close to orthonormal and hence may be shown to form a Riesz basis of $L^2(0,1)$ (Mihailov [9]). Consequently, the set of functions

$$\psi_k(x) = \overline{\phi_k(x)},$$

which are eigenfunctions of the problem with q_0 replaced by $\overline{q_0}$, also form a Riesz basis of $L^2(0,1)$. Expanding f in a generalized Fourier series

$$f = \sum_{k=1}^{\infty} \gamma_k \psi_k, \tag{17}$$

the coefficients γ_k are given by

$$\gamma_k = \frac{(f, \phi_k)}{(\psi_k, \phi_k)} = \frac{\alpha_k}{(\psi_k, \phi_k)} = \alpha_k(1 + o(1)). \tag{18}$$

Consequently

$$\|f\|_2 \leq C \left(\sum_{k=1}^{\infty} |\alpha_k|^2 \right)^{1/2} \leq C(Q)(1 + \|f\|_2) \left\| \{a_k/k\}_{k=1}^{\infty} \right\|_2.$$

Provided the norm

$$\|\{a_k/k\}_{k=1}^{\infty}\|_2$$

is sufficiently small (less than $1/(2C(Q))$, for instance) this establishes a bound on $\|f\|_2$. From (16) and (18) it now follows that for some constant $C = C(Q)$,

$$|\gamma_k| \leq C(Q)\frac{a_k}{k}. \tag{19}$$

The functions ψ_k may be shown to satisfy a bound

$$\sup_{k \in \mathbb{N}} \|\psi_k\|_\infty < +\infty. \tag{20}$$

As an immediate corollary of (17), (19) and (20) we have the following result.

Lemma 3.4. *For some constant $C = C(Q)$,*

$$\|f\|_\infty \le C(Q) \left\| \{a_k/k\}_{k=1}^\infty \right\|_1 . \tag{21}$$

We now seek a similar bound on $\|G\|_\infty$, where $G(\xi) = \int_0^\xi g(s)ds$.

Differentiating (7) with respect to x and evaluating at $x = 1$,

$$s'(\lambda, 1) = s_0'(\lambda, 1) + \int_0^1 g(t)s_0(\lambda, t)dt$$

where the boundary condition $K(1,1) = 0$ has been used - this being an immediate consequence of the fact that $2K(x,x) = \int_0^x (q(s) - q_0(s))ds$ and the assumption that q and q_0 have the same mean value. Next observe that

$$s'(\mu_k(q_0), 1) = \int_0^1 g(t)s_0(\mu_k(q_0), t)dt = \frac{(g, \omega_k)}{\sqrt{2}k\pi},$$

where $\omega_k = \sqrt{2}k\pi\overline{s_0(\mu_k(q_0), \cdot)}$. Also,

$$0 = s_0'(\mu_k(q), 1) + \int_0^1 g(t)s_0(\mu_k(q), t)dt.$$

Defining $\beta_k = (g, \omega_k)$ we obtain

$$\frac{\beta_k}{\sqrt{2}\ k\pi} = \int_0^1 g(t)[s_0(\mu_k(q_0), t) - s_0(\mu_k(q), t)]dt + [s_0'(\mu_k(q_0), 1) - s_0'(\mu_k(q), 1)].$$

Recalling that s_0' is $s_0^{(0,1)}$, we use the results of Lemmas 3.2 and 3.3 to obtain, by reasoning similar to that used to obtain (16),

$$|\beta_k| \le C(Q)\ k \left[\frac{b_k}{k^2}\|g\|_2 + \frac{b_k}{k}\right] \tag{22}$$

where $b_k = |\mu_k(q) - \mu_k(q_0)|$. The quantity $\|g\|_2$ may be bounded a priori by the same reasoning as we used to bound $\|f\|_2$, giving

$$|\beta_k| \le C(Q)b_k. \tag{23}$$

As in the arguments which we followed for dealing with f, the functions ω_k form a Riesz basis, as do the functions $\varphi_k = \overline{\omega_k}$. Expanding G as

$$G(\xi) = \sum_{k=1}^\infty \eta_k\varphi_k(\xi), \tag{24}$$

the coefficients in (24) are given by

$$\eta_k = \frac{(G, \omega_k)}{(\phi_k, \omega_k)} = \frac{\beta_k}{(\phi_k, \omega_k)} = \beta_k(1 + o(1)).$$

In view of (23) we therefore have

$$|\eta_k| \le C(Q)b_k.$$

Now

$$|G(\xi)| = \left|\int_0^\xi g(s)ds\right| = |(\chi_{[0,\xi]}, g)| \le \sum_{k=1}^\infty |\eta_k||(\chi_{[0,\xi]}, \varphi_k)|.$$

The asymptotic estimates which yield

$$\varphi_k(x) \sim \sqrt{2}\sin(\sqrt{\mu_k(q_0)}x)$$

also give

$$(\chi_{[0,\xi]}, \varphi_k) = \int_0^\xi \varphi_k(x)dx \sim \frac{\sqrt{2}}{\sqrt{\mu_k(q_0)}}(1 - \cos(\sqrt{\mu_k(q_0)}\xi))$$

which is an $O(1/k)$ quantity. Hence we have

Lemma 3.5. *The function G defined by $G(\xi) = \int_0^\xi g(s)ds$ satisfies*

$$\|G\|_\infty \le C(Q) \left\|\{b_k/k\}_{k=1}^\infty\right\|_1 .$$

We can now prove Theorems 2.1 and 2.2, at least for the case of simple eigenvalues.

Proof of Theorems 2.1 and 2.2. In view of Theorem 3.1 and the remarks which follow it, for all $x \in [0,1]$

$$\left|\int_0^x (q-q_0)(s)ds\right| \le 4\exp(Q)\left\{\|f\|_\infty + \|G\|_\infty\right\}.$$

Lemmas 3.4 and 3.5 now yield the result of Theorem 2.2. In order to obtain the proof of Theorem 2.1, we make the estimates

$$\begin{aligned}
\left\|\{a_j/j\}_{j=1}^\infty\right\|_1 &= \sum_{j=1}^N \frac{a_j}{j} + \sum_{j=N+1}^\infty \frac{a_j}{j} \\
&\le \varepsilon \sum_{j=1}^N \frac{1}{j} + \left(\sum_{j=N+1}^\infty |a_j|^2\right)^{1/2} \left(\sum_{j=N+1}^\infty j^{-2}\right)^{1/2} \\
&\le \varepsilon \log(N+1) + \frac{\|(a_j)\|_2}{N^{1/2}}
\end{aligned}$$

and similarly for $\left\|\{b_j/j\}_{j=1}^\infty\right\|_1$. These yield the result of Theorem 2.1.

Multiple eigenvalues. The case of multiple eigenvalues involves various technicalities which are described in detail in [4]. Here we only mention them briefly. Since the problem can possess at most finitely many multiple eigenvalues we may assume without loss of generality that any multiple eigenvalue is one of the first N which are approximated with accuracy ε.

Let $\lambda_\kappa(q_0)$ be an eigenvalue of multiplicity $\nu > 0$. We may assume that there are ν eigenvalues $\lambda_\kappa(q), \ldots, \lambda_{\kappa+\nu-1}(q)$, counted according to algebraic multiplicity and therefore not necessarily distinct, in a disc of centre $\lambda_\kappa(q_0)$ and radius ε. For each t, let $p(\lambda,t)$ and $p_0(\lambda,t)$ be, respectively, the unique polynomials of degree at most $(\nu-1)$ interpolating $s(\lambda,t)$ and $s_0(\lambda,t)$ at the points $\lambda_\kappa(q), \ldots, \lambda_{\kappa+\nu-1}(q)$. From (7) we know that

$$p(\lambda,1) = p_0(\lambda,1) + \int_0^1 f(t)p_0(\lambda,t)dt$$

for $\lambda = \lambda_\kappa(q), \ldots, \lambda_{\kappa+\nu-1}(q)$ and hence for all λ, since both sides of the equation are polynomials of degree at most $\nu-1$. We can therefore differentiate this formula $\nu-j-1$ times, for $j = 0, \ldots, \nu-1$, and obtain

$$p^{(\nu-1-j,0)}(\lambda,1) = p_0^{(\nu-1-j,0)}(\lambda,1) + \int_0^1 f(t)p_0^{(\nu-1-j,0)}(\lambda,t)dt. \tag{25}$$

We now observe that since $s(\lambda,1) = 0$ at all of the points $\lambda_\kappa(q), \ldots, \lambda_{\kappa+\nu-1}(q)$, the function $p(\lambda,1)$ is identically zero. Thus the left hand side of (25) is identically zero, and in particular

$$0 = p_0^{(\nu-1-j,0)}(\lambda_\kappa(q_0),1) + \int_0^1 f(t)p_0^{(\nu-1-j,0)}(\lambda_\kappa(q_0),t)dt. \tag{26}$$

The eigenfunction expansion of f is now replaced by an expansion in terms of eigen- and associated functions. The part of this expansion associated with the root subspace of $\lambda_\kappa(q_0)$ has the form

$$\sum_{j=0}^{\nu-1} \alpha_{\kappa+j}\phi_{\kappa+j}(x), \tag{27}$$

where

$$\phi_{\kappa+j}(x) = \sum_{k=0}^j \frac{\gamma_{\kappa+j-k}}{k!} s_0^{(k,0)}(\lambda_\kappa(q_0),x),$$

and the coefficients $\gamma_{\kappa+j-k}$ are to be chosen. Define functions

$$\psi_{\kappa+j}(x) = \frac{\kappa\pi}{(\nu-j-1)!}\overline{s_0^{(\nu-j-1,0)}(\lambda_\kappa(q_0),x)}.$$

The coefficients $\gamma_{\kappa+j-k}$ can be chosen so that

$$(\psi_k, \phi_j) = \delta_{k,j}.$$

When this is done, the coefficients $\alpha_{\kappa+j}$ in (27) are given by

$$\alpha_{\kappa+j} = (f, \psi_{\kappa+j}) = \frac{\kappa\pi}{(\nu-1-j)!}\int_0^1 f(t)s_0^{(\nu-1-j,0)}(\lambda_\kappa(q_0),t)dt, \quad j = 0,1,\dots,\nu-1.$$

By virtue of (26) and the fact that $s_0^{(\nu-1-j,0)}(\lambda_j(q_0),1) = 0$ we can write this equation in terms of $R(\lambda,t) := s_0(\lambda,t) - p_0(\lambda,t)$ as

$$\alpha_{\kappa+j} = \frac{\kappa\pi}{(\nu-1-j)!}\int_0^1 f(t)R^{(\nu-1-j,0)}(\lambda_\kappa(q_0),t)dt + R^{(\nu-1-j,0)}(\lambda_\kappa(q_0),1).$$

In [4] we use a result of Markushevich to establish the estimate

$$|R^{(\nu-j-1)}(\lambda_k(q_0),t)| \le C2^\nu\varepsilon^{j+1} \sup_{|\lambda-\lambda_\kappa(q_0)|=1}\ \sup_{t\in[0,1]} |s_0(\lambda,t)|.$$

The term $\sup_{|\lambda-\lambda_\kappa(q_0)|=1}\ \sup_{t\in[0,1]} |s_0(\lambda,t)|$ is bounded by a constant, in view of the asymptotic results established for the solution s_0 of the ODE. Thus the multiple eigenvalue contributes only an $O(\varepsilon)$ term to the coefficient $\alpha_{\kappa+j}$ in the expansion of f. Again in view of L^∞-bounds on the eigen- and associated functions, this contributes only an $O(\varepsilon)$ term to $\|f\|_\infty$.

Similar arguments also hold for G when $\mu_\kappa(q_0)$ is an eigenvalue of multiplicity greater than 1.

4. Extensions

We consider the possibility of improving the result in Theorem 2.1 in two ways: by strengthening the norm, and by improving the factor of $1/\sqrt{N}$ in the error bound to something smaller.

The norm in Theorems 2.1 and 2.2 can be strengthened if one is prepared to make a-priori assumptions about the boundedness of $q-q_0$ in some stronger Sobolev space. For instance, Theorem 2.1 can be strengthened as follows.

Theorem 4.1. *Suppose that q and q_0 are complex-valued functions in in $L^2(0,1)$ with the same mean value. Suppose also that $q-q_0$ lies in a bounded set in the Sobolev space $H^n(0,1)$. Let $a_j = |\lambda_j(q) - \lambda_j(q_0)|$ and $b_j = |\mu_j(q) - \mu_j(q_0)|$. Let $\varepsilon_0 \ge 0$ and $N_0 \in \mathbb{N}$ be fixed. Then for each $-1 \le r \le n$ there exists a constant C depending only on ε_0, N_0, r and q_0 such that the following statement is true.*

If $0 \le \varepsilon \le \varepsilon_0$, $N \ge N_0$, and $\max\{a_1,...,a_N,b_1,...,b_N\} \le \varepsilon$ then

$$\|q-q_0\|_{H^r} \le C\left[\varepsilon\log N + \frac{\|a\|_2 + \|b\|_2}{\sqrt{N}}\right]^{(n-r)/(n+1)}.$$

Proof The hypothesis that $q-q_0$ is bounded in $H^n(0,1)$ means that q lies in a bounded set in $L^2(0,1)$ determined by q_0, since $q_0 \in L^2(0,1)$. Thus the term $\exp(\|q\|_2)$ appearing in Theorem 2.1) can be absorbed into the constant C. The result is then immediate from standard results in interpolation space theory, and in particular the inequality

$$\|f\|_{H^{(1-\theta)r+\theta s}} \le C\|f\|_{H^r}^{1-\theta}\|f\|_{H^s}^{\theta},$$

for $0 \le \theta \le 1$ (see, e.g., McLean [7]). □

Notice that q and q_0 are not each required to be in H^n for this result: it is enough that their difference lie in H^n. The technique of Rundell and Sacks [11] for solving the inverse problem by finding some approximation q to q_0 has the property that $q-q_0$ is smoother than q_0, and so this improved error bound is available.

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