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# Toward a Characterization of Elliptic Solutions of Hierarchies of Soliton Equations

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Abstract. The current status of an explicit characterization of all elliptic algebro-geometric solutions of hierarchies of soliton equations is discussed and the case of the KdV hierarchy is considered in detail. More precisely, we review our recent result that an elliptic function $q$ is a solution of some equation of the stationary KdV hierarchy, if and only if the associated differential equation $\psi''(E,z)+q(z)\psi(E,z)=E\psi(E,z)$ has a meromorphic fundamental system for every complex value of the spectral parameter $E$.

This result also provides an explicit condition under which a classical theorem of Picard holds. This theorem guarantees the existence of solutions which are elliptic of the second kind for second-order ordinary differential equations with elliptic coefficients associated with a common period lattice. The fundamental link between Picard's theorem and elliptic algebro-geometric solutions of completely integrable hierarchies of nonlinear evolution equation is the principal new aspect of our approach.

In addition, we describe most recent attempts to extend this circle of ideas to $n$-th-order scalar differential equations and first-order $n \times n$ systems of differential equations with elliptic functions as coefficients associated with Gelfand-Dickey and matrix-valued hierarchies of soliton equations.

## 1. Introduction

The principal purpose of this review is to describe the basic ideas underlying an efficient characterization of elliptic algebro-geometric solutions of general hierarchies of soliton equations. Since at this time the only case worked out in all details is that of the KdV hierarchy, we will focus to a large extent on this case and turn in our final two sections to possible extensions to the Gelfand-Dickey and matrix-valued hierarchies.

Before describing our approach in some detail, we shall give a brief account of the history of the problem involved. This theme dates back to a 1940 paper of Ince [**65**] who studied what is presently called the Lamé–Ince potential

$$q(x) = -g(g+1)\wp(x+\omega_3),\ g \in \mathbb{N},\ x \in \mathbb{R} \tag{1.1}$$

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in connection with the second-order ordinary differential equation

$$\psi''(E,x) + q(x)\psi(E,x) = E\psi(E,x),\ E \in \mathbb{C}. \tag{1.2}$$

Here $\wp(x) = \wp(x;\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods $2\omega_1$ and $2\omega_3$ ($\operatorname{Im}(\omega_3/\omega_1) \neq 0$). In the special case where $\omega_1$ is real and $\omega_3$ is purely imaginary, the potential $q(x)$ in (1.1) is real-valued and Ince's striking result [**65**], in modern spectral theoretic terminology, yields that the spectrum of the unique self-adjoint operator associated with the differential expression $L_2 = d^2/dx^2 + q(x)$ in $L^2(\mathbb{R})$ exhibits finitely many bands (respectively gaps), that is,

$$\sigma(L_2) = (-\infty, E_{2g}] \cup \bigcup_{m=1}^{g} \left[E_{2m-1}, E_{2m-2}\right],\ E_{2g} < E_{2g-1} < \ldots < E_0. \tag{1.3}$$

What we call the Lamé–Ince potential has, in fact, a long history and many investigations of it precede Ince's work [**65**]. Without possibly trying to be complete we refer the interested reader, for instance, to [**3**], [**4**], Sect. 59, [**6**], Ch. IX, [**9**], Sect. 3.6.4, [**18**], Sects. 135–138, [**19**], [**20**], [**22**], [**54**], [**63**], p. 494–498, [**64**], p. 118–122, 266–418, 475–478, [**66**], p. 378–380, [**69**], [**71**], p. 265–275, [**87**], [**88**], [**118**], [**120**], [**124**], Ch. XXIII as pertinent publications before and after Ince's fundamental paper.

Following the traditional terminology, any real-valued potential $q$ that gives rise to a spectrum of the type (1.3) is called an algebro-geometric potential. The proper extension of this notion to general complex-valued meromorphic potentials $q$ and its connection with stationary solutions of the KdV hierarchy on the basis of elementary algebro-geometric concepts is then obtained as follows. Let $L_2(t)$ be the second-order differential expression

$$L_2(t) = \frac{d^2}{dx^2} + q(x,t),\ (x,t) \in \mathbb{R}^2, \tag{1.4}$$

where $q$ depends on the additional (deformation) parameter $t$. It is well known (see, e.g., Ohmiya [**87**], Schimming [**100**], Wilson [**125**], [**126**]) that one can find coefficients $p_j(x,t)$ in

$$P_{2g+1}(t) = \frac{d^{2g+1}}{dx^{2g+1}} + p_{2g}(x,t)\frac{d^{2g}}{dx^{2g}} + \cdots + p_0(x,t), \tag{1.5}$$

in such a way that $[P_{2g+1}, L_2]$ is a multiplication operator. The coefficients $p_j$ are then certain differential polynomials in $q$, that is, polynomials in $q$ and its $x$-derivatives. The pair $(P_{2g+1}, L_2)$ is called a Lax pair, and the equation

$$\frac{d}{dt}L_2 = \left[P_{2g+1}, L_2\right], \quad \text{that is,} \quad q_t = \left[P_{2g+1}, L_2\right] \tag{1.6}$$

is a nonlinear evolution equation for $q$. The collection of all such equations for all possible choices of $P_{2g+1}$, $g \in \mathbb{N}_0$ is then called the KdV hierarchy (see Section 2 for more details). Due to the commutator structure in (1.6), solutions $q(.,t)$ of the nonlinear evolution equations of the KdV hierarchy represent isospectral deformations of $L_2(0)$. In this context, $q(x,t)$ is called an algebro-geometric solution of the KdV equation if it satisfies one of the stationary higher-order equations $[P_{2g+1}, L_2] = 0$ for some $g \geq 0$ for some (and hence for all) $t \in \mathbb{R}$

Novikov [**85**], Dubrovin [**30**], Its and Matveev [**68**], and McKean and van Moerbeke [**79**] then showed that a real-valued smooth potential $q$ is an algebro-geometric potential if and only if it satisfies one of the higher-order stationary (i.e.,

$t$-independent) KdV equations. Because of these facts it is common to call any complex-valued meromorphic function $q$ an algebro-geometric potential if $q$ satisfies one (and hence infinitely many) of the equations of the stationary KdV hierarchy. Therefore, without loss of generality, we mostly focus on stationary solutions in the remainder of this review.

The stationary KdV hierarchy, characterized by $q_t = 0$ or $[P_{2g+1}, L_2] = 0$, is intimately connected with the question of commutativity of ordinary differential expressions. Thus, if $[P_{2g+1}, L] = 0$, a celebrated theorem of Burchnall and Chaundy [**16**], [**17**] implies that $P_{2g+1}$ and $L_2$ satisfy an algebraic relation of the form

$$P_{2g+1}^2 = \prod_{m=0}^{2g} (L_2 - E_m), \ \{E_m\}_{m=0}^{2g} \subset \mathbb{C}. \tag{1.7}$$

The locations $E_m$ of the (finite) branch points and singular points of the associated hyperelliptic curve

$$F^2 = \prod_{m=0}^{2g} (E - E_m) \tag{1.8}$$

are precisely the band (gap) edges of the spectral bands of $L_2$ (see (1.3)) whenever $q(x)$ is real-valued and smooth for $x \in \mathbb{R}$ (with appropriate generalizations to the complex-valued case, see Section 2). It is the (possibly singular) hyperelliptic compact Riemann surface $K_g$ of (arithmetic) genus $g$, obtained upon one-point compactification of the curve (1.8), which signifies that $q$ in $L_2 = d^2/dx^2 + q(x)$ represents an algebro-geometric potential.

While these considerations pertain to general solutions of the stationary KdV hierarchy, we now concentrate on the additional restriction that $q$ be an elliptic function (i.e., meromorphic and doubly periodic) and hence return to our main subject, elliptic algebro-geometric potentials $q$ for $L_2 = d^2/dx^2 + q(x)$, or, equivalently, elliptic solutions of the stationary KdV hierarchy. Ince's remarkable algebro-geometric result (1.3) remained the only explicit elliptic algebro-geometric example until the KdV flow $q_t = \dfrac{1}{4} q_{xxx} + \dfrac{3}{2} q q_x$ with the initial condition $q(x, 0) = -6\wp(x)$ was explicitly integrated by Dubrovin and Novikov [**34**] in 1975 (see also [**36**], [**37**], [**38**], [**67**]), and found to be of the type

$$q(x, t) = -2 \sum_{j=1}^{3} \wp(x - x_j(t)) \tag{1.9}$$

for appropriate $\{x_j(t)\}_{1 \le j \le 3}$. As observed above, all potentials $q(\cdot, t)$ in (1.9) are isospectral to $q(\cdot, 0) = -6\wp(\cdot)$. Given these results it was natural to ask for a systematic account of all elliptic solutions of the KdV hierarchy, a problem posed, for instance, in [**86**], p. 152.

In 1977, Airault, McKean and Moser, in their seminal paper [**2**], presented the first systematic study of the isospectral torus $I_{\mathbb{R}}(q_0)$ of real-valued smooth potentials $q_0(x)$ of the type

$$q_0(x) = -2 \sum_{j=1}^{M} \wp(x - x_j) \tag{1.10}$$

with an algebro-geometric spectrum of the form (1.3). Among a variety of results they proved that any element $q$ of $I_{\mathbb{R}}(q_0)$ is an elliptic function of the type (1.10) (with different $x_j$) with $M$ constant throughout $I_{\mathbb{R}}(q_0)$ and that $\dim I_{\mathbb{R}}(q_0) \leq M$. In particular, if $q_0$ evolves according to any equation of the KdV hierarchy it remains an elliptic algebro-geometric potential. The potential (1.10) is intimately connected with completely integrable many-body systems of the Calogero-Moser-type [**19**], [**84**] (see also [**20**], [**22**]). This connection with integrable particle systems was subsequently exploited by Krichever [**74**] in his fundamental construction of elliptic algebro-geometric solutions of the Kadomtsev-Petviashvili equation. In particular, he explicitly determined the underlying algebraic curve $\Gamma$ and characterized the Baker-Akhiezer function associated with it in terms of elliptic functions as well as the corresponding theta function of $\Gamma$. The next breakthrough occurred in 1988 when Verdier [**119**] published new explicit examples of elliptic algebro-geometric potentials. Verdier's examples spurred a flurry of activities and inspired Belokolos and Enol'skii [**11**], Smirnov [**105**], and subsequently Taimanov [**110**] and Kostov and Enol'skii [**70**] to find further such examples by combining the reduction process of abelian integrals to elliptic integrals (see [**7**], [**8**], [**9**], Ch. 7, [**10**]) with the aforementioned techniques of Krichever [**74**], [**75**]. This development finally culminated in a series of recent results of Treibich and Verdier [**115**], [**116**], [**117**] where it was shown that a general complex-valued potential of the form

$$q(x) = -\sum_{j=1}^{4} d_j \ \wp(x - \omega_j) \tag{1.11}$$

($\omega_2 = \omega_1 + \omega_3,\ \omega_4 = 0$) is an algebro-geometric potential if and only if $d_j/2$ are triangular numbers, that is, if and only if

$$d_j = g_j(g_j + 1) \text{ for some } g_j \in \mathbb{Z},\ 1 \leq j \leq 4. \tag{1.12}$$

We shall from now on refer to potentials of the type

$$q(x) = -\sum_{j=1}^{4} g_j(g_j+1)\wp(x - \omega_j),\ g_j \in \mathbb{Z},\ 1 \leq j \leq 4 \tag{1.13}$$

as Treibich-Verdier potentials. The methods of Treibich and Verdier are based on hyperelliptic tangent covers of the torus $\mathbb{C}/\Lambda$ ($\Lambda$ being the period lattice generated by $2\omega_1$ and $2\omega_3$). The state of the art of elliptic algebro-geometric solutions up to 1993 was recently reviewed in issues 1 and 2 of volume 36 of Acta Applicandae Math., see, for instance, [**12**], [**40**], [**76**], [**107**], [**111**], [**114**] and also in [**13**], [**24**], [**27**], [**39**], [**61**], [**98**], [**108**], [**113**]. In addition to these investigations on elliptic solutions of the KdV hierarchy, the study of other soliton hierarchies, such as the modified KdV hierarchy, nonlinear Schrödinger hierarchy, and Boussinesq hierarchy has also begun. We refer, for instance, to [**21**], [**35**], [**54**], [**55**], [**77**], [**80**], [**82**], [**96**], [**97**], [**103**], [**104**], [**106**], [**109**].

Despite the efforts described thus far, an efficient characterization of all elliptic solutions of the KdV hierarchy remained an open problem until 1994. Around 1992 we became aware of this problem and started to develop our own approach toward its solution. In contrast to all existing basically algebro-geometric approaches in this area, we realized early on that the most powerful analytic tool in this context, a theorem of Picard (Theorem 4.1) concerning the existence of solutions which are elliptic of the second kind of ordinary differential equations with elliptic coefficients, had not been applied at all. As we have recently shown in [**59**] (see also [**58**]), Picard's theorem combined with Floquet theoretic results indeed provides a very simple and efficient characterization of all elliptic algebro-geometric solutions of the KdV hierarchy. Moreover, for reflection symmetric elliptic algebro-geometric potentials $q$ (i.e., $q(z) = q(2z_0 - z)$ for some $z_0 \in \mathbb{C}$) including Lamé-Ince and Treibich-Verdier potentials, our approach reduces the computation of the branch points and singular points of the underlying hyperelliptic curve $K_g$ to certain (constrained) linear algebraic eigenvalue problems as shown in [**54**], [**55**], and [**56**].

Since the main hypothesis in Picard's theorem for a second-order differential equation of the form

$$\psi''(z) + q(z)\psi(z) = E\psi(z), \ E \in \mathbb{C} \tag{1.14}$$

with an elliptic potential $q$ assumes the existence of a fundamental system of solutions meromorphic in $z$ for each value of the spectral parameter $E \in \mathbb{C}$, we call any such elliptic function $q$ which gives rise to this property a **Picard potential**. The principal result, a characterization of all elliptic algebro-geometric solutions of the stationary KdV hierarchy, then reads as follows:

THEOREM 1.1. *([**59**]) $q$ is an elliptic algebro-geometric potential if and only if $q$ is a Picard potential (i.e., if and only if for each $E \in \mathbb{C}$ every solution of $\psi''(z) + q(z)\psi(z) = E\psi(z)$ is meromorphic with respect to $z$).*

In particular, Theorem 1.1 sheds new light on Picard's theorem since it identifies the elliptic coefficients $q$ for which there exists a meromorphic fundamental system of solutions of (1.14) precisely as the elliptic algebro-geometric solutions of the stationary KdV hierarchy. Moreover, we stress its straightforward applicability based on an elementary Frobenius-type analysis which decides whether or not (1.14) has a meromorphic fundamental system for each $E \in \mathbb{C}$. In addition, we might mention the obvious connections between this result and the Weierstrass theory of reduction of abelian to elliptic integrals.

The proof of Theorem 1.1 in Section 4 (Theorem 4.7) relies on two main ingredients: A purely Floquet theoretic part to be discussed in Section 3 and an elliptic function part sketched in Section 4.

The result embodied by Theorem 1.1 in the special context of the KdV hierarchy, uncovers a new general principle in connection with elliptic algebro-geometric solutions of completely integrable systems: The existence of such solutions appears to be in a one-to-one correspondence with the existence of a meromorphic (with respect to $z$) fundamental system of solutions for the underlying linear Lax differential expression (for all values of the corresponding spectral parameter $E$).

Having dealt with the second-order Lax differential expression $L_2 = d^2/dx^2 + q$ underlying the KdV hierarchy, it is natural to seek extensions to $n$-th-order Lax differential expressions $L_n$ associated with the Gelfand-Dickey hierarchy and more generally, to matrix-valued hierarchies of soliton equations. At present our results in these directions are promising but far from being complete. We provide a recent generalization of Picard's theorem to first-order $n \times n$ systems of differential equations in Section 5 and devote Section 6 to partial progress in the context of $n$-th-order scalar differential expressions with elliptic coefficients.

## 2. The KdV Hierarchy and Hyperelliptic Curves

In this section we review basic facts on the stationary KdV hierarchy. Since this material is well-known (see, e.g., [**5**], [**23**], [**26**], Ch. 12, [**28**], [**29**], [**49**], [**53**], [**87**], [**100**], [**101**], [**126**]), we confine ourselves to a brief account. Assuming $q \in$

$C^\infty(\mathbb{R})$ or $q$ meromorphic in $\mathbb{C}$ (depending on the particular context in which one is interested) and hence either $x \in \mathbb{R}$ or $x \in \mathbb{C}$, consider the recursion relation

$$\hat{f}'_{j+1}(x) = \frac{1}{4}\hat{f}_j^{'''}(x) + q(x)\hat{f}'_j(x) + \frac{1}{2}q'(x)\hat{f}_j(x), \ 0 \le j \le g, \ \hat{f}_0(x) = 1 \tag{2.1}$$

and the associated differential expressions (Lax pair)

$$L_2 = \frac{d^2}{dx^2} + q(x), \tag{2.2}$$

$$\hat{P}_{2g+1} = \sum_{j=0}^{g} \left[-\frac{1}{2}\hat{f}'_j(x) + \hat{f}_j(x)\frac{d}{dx}\right] L_2^{g-j}, \ g \in \mathbb{N}_0 \tag{2.3}$$

(here $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). One can show that

$$\left[\hat{P}_{2g+1}, L_2\right] = 2\hat{f}'_{g+1} = \frac{1}{2}\hat{f}'''_g(x) + 2q(x)\hat{f}'_g(x) + q'(x)\hat{f}_g(x) \tag{2.4}$$

($[\cdot,\cdot]$ the commutator symbol) and explicitly computes from (2.1),

$$\hat{f}_0 = 1, \ \hat{f}_1 = \frac{1}{2}q + c_1, \ \hat{f}_2 = \frac{1}{8}q^{''} + \frac{3}{8}q^2 + \frac{c_1}{2}q + c_2, \quad \text{etc.}, \tag{2.5}$$

where the $c_j$ are integration constants. Using the convention that the corresponding homogeneous quantities obtained by setting $c_\ell = 0$ for $\ell = 1, 2, \dots$ are denoted by $f_j$, that is, $f_j = \hat{f}_j(c_\ell \equiv 0)$, the (homogeneous) stationary KdV hierarchy is then defined as the sequence of equations

$$\mathrm{KdV}_g(q) = 2f'_{g+1} = 0, \ g \in \mathbb{N}_0. \tag{2.6}$$

Explicitly, this yields

$$\mathrm{KdV}_0(q) = q_x = 0, \quad \mathrm{KdV}_1(q) = \frac{1}{4}q^{'''} + \frac{3}{2}qq' = 0, \quad \text{etc.} \tag{2.7}$$

The corresponding non-homogeneous version of $\mathrm{KdV}_g(q) = 0$ is then defined by

$$\hat{f}'_{g+1} = \sum_{j=0}^{g} c_{g-j} f'_{j+1} = 0, \tag{2.8}$$

where $c_0 = 1$ and $c_1, ..., c_g$ are arbitrary complex constants.

If one assigns to $q^{(\ell)} = d^\ell q/dx^\ell$ the degree $\deg(q^{(\ell)}) = \ell + 2, \ \ell \in \mathbb{N}_0$, then the homogeneous differential polynomial $f_j$ with respect to $q$ turns out to have degree $2j$, that is,

$$\deg(f_j) = 2j, \ j \in \mathbb{N}_0. \tag{2.9}$$

Next, introduce the polynomial $\hat{F}_g(E, x)$ in $E \in \mathbb{C}$,

$$\hat{F}_g(E, x) = \sum_{j=0}^{g} \hat{f}_{g-j}(x)E^j. \tag{2.10}$$

Since $\hat{f}_0(x) = 1$,

$$\hat{R}_{2g+1}(E, x) = (E - q(x))\hat{F}_g(E, x)^2 - \frac{1}{2}\hat{F}''_g(E, x)\hat{F}_g(E, x) + \frac{1}{4}\hat{F}'_g(E, x)^2 \tag{2.11}$$

is a monic polynomial in $E$ of degree $2g+1$. However, equations (2.1) and (2.8) imply that

$$\frac{1}{2}\hat{F}_g^{\prime\prime\prime} - 2(E-q)\hat{F}_g' + q'\hat{F}_g = 0 \tag{2.12}$$

and this shows that $\hat{R}_{2g+1}(E,x)$ is in fact independent of $x$. Hence it can be written as

$$\hat{R}_{2g+1}(E) = \prod_{m=0}^{2g}(E-\hat{E}_m),\ \{\hat{E}_m\}_{m=0}^{2g} \subset \mathbb{C}. \tag{2.13}$$

By (2.4) the non-homogeneous KdV equation (2.8) is equivalent to the commutativity of $L_2$ and $\hat{P}_{2g+1}$. This shows that

$$[\hat{P}_{2g+1}, L_2] = 0, \tag{2.14}$$

and therefore, if $L_2\psi = E\psi$, this implies that $\hat{P}_{2g+1}^2\psi = \hat{R}_{2g+1}(E)\psi$. Thus $[\hat{P}_{2g+1}, L_2] = 0$ implies

$$\hat{P}_{2g+1}^2 = \hat{R}_{2g+1}(L_2) = \prod_{m=0}^{2g}(L_2 - \hat{E}_m), \tag{2.15}$$

a celebrated theorem by Burchnall and Chaundy [**16**], [**17**] (see, e.g., [**27**], [**47**], [**61**], [**98**], [**126**] for more recent accounts).

In the second part of Section 3 we will need the converse of the above procedure. It is given by

LEMMA 2.1. *([**59**]) Assume that $\hat{F}_g(E,x)$, given by (2.10) with $\hat{f}_0(x) = 1$, is twice differentiable with respect to $x$, and that*

$$(E-q(x))\hat{F}_g(E,x)^2 - \frac{1}{2}\hat{F}_g''(E,x)\hat{F}_g(E,x) + \frac{1}{4}\hat{F}_g'(E,x)^2 \tag{2.16}$$

*is independent of $x$. Then $q \in C^\infty(\mathbb{R})$. Also the functions $\hat{f}_j(x)$ are infinitely often differentiable and satisfy the recursion relations (2.1) for $j = 0, ..., g-1$. Moreover, $\hat{f}_g$ satisfies*

$$\frac{1}{4}\hat{f}_g^{\prime\prime\prime}(x) + q(x)\hat{f}_g'(x) + \frac{1}{2}q'(x)\hat{f}_g(x) = 0, \tag{2.17}$$

*that is, the differential expression $\hat{P}_{2g+1}$ given in (2.3) commutes with the expression $L_2 = d^2/dx^2 + q$.*

Equation (2.15) illustrates the intimate connection between the stationary KdV equation $\hat{f}_{g+1}' = 0$ in (2.8) and the compact (possibly singular) hyperelliptic curve $K_g$ of (arithmetic) genus $g$ obtained upon one-point compactification of the curve

$$F^2 = \hat{R}_{2g+1}(E) = \prod_{m=0}^{2g}(E-\hat{E}_m). \tag{2.18}$$

The above formalism leads to the following standard definition.

DEFINITION 2.2. Any solution $q$ of one of the stationary KdV equations (2.8) is called an **algebro-geometric potential** associated with the KdV hierarchy.

Algebro-geometric potentials $q$ can be expressed in terms of the Riemann theta function or through $\tau$-functions associated with the curve $K_g$ (see, e.g., [**68**], [**102**]).

## 3. Floquet Theory and Algebro-Geometric Potentials

In the first part of this section we discuss Floquet theory in connection with a complex-valued non-constant periodic potential $q$. In the second part we recall a criterion for $q$ to be an algebro-geometric potential in terms of Floquet solutions.

Suppose

$$q \in L^1_{\text{loc}}(\mathbb{R}),\ q(x+\Omega) = q(x),\ x \in \mathbb{R} \tag{3.1}$$

for some $\Omega > 0$ and let $\mathcal{L}(E)$ be the (two-dimensional) space of solutions of $L_2 y = Ey$. Then $T(E)$, the restriction of the operator defined by $y \mapsto y(\cdot+\Omega)$ to $\mathcal{L}(E)$, commutes with the corresponding restriction of $L_2$ and hence maps $\mathcal{L}(E)$ to itself. The eigenvalues and eigenfunctions of $T(E)$ are called Floquet multipliers and Floquet solutions of $L_2 y = Ey$. On $\mathcal{L}(E)$ we introduce the basis $c(E,x,x_0)$ and $s(E,x,x_0)$ defined by

$$c(E,x_0,x_0) = s'(E,x_0,x_0) = 1,\ c'(E,x_0,x_0) = s(E,x_0,x_0) = 0. \tag{3.2}$$

Using this basis the operator $T(E)$ is represented by the so called monodromy matrix

$$\begin{pmatrix} c(E,x_0+\Omega,x_0) & s(E,x_0+\Omega,x_0) \\ c'(E,x_0+\Omega,x_0) & s'(E,x_0+\Omega,x_0) \end{pmatrix}. \tag{3.3}$$

Since $\det(T(E)) = 1$ the Floquet multipliers $\rho_\pm(E)$ are given by

$$\rho_\pm(E) = \Delta(E) \pm \sqrt{\Delta(E)^2 - 1}, \tag{3.4}$$

where $\Delta(E)$ denotes the Floquet discriminant,

$$\Delta(E) = \frac{1}{2}\operatorname{tr}(T(E)) = [c(E,x_0+\Omega,x_0) + s'(E,x_0+\Omega,x_0)]/2. \tag{3.5}$$

For each $E \in \mathbb{C}$ there exists at least one nontrivial Floquet solution. In fact, since together with $\rho(E)$, $1/\rho(E)$ is also a Floquet multiplier, there are two linearly independent Floquet solutions for a given $E$ provided $\rho(E)^2 \neq 1$. Floquet solutions can be expressed in terms of the fundamental system $c(E,x,x_0)$ and $s(E,x,x_0)$ by

$$\psi_\pm(E,x,x_0) = c(E,x,x_0) + \frac{\rho_\pm(E) - c(E,x_0+\Omega,x_0)}{s(E,x_0+\Omega,x_0)} s(E,x,x_0), \tag{3.6}$$

if $s(E,x_0+\Omega,x_0) \neq 0$, or by

$$\tilde{\psi}_\pm(E,x,x_0) = s(E,x,x_0) + \frac{\rho_\pm(E) - s'(E,x_0+\Omega,x_0)}{c'(E,x_0+\Omega,x_0)} c(E,x,x_0), \tag{3.7}$$

if $c'(E,x_0+\Omega,x_0) \neq 0$. If both $s(E,x_0+\Omega,x_0)$ and $c'(E,x_0+\Omega,x_0)$ are equal to zero, then $s(E,x,x_0)$ and $c(E,x,x_0)$ are linearly independent Floquet solutions.

Associated with the second-order differential expression $L_2 = d^2/dx^2 + q(x)$ we consider the densely defined closed linear operators $H$, $H_D(x_0)$, $H(\beta,x_0)$, $\beta \in \mathbb{C}$, and $H(\theta)$, $\theta \in \mathbb{C}$. While $H$ will be an operator in $L^2(\mathbb{R})$, the others will be defined in $L^2(I(x_0))$, where $I(x_0) = (x_0, x_0+\Omega)$ for some $x_0 \in \mathbb{R}$. Specifically, the operators are given as restrictions of the expression $L_2$ to the following domains:

$$\mathcal{D}(H) = \{g \in L^2(\mathbb{R}) : g, g' \in AC_{\text{loc}}(\mathbb{R}), (g'' + qg) \in L^2(\mathbb{R})\}, \tag{3.8}$$

$$\mathcal{D}(H_D(x_0)) = \{g \in \mathcal{D}_{x_0} : g(x_0) = g(x_0+\Omega) = 0\}, \tag{3.9}$$

$$\mathcal{D}(H(\beta,x_0)) = \{g \in \mathcal{D}_{x_0} : U_1(\beta,g)(x_0) = U_1(\beta,g)(x_0+\Omega) = 0\}, \tag{3.10}$$

$$\mathcal{D}(H(\theta)) = \{g \in \mathcal{D}_{x_0} : U_2(\theta,g)(x_0) = U_2(\theta,g)'(x_0) = 0\}, \tag{3.11}$$

where

$$\mathcal{D}_{x_0} = \{g \in L^2(I(x_0)) : g, g' \in AC([x_0, x_0 + \Omega]), (g'' + qg) \in L^2(I(x_0))\} \tag{3.12}$$

and $U_1(\beta, y) = y' + \beta y$ and $U_2(\theta, y) = y(\cdot + \Omega) - e^{i\theta} y(\cdot)$. Here $AC(\cdot)$ ($AC_{\mathrm{loc}}(\cdot)$) denotes the set of (locally) absolutely continuous functions. Next we denote the purely discrete spectra of $H_D(x_0)$, $H(\beta, x_0)$, and $H(\theta)$ by $\sigma(H_D(x_0)) = \{\mu_n(x_0)\}_{n\in\mathbb{N}}$, $\sigma(H(\beta, x_0)) = \{\lambda_n(\beta, x_0)\}_{n\in\mathbb{N}_0}$ and $\sigma(H(\theta)) = \{E_n(\theta)\}_{n\in\mathbb{N}_0}$, respectively. While $H(\theta)$ depends on $x_0$ its spectrum does not. We agree that here, as well as in the rest of the paper, all point spectra (i.e., sets of eigenvalues) are recorded in such a way that all eigenvalues are consistently repeated according to their algebraic multiplicity unless explicitly stated otherwise.

The eigenvalues of $H_D(x_0)$ are called Dirichlet eigenvalues with respect to the interval $[x_0, x_0 + \Omega]$. The eigenvalues of $H(\theta)$ are precisely those values $E$ where $T(E)$ has eigenvalues $\rho = e^{\pm i\theta}$. The eigenvalues $E_n(0)$ ($E_n(\pi)$) of $H(0)$ ($H(\pi)$) are called the periodic (antiperiodic) eigenvalues associated with $q$. Note that the (anti)periodic eigenvalues $E_n(0)$ ($E_n(\pi)$) are the zeros of $\Delta(\cdot)-1$ ($\Delta(\cdot)+1$) and that their algebraic multiplicities coincide with the orders of the respective zeros (see, e.g., [**57**]). In the following we denote the zeros of $\Delta(E)^2 - 1$ by $E_n$, $n \in \mathbb{N}_0$. They are repeated according to their multiplicity and are related to the (anti)periodic eigenvalues via

$$E_{4n} = E_{2n}(0), \quad E_{4n+1} = E_{2n}(\pi), \quad E_{4n+2} = E_{2n+1}(\pi), \quad E_{4n+3} = E_{2n+1}(0) \tag{3.13}$$

for $n \in \mathbb{N}_0$. We also introduce

$$p(E) = \mathrm{ord}_E(\Delta(\cdot)^2 - 1), \tag{3.14}$$

the order of $E$ as a zero of $\Delta(\cdot)^2 - 1$ ($p(E) = 0$ if $\Delta(E)^2 - 1 \neq 0$).

Similarly, the eigenvalues of $H_D(x_0)$ and $H(\beta, x_0)$ are the zeros of the functions $s(\cdot, x_0 + \Omega, x_0)$ and $h(\cdot, \beta, x_0) = (\beta^2 s + \beta(s' - c) - c')(\cdot, x_0 + \Omega, x_0)$, respectively. Again their algebraic multiplicities coincide precisely with the multplicities of the respective zeros (see, e.g., [**57**]). These multiplicities depend in general on $x_0$. We introduce the notation

$$d(E, x_0) = \mathrm{ord}_E(s(\cdot, x_0 + \Omega, x_0)), \tag{3.15}$$

$$r(E, \beta, x_0) = \mathrm{ord}_E(h(\cdot, \beta, x_0)), \tag{3.16}$$

and remark that $d(E, x_0)$ and $r(E, \beta, x_0)$ are combinations of movable and immovable parts. Specifically, define $d_i(E) = \min\{d(E, x_0) : x_0 \in \mathbb{R}\}$, $r_i(E, \beta) = \min\{r(E, \beta, x_0) : x_0 \in \mathbb{R}\}$ and $d_m(E, x_0)$ and $r_m(E, x_0)$ by

$$d(E, x_0) = d_i(E) + d_m(E, x_0), \tag{3.17}$$

$$r(E, \beta, x_0) = r_i(E, \beta) + r_m(E, \beta, x_0). \tag{3.18}$$

If $d_i(E) > 0$ then $E$ is a Dirichlet eigenvalue irrespective of the value of $x_0$ and we will call $E$ an immovable Dirichlet eigenvalue. Otherwise, if $d_i(E) = 0$ but $d(E, x_0) > 0$ we call $E$ a movable Dirichlet eigenvalue. (Note that here we use a notation different from the one in [**59**], in particular, the multiplicties $d$, $d_i$, and $d_m$ now refer to Dirichlet eigenvalues while the multiplicities $p$ refer to periodic or antiperiodic eigenvalues).

The functions $c(\cdot, x, x_0)$ and $s(\cdot, x, x_0)$ and their $x$-derivatives are entire functions of order $1/2$ for every choice of $x$ and $x_0$. This and their asymptotic behavior as

$|E|$ tends to infinity is obtained via Volterra integral equations. Invoking Rouché's theorem then yields the following facts:

1. The zeros $\mu_n(x_0)$ of $s(E, x_0+\Omega, x_0)$ and the zeros $\lambda_n(\beta, x_0)$ of $h(\cdot, \beta, x_0)$ are simple for $n \in \mathbb{N}$ sufficiently large.
2. The zeros $E_n$ of $\Delta(E)^2 - 1$ are at most double for $n \in \mathbb{N}$ large enough.
3. $\mu_n(x_0)$, $\lambda_n(\beta, x_0)$, and $E_n$ can be arranged (and will be subsequently) such that they have the following asymptotic behavior as $n$ tends to infinity:

$$\mu_n(x_0) = -\frac{n^2\pi^2}{\Omega^2} + O(1), \tag{3.19}$$

$$\lambda_n(\beta, x_0) = -\frac{n^2\pi^2}{\Omega^2} + O(1), \tag{3.20}$$

$$E_{2n-1}, E_{2n} = -\frac{n^2\pi^2}{\Omega^2} + O(1). \tag{3.21}$$

The Hadamard factorization of $s(E, x_0 + \Omega, x_0)$ therefore reads

$$s(E, x_0 + \Omega, x_0) = c_1(x_0) \prod_{n=1}^{\infty} \left(1 - \frac{E}{\mu_n(x_0)}\right) = F_D(E, x_0) D(E), \tag{3.22}$$

where all those factors which do not depend on $x_0$ are collected in $D(E)$. Here we assume that none of the eigenvalues is equal to zero; otherwise, obvious modifications have to be used.

For more details on algebraic versus geometric multiplicities of eigenvalue problems of the type of $H_D(x_0)$ and $H(\theta)$ see, for instance, [**57**].

It was shown by Rofe-Beketov [**99**] that the spectrum of $H$ is equal to the conditional stability set of $L_2$, that is, the set of all spectral parameters $E$ for which a nontrivial bounded solution of $L_2\psi = E\psi$ exists. Hence

$$\sigma(H) = \bigcup_{\theta \in [0, 2\pi]} \sigma(T(\theta)) = \bigcup_{n \in \mathbb{N}_0} \sigma_n, \text{ where } \sigma_n = \bigcup_{\theta \in [0, \pi]} E_n(\theta). \tag{3.23}$$

We note that in the general case where $q$ is complex-valued some of the spectral arcs $\sigma_n$ may cross each other, see, for instance, [**57**] and [**91**] for explicit examples.

The Green's function $G(E, x, x')$ of $H$, that is, the integral kernel of the resolvent of $H$,

$$G(E, x, x') = (H - E)^{-1}(x, x'), \; E \in \mathbb{C} \backslash \sigma(H), \; x, x' \in \mathbb{R}, \tag{3.24}$$

is explicitly given by

$$G(E, x, x') = W(f_-(E, x), f_+(E, x))^{-1} \begin{cases} f_+(E, x) f_-(E, x'), \; x \geq x' \\ f_-(E, x) f_+(E, x'), x \leq x' \end{cases}. \tag{3.25}$$

Here $f_\pm(E, \cdot)$ solve $L_2 f = Ef$ and are chosen such that

$$f_\pm(E, \cdot) \in L^2((R, \pm\infty)), \; E \in \mathbb{C} \backslash \sigma(H), \; R \in \mathbb{R}, \tag{3.26}$$

with $W(f, g) = fg' - f'g$ the Wronskian of $f$ and $g$.

Equation (3.25) implies that the diagonal Green's function is twice differentiable and satisfies the nonlinear second-order differential equation (see, e.g., [**48**], [**83**])

$$4(E - q(x)) G(E, x, x)^2 - 2G(E, x, x) G''(E, x, x) + G'(E, x, x)^2 = 1 \tag{3.27}$$

(the primes denoting derivatives with respect to $x$).

It follows from (3.23) that $|\rho(E)| \neq 1$ unless $E \in \sigma(H)$. Therefore, if $E \notin \sigma(H)$ there is precisely one Floquet solution in $L^2((-\infty, R))$ and one in $L^2((R, \infty))$. Letting $\rho_\pm(E) = e^{\pm i\theta}$ with $\mathrm{Im}(\theta) > 0$ we obtain $|\rho_+(E)| < 1 < |\rho_-(E)|$. Hence $f_+(E,x) = \psi_+(E,x,x_0)$ and $f_-(E,x) = \psi_-(E,x,x_0)$. Since $\psi_\pm(E,x_0,x_0) = 1$, equations (3.4) and (3.6) imply

$$W(f_-(E,\cdot), f_+(E,\cdot)) = \frac{e^{i\theta} - e^{-i\theta}}{s(E, x_0+\Omega, x_0)} = -2\frac{[\Delta(E)^2 - 1]^{1/2}}{s(E, x_0+\Omega, x_0)}. \tag{3.28}$$

The sign of the square root was chosen such that $[\Delta(E)^2 - 1]^{1/2}$ is asymptotically equal to $\rho_-(E)/2$ for large positive $E$. Equation (3.28) implies (see also [**50**])

$$G(E, x_0, x_0) = -\frac{s(E, x_0+\Omega, x_0)}{2[\Delta(E)^2 - 1]^{1/2}}. \tag{3.29}$$

Theorem 3.1. *([**59**], [**123**]) If $q$ is a locally integrable periodic function on $\mathbb{R}$ then $p(E) - 2d_i(E) \geq 0$ for all $E \in \mathbb{C}$.*

Proof. Equations (3.22), (3.27), and (3.29) show that

$$4(E - q(x))F_D(E,x)^2 - 2F_D(E,x)F_D''(E,x) + F_D'(E,x)^2 = \frac{4(\Delta(E)^2 - 1)}{D(E)^2}. \tag{3.30}$$

Since the left hand side is entire the claim follows immediately from the definitions of the numbers $p(E)$ and $d_i(E)$. □

A somewhat bigger effort allows one to prove also

Theorem 3.2. *([**59**], [**123**]) If $q$ is a locally integrable periodic function on $\mathbb{R}$ then $d_i(E) = r_i(E, \beta)$ unless $q$ is a constant and $E = q + \beta^2$. Moreover, if $d_i(E) > 0$ then there exist two linearly independent Floquet solutions of $L_2 y = Ey$. Finally, $p(E) - 2d_i(E) > 0$ if and only if there exists an $x_0 \in \mathbb{R}$ such that $W(z, x_0)$, the Wronskian of the Floquet solutions $\psi_\pm$ given by* (3.6)*, tends to zero as $z$ tends to $E$.*

Hence, if there are not two linearly independent Floquet solutions for $L_2 y = Ey$ then $\rho^2 = 1$ and $p(E) > 0$ but $d_i(E) = 0$ and thus $p(E) - 2d_i(E) > 0$ at all such points.

Nowhere in this section did we use thus far that $q$ is an algebro-geometric potential. Next we give necessary and sufficient conditions for this in terms of properties of multiplicities of eigenvalues of (anti)periodic boundary value problems on one hand and the Dirichlet problem on the other hand. We begin with

Definition 3.3. The number $\mathrm{def}(L_2) = \sum_{E\in\mathbb{C}}(p(E) - 2d_i(E))$ is called the Floquet defect. The number $\sum_{E\in\mathbb{C}} d_m(E, x_0)$ will be called the number of movable Dirichlet eigenvalues; similarly, $\sum_{E\in\mathbb{C}} r_m(E, \beta, x_0)$ denotes the number of movable eigenvalues of $H(\beta, x_0)$.

Note that by Theorem 3.1, $\mathrm{def}(L_2)$ is either infinite or else a nonnegative integer. If it is finite then $\mathrm{def}(L_2) = \deg(4(\Delta^2 - 1)/D^2)$. Both, $\mathrm{def}(L_2)$ and the number of movable Dirichlet eigenvalues are in general infinite.

Theorem 3.4. *([**59**], [**123**]) Assume that $q$ is a locally integrable, periodic function of period $\Omega > 0$ on $\mathbb{R}$. Then the following statements are equivalent:*
*1. The Floquet defect* $\mathrm{def}(L_2)$ *equals* $2g+1$*.*

*2. The number of movable Dirichlet eigenvalues equals $g$.*
*3. There exists a monic differential expression $\hat{P}_{2g+1}$ of odd order $2g+1$ which commutes with $L_2$ but none of smaller odd order, i.e., $q$ is an algebro-geometric potential.*

In particular, $\operatorname{def}(L_2)$ is either odd or infinite.

SKETCH OF PROOF. If $\operatorname{def}(L_2)$ is finite, asymptotic considerations show that only finitely many Dirichlet eigenvalues can be movable. Hence $F_D(\cdot, x_0)$ is a polynomial, say of degree $\hat{g}$. By equation (3.30) $4(\Delta^2-1)/D^2$ is a polynomial of degree $2\hat{g}+1$. Hence $\hat{g}=g$. This shows the equivalence of the first two statements.

Next one shows that the leading coefficent of $F_D(\cdot, x_0)$ is independent of $x_0$. The third statement follows then from the second using Lemma 2.1. To prove that the third statement implies the other two one has to show that the zeros of the function $\hat{F}_g(\cdot, x_0)$ in (2.10) are precisely the movable Dirichlet eigenvalues. This follows from applying $\hat{P}_{2g+1}$ as given in (2.3) succesively to the generalized Dirichlet eigenfunctions. $\square$

THEOREM 3.5. *(**[59]**, **[123]**) Assume that $q$ is a non-constant, locally integrable, periodic function of period $\Omega > 0$ on $\mathbb{R}$ and that any (and hence all) of the three statements in Theorem 3.4 is satisfied. Then the following statements hold.*
*1. The number of movable eigenvalues of $H(\beta, x_0)$ equals $g+1$, i.e.,*

$$\sum_{E\in\mathbb{C}} r_m(E,\beta,x_0) = g+1. \tag{3.31}$$

*2. $q \in C^\infty(\mathbb{R})$.*
*3. The differential expression $\hat{P}_{2g+1}$ satisfies the Burchnall-Chaundy relation*

$$\hat{P}_{2g+1}^2 = \prod_{z\in\mathbb{C}} (L-z)^{p(z)-2d_i(z)}. \tag{3.32}$$

*4. The diagonal Green's function $G(\cdot, x, x)$ of $H$ is continuous on $\mathbb{C}\backslash\{z : p(z) - 2d_i(z) > 0\}$ and is of the type*

$$G(E,x,x) = -\frac{1}{2}\frac{\prod_{z\in\mathbb{C}}(E-z)^{d_m(z,x)}}{\prod_{z\in\mathbb{C}}(E-z)^{p(z)-2d_i(z)}}. \tag{3.33}$$

*5. The spectrum of $H$ consists of finitely many bounded spectral arcs $\tilde{\sigma}_n$, $1 \le n \le \tilde{g}$ for some $\tilde{g} \le g$ and one unbounded (semi-infinite) arc $\tilde{\sigma}_\infty$ which tends to $-\infty + <q>$, with $<q> = \Omega^{-1}\int_{x_0}^{x_0+\Omega} q(x)dx$, that is,*

$$\sigma(H) = \left(\bigcup_{n=1}^{\tilde{g}} \tilde{\sigma}_n\right) \cup \tilde{\sigma}_\infty, \tag{3.34}$$

*where each $\tilde{\sigma}_n$ and $\tilde{\sigma}_\infty$ is a union of some of the spectral arcs $\sigma_n$ in (3.23).*

Note that the set $B$ of values of $E$ where $p(E) - 2d_i(E) > 0$ contains $B_1$, the set of all those points where less than two linearly independent Floquet solutions exist. For $B\backslash B_1$ to be nonempty, it is necessary that $p(z) \ge 3$ for some (anti)periodic eigenvalue $z$. While it seems difficult to construct an explicit example where $B\backslash B_1 \neq \emptyset$, the very existence of this phenomenon has first been noted in **[59]**. References **[45]**, **[46]**, **[62]**, **[89]**, **[90]** treat potentials with $p(E) \le 2$

and references [**14**], [**15**] require that algebraic and geometric multiplicities of all (anti)periodic eigenvalues coincide and hence also that $p(E) \leq 2$. Generically one has $p(E) - 2d_i(E) = 1$ if this is positive at all and $B = B_1$ (cf. [**112**]).

Remark 3.6 (Singularity structure of the Green's function). As Theorem 3.5 shows, it is precisely the multiplicity $p - 2d_i$ of the branch and singular points in the Burchnall-Chaundy polynomial (3.32) which determines the singularity structure of the diagonal Green's function $G(E, x, x)$ of $H$. Moreover, since (see, e.g., [**83**])

(3.35)

$$G(E, x, x') = [G(E, x, x)G(E, x', x')]^{1/2} \exp[-\frac{1}{2} \int_{\min(x,x')}^{\max(x,x')} G(E, s, s)^{-1} ds],$$

this observation extends to the off-diagonal Green's function $G(E, x, x')$ of $H$ as well.

Remark 3.7 (Inverse square singularities). The case of the Lamé-Ince potential, where $q$ has singularities of the form $-g(g + 1)/x^2$, indicates the necessity to consider also potentials with such singularities. This is possible by modifying the usual approach via Volterra integral equations which are used to obtain the asymptotic properties (3.19)–(3.21) of the corresponding eigenvalue distributions. One obtains essentially the same results as in the present review, the only difference being that the conditional stability set cannot be interpreted as the spectrum of an operator in $L^2(\mathbb{R})$. This approach has been worked out in detail in [**122**] and [**123**].

Remark 3.8 (Finite-band potentials). For real-valued potentials Novikov [**85**] and Dubrovin [**30**] showed that $q$ is an algebro-geometric potential if and only if the spectrum of the operator $H$ consists of only finitely many bands. This is no longer true for complex-valued potentials. In fact, for $q = \mathrm{e}^{ix}$ one infers $\sigma(H) = (-\infty, 0]$ but every Dirichlet eigenvalue is movable (see [**123**]).

## 4. A Characterization of Elliptic Solutions of the KdV Hierarchy

In this section we discuss the principal result in [**59**], an explicit characterization of all elliptic algebro-geometric solutions of the KdV hierarchy. One of the two key ingredients in our main Theorem 4.7 (the other being Theorem 3.4) is a systematic use of a powerful theorem of Picard (see Theorem 4.1 below) concerning the existence of solutions which are elliptic functions of the second kind of ordinary differential equations with elliptic coefficients.

We start with Picard's theorem.

Theorem 4.1. *([**93**] – [**95**], see also [**4**], p. 182–187, [**66**], p. 375–376) Let $q_m$, $0 \leq m \leq n$ be elliptic functions with a common period lattice spanned by the fundamental periods $2\omega_1$ and $2\omega_3$. Consider the differential equation*

$$\sum_{m=0}^{n} q_m(z)\psi^{(m)}(z) = 0, \; q_n(z) = 1, \; z \in \mathbb{C} \tag{4.1}$$

*and assume that (4.1) has a meromorphic fundamental system of solutions. Then there exists at least one solution $\psi_0$ which is elliptic of the second kind, that is, $\psi_0$ is meromorphic and*

$$\psi_0(z + 2\omega_j) = \rho_j\psi_0(z), \; j = 1, 3 \tag{4.2}$$

*for some constants $\rho_1, \rho_3 \in \mathbb{C}$. If in addition, the characteristic equation corresponding to the substitution $z \to z+2\omega_1$ or $z \to z+2\omega_3$ (see [**66**], p. 358, 376) has distinct roots then there exists a fundamental system of solutions of (4.1) which are elliptic functions of the second kind.*

The characteristic equation associated with the substitution $z \to z+2\omega_j$ alluded to in Theorem 4.1 is given by

$$\det[A - \rho I] = 0, \tag{4.3}$$

where

$$\phi_\ell(z + 2\omega_j) = \sum_{m=1}^{n} a_{\ell,m}\phi_m(z), \ A = (a_{\ell,m})_{1\le \ell,m\le n} \tag{4.4}$$

and $\phi_1, ...., \phi_n$ is any fundamental system of solutions of (4.1).

What we call Picard's theorem following the usual convention in [**4**], p. 182–185, [**18**], p. 338–343, [**63**], p. 536–539, [**71**], p. 181–189, appears, however, to have a longer history. In fact, Picard's investigations [**93**], [**94**], [**95**] were inspired by earlier work of Hermite in the special case of Lamé's equation [**64**], p. 118–122, 266–418, 475–478 (see also [**9**], Sect. 3.6.4 and [**124**], p. 570–576). Further contributions were made by Mittag-Leffler [**81**], and Floquet [**42**], [**43**], [**44**]. Detailed accounts on Picard's differential equation can be found in [**63**], p. 532–574, [**71**], p. 198–288.

In this context it seems appropriate to recall the well-known fact (see, e.g., [**4**], p. 185–186) that $\psi_0$ is elliptic of the second kind if and only if it is of the form

$$\psi_0(z) = Ce^{\lambda z}\prod_{j=1}^{m}[\sigma(z-a_j)/\sigma(z-b_j)] \tag{4.5}$$

for suitable $m \in \mathbb{N}$ and $C, \lambda, a_j, b_j \in \mathbb{C}$, $1 \le j \le m$. Here $\sigma(z)$ is the Weierstrass sigma function associated with the period lattice $\Lambda$ spanned by $2\omega_1, 2\omega_3$ (see [**1**], Ch. 18).

Picard's Theorem 4.1, restricted to the second-order case

$$\psi''(z) + q(z)\psi(z) = E\psi(z), \tag{4.6}$$

motivates the following definition.

Definition 4.2. Let $q$ be an elliptic function. Then $q$ is called a **Picard potential** if and only if the differential equation (4.6) has a meromorphic fundamental system of solutions (with respect to $z$) for each value of the spectral parameter $E \in \mathbb{C}$.

For completeness we recall the following result.

Theorem 4.3. *([**56**]) (i) Any non-constant Picard potential $q$ has a representation of the form*

$$q(z) = C - \sum_{j=1}^{m} s_j(s_j+1)\wp(z-b_j) \tag{4.7}$$

*for suitable $m, s_j \in \mathbb{N}$ and $C, b_j \in \mathbb{C}$, $1 \le j \le m$, where the $b_j$ are pairwise distinct $\mathrm{mod}(\Lambda)$ and $\wp(z)$ denotes the Weierstrass $\wp$-function associated with the period lattice $\Lambda$ ([**1**], Ch. 18).*

*(ii) Let $q(z)$ be given as in (4.7). If $\psi'' + q\psi = E\psi$ has a meromorphic fundamental system of solutions for a number of distinct values of $E$ which exceeds* $\max\{s_1, \dots, s_m\}$, *then $q$ is a Picard potential.*

We emphasize that while any Picard potential is necessarily of the form (4.7), a potential $q$ of the type (4.7) is a Picard potential only if the constants $b_j$ satisfy a series of additional intricate constraints, see, for instance, Section 3.2 in [**56**].

The following result indicates the connection between Picard potentials and elliptic algebro-geometric potentials.

THEOREM 4.4. *(Its and Matveev* [**68**]*, Krichever* [**72**], [**73**]*, Segal and Wilson* [**102**]*) Every elliptic algebro-geometric potential $q$ is a Picard potential.*

SKETCH OF PROOF. For nonsingular curves $K_g : F^2 = \prod_{j=0}^{2g}(E - \hat{E}_j)$ associated with $q$ (see (2.18)), where $\hat{E}_\ell \neq \hat{E}_{\ell'}$ for $\ell \neq \ell'$, Theorem 4.4 is obvious from the standard representation of the Baker-Akhiezer function in terms of the Riemann theta function of $K_g$ ([**32**], [**68**], [**72**], [**73**]). For singular curves $K_g$ the result follows from the $\tau$-function representation of the Floquet solutions $\psi_\pm(E,x)$ associated with $q$

$$\psi_\pm(E,x) = e^{\pm k(E)x}\tau_\pm(E,x)/\tau(x), \tag{4.8}$$

where

$$q(x) = C + 2\{\ln[\tau(x)]\}'' \tag{4.9}$$

and from the fact that $\tau(x)$ and $\tau_\pm(E,x)$ are entire with respect to $x$ (cf. [**102**]). $\square$

Naturally, one is tempted to conjecture that the converse of Theorem 4.4 is true as well. The rest of this section will explain our proof of this conjecture in [**59**].

We start with a bit of notation. Let $q(z)$ be an elliptic function with fundamental periods $2\omega_1, 2\omega_3$ and assume, without loss of generality, that $\mathrm{Re}(\omega_1) > 0$, $\mathrm{Re}(\omega_3) \geq 0$, $\mathrm{Im}(\omega_3/\omega_1) > 0$. The fundamental period parallelogram then consists of the points $z = 2\omega_1 s + 2\omega_3 t$, where $0 \leq s, t < 1$.

We introduce

$$e^{i\phi} = \frac{\omega_3}{\omega_1}\left|\frac{\omega_1}{\omega_3}\right|, \ \phi \in (0,\pi), \tag{4.10}$$

and

$$t_j = \omega_j/|\omega_j|, \ j = 1,3 \tag{4.11}$$

and define

$$q_j(x) := t_j^2 q(t_j x + z_0), \ j = 1,3 \tag{4.12}$$

for a $z_0 \in \mathbb{C}$ which we choose in such a way that no pole of $q_j, j = 1,3$ lies on the real axis. (This is equivalent to the requirement that no pole of $q$ lies on the line through the points $z_0$ and $z_0 + 2\omega_1$ or on the line through $z_0$ and $z_0 + 2\omega_3$. Since $q$ has only finitely many poles in the fundamental period parallelogram this can always be achieved.) For such a choice of $z_0$ we infer that $q_j(x)$ are real-analytic and periodic of period $\Omega_j = 2|\omega_j|, \ j = 1,3$. Comparing the differential equations

$$\psi''(z) + q(z)\psi(z) = E\psi(z) \tag{4.13}$$

and

$$w''(x) + q_j(x)w(x) = \lambda w(x),\ j = 1, 3, \tag{4.14}$$

connected by the variable transformation

$$z = t_j x + z_0,\ \psi(z) = w(x), \tag{4.15}$$

one concludes that $w$ is a solution of (4.14) if and only if $\psi$ is a solution of (4.13) with

$$\lambda = t_j^2 E,\ j = 1, 3. \tag{4.16}$$

Next, consider $\tilde{q} \in C^0(\mathbb{R})$ of period $\tilde{\Omega} > 0$ and let $\tilde{c}(\lambda, x), \tilde{s}(\lambda, x)$ be the corresponding fundamental system of solutions of $\tilde{w}'' + \tilde{q}\tilde{w} = \lambda\tilde{w}$ defined by

$$\tilde{c}(\lambda, 0) = \tilde{s}'(\lambda, 0) = 1, \quad \tilde{c}'(\lambda, 0) = \tilde{s}(\lambda, 0) = 0. \tag{4.17}$$

The corresponding Floquet discriminant is now given by

$$\tilde{\Delta}(\lambda) = [\tilde{c}(\lambda, \tilde{\Omega}) + \tilde{s}(\lambda, \Omega)]/2 \tag{4.18}$$

and Rouché's theorem then yields

$$\tilde{\Delta}(\lambda) = \cos[i\tilde{\Omega}\lambda^{1/2}(1 + O(\lambda^{-1}))] \tag{4.19}$$

as $|\lambda|$ tends to infinity.

Lemma 4.5. *Let $\tilde{\lambda}_n$ be a periodic or antiperiodic eigenvalue of $\tilde{q}$. Then there exists an $m \in \mathbb{Z}$ such that*

$$\left|\tilde{\lambda}_n + m^2\pi^2\tilde{\Omega}^{-2}\right| \le \tilde{C} \tag{4.20}$$

*for some $\tilde{C} > 0$ independent of $n \in \mathbb{N}_0$. In particular, all periodic and antiperiodic eigenvalues $\tilde{\lambda}_n$, $n \in \mathbb{N}_0$ of $\tilde{q}$ are contained in a half-strip $\tilde{S}$ given by*

$$\tilde{S} = \{\lambda \in \mathbb{C} |\, |\mathrm{Im}(\lambda)| \le \tilde{C},\ \mathrm{Re}(\lambda) \le \tilde{M}\} \tag{4.21}$$

*for some $\tilde{M} \in \mathbb{R}$.*

In order to apply Lemma 4.5 to $q_1$ and $q_3$ we note that according to (4.19),

$$\Delta_j(\lambda) = \cos[i\Omega_j\lambda^{1/2}(1 + O(\lambda^{-1}))],\ j = 1, 3 \tag{4.22}$$

as $|\lambda|$ tends to infinity, where, in obvious notation, $\Delta_j(\lambda)$ denotes the discriminant of $q_j(x)$, $j = 1, 3$. Next, denote by $\lambda_{j,n}$ an $\Omega_j$-(anti)periodic eigenvalue of $w'' + q_j w = \lambda w$. Then $E_{j,n} = t_j^{-2}\lambda_{j,n}$ is a $2\omega_j$-(anti)periodic eigenvalue of $\psi'' + q\psi = E\psi$ and vice versa. Hence Lemma 4.5 immediately yields the following result.

Lemma 4.6. *Let $j = 1$ or $3$. Then all $2\omega_j$-(anti)periodic eigenvalues $E_{j,n}$, $n \in \mathbb{N}_0$ associated with $q$ lie in the half-strip $S_j$ given by*

$$S_j = \{E \in \mathbb{C} : |\,\mathrm{Im}(t_j^2 E)| \le C_j,\ \mathrm{Re}(t_j^2 E) \le M_j\} \tag{4.23}$$

*for suitable constants $C_j > 0, M_j \in \mathbb{R}$. The angle between the axes of the strips $S_1$ and $S_3$ equals $2\phi \in (0, 2\pi)$.*

Lemmas 4.5 and 4.6 apply to any elliptic potential whether or not they are algebro-geometric. In our final step we shall now invoke Picard's Theorem 4.1 to obtain our characterization of elliptic algebro-geometric potentials.

THEOREM 4.7. *$q$ is an elliptic algebro-geometric potential if and only if $q$ is a Picard potential (i.e., if and only if for each $E \in \mathbb{C}$ every solution of $\psi''(z) + q(z)\psi(z) = E\psi(z)$ is meromorphic with respect to $z$).*

PROOF. By Theorem 4.4 it remains to prove that a Picard potential is algebro-geometric. Hence we assume in the following that $q$ is a Picard potential. Since all $2\omega_j$-(anti)periodic eigenvalues $E_{j,n}$ of $q$ yield zeros $\lambda_{j,n} = t_j^2 E_{j,n}$ of the entire functions $\Delta_j(\lambda)^2 - 1$, the $E_{j,n}$ have no finite limit point. Next we choose $R > 0$ large enough such that the exterior of the closed disk $\overline{D(0,R)}$ centered at the origin of radius $R > 0$ contains no intersection of $S_1$ and $S_3$ (defined in (4.23)), that is,

$$(\mathbb{C}\backslash\overline{D(0,R)}) \cap (S_1 \cap S_3) = \emptyset. \tag{4.24}$$

Let $\rho_{j,\pm}(\lambda)$ be the Floquet multipliers of $q_j(x)$, that is, the solutions of

$$\rho_j^2 - 2\Delta_j\rho_j + 1 = 0, \ j = 1, 3. \tag{4.25}$$

Then (4.24) implies that for $E \in \mathbb{C}\backslash\overline{D(0,R)}$, at most one of the numbers $\rho_1(t_1 E)$ and $\rho_3(t_3 E)$ can be in $\{-1, 1\}$. In particular, at least one of the characteristic equations corresponding to the substitution $z \to z + 2\omega_1$ or $z \to 2\omega_3$ (cf. (4.3) and (4.4)) has two distinct roots. Since by hypothesis $q$ is a Picard potential, Picard's Theorem 4.1 applies and guarantees for all $E \in \mathbb{C}\backslash\overline{D(0,R)}$ the existence of two linearly independent solutions $\psi_1(E,z)$ and $\psi_2(E,z)$ of $\psi'' + q\psi = E\psi$ which are elliptic of the second kind. Then $w_{j,k}(x) = \psi_k(t_j x + z_0)$, $k = 1, 2$ are linearly independent Floquet solutions associated with $q_j$. Therefore the points $\lambda$ for which $w'' + q_j w = \lambda w$ has only one Floquet solution are necessarily contained in $\overline{D(0,R)}$ and hence finite in number. This is true for both $j = 1$ and $j = 3$. Applying Theorem 3.4 then proves that both $q_1$ and $q_3$ are algebro-geometric potentials.

By (2.8) (in slight abuse of notation)

$$\sum_{k=0}^{g} c_{g-k} \frac{df_{k+1}(q_1(x))}{dx} = 0, \tag{4.26}$$

where $g \in \mathbb{N}_0$, $f_{k+1}$, $k = 0, ..., g$, are differential polynomials in $q_1$ homogeneous of degree $2k + 2$ (cf. (2.9)), and $c_k$, $k = 0, ..., g$ are complex constants. Since

$$q_1^{(\ell)}(x) = t_1^{\ell+2} q^{(\ell)}(z), \tag{4.27}$$

(where $z = t_1 x + z_0$) we obtain

$$\sum_{k=0}^{g} c_{g-k} t_1^{2k+3} \frac{df_{k+1}(q(z))}{dz} = 0, \tag{4.28}$$

that is, $q$ is an algebro-geometric potential as well. A similar argument would have worked using the relationship between $q_3$ and $q$. In particular, the order of the operators commuting with $d^2/dz^2 + q(z)$, $d^2/dx^2 + q_1(x)$, and $d^2/dx^2 + q_3(x)$, respectively, is the same in all cases, namely $2g + 1$. □

We add a series of remarks further illustrating the significance of Theorem 4.7.

REMARK 4.8 (Complementing Picard's theorem). First we note that Theorem 4.7 extends and complements Picard's Theorem 4.1 in the sense that it determines the elliptic functions which satisfy the hypothesis of the theorem precisely as (elliptic) algebro-geometric solutions of the stationary KdV hierarchy.

Remark 4.9 (Characterizing elliptic algebro-geometric potentials). While an explicit proof of the algebro-geometric property of $q$ is, in general, highly nontrivial (see, e.g., the references cited in connection with special cases such as the Lamé-Ince and Treibich-Verdier potentials in Remark 4.11 below), the fact of whether or not $\psi''(z)+q(z)\psi(z)=E\psi(z)$ has a fundamental system of solutions meromorphic in $z$ for a finite (but sufficiently large) number of values of the spectral parameter $E\in\mathbb{C}$ can be decided by means of an elementary Frobenius-type analysis (see, e.g., [**54**] and [**55**]). Theorem 4.7 appears to be the only effective tool to identify general elliptic algebro-geometric solutions of the KdV hierarchy.

Remark 4.10 (Reduction of abelian integrals). Theorem 4.7 is also relevant in the context of the Weierstrass theory of reduction of abelian to elliptic integrals, a subject that attracted considerable interest, see, for instance, [**7**], [**8**], [**9**], Ch. 7, [**10**], [**11**], [**21**], [**36**], [**37**], [**38**], [**67**], [**70**], [**74**], [**77**], [**104**], [**105**], [**110**]. In particular, the theta functions corresponding to the hyperelliptic curves derived from the Burchnall-Chaundy polynomials (2.15), associated with Picard potentials, reduce to one-dimensional theta functions.

Remark 4.11 (Computing genus and branch points). Even though Theorem 4.7 characterizes all elliptic algebro-geometric potentials as Picard potentials, it does not yield an effective way to compute the underlying hyperelliptic curve $K_g$; in particular, its proof provides no means to compute the branch and singular points nor the (arithmetic) genus $g$ of $K_g$. To the best of our knowledge $K_g$ has been computed only for Lamé-Ince potentials and certain Treibich-Verdier potentials (see, e.g., [**6**], [**11**], [**70**], [**86**], [**105**], [**110**], [**118**], [**120**], [**124**]). Even the far simpler task of computing $g$ previously had only been achieved in the case of Lamé-Ince potentials (see [**65**] and [**115**] for the real and complex-valued case, respectively). In [**54**], [**55**], and [**56**] we have treated these problems for Lamé-Ince, Treibich-Verdier, and reflection symmetric elliptic algebro-geometric potentials, respectively. In particular, in [**55**] we computed $g$ for all Treibich-Verdier potentials and in [**56**] we reduced the computation of the branch and singular points of $K_g$ for any reflection symmetric elliptic algebro-geometric potential to the solution of constraint linear algebraic eigenvalue problems. We refrain from reproducing a detailed discussion of this matter here, instead we just recall an example taken from [**55**] which indicates some of the subtleties involved: Consider the potentials

$$q_4(z)=-20\wp(z-\omega_j)-12\wp(z-\omega_k), \tag{4.29}$$

$$\hat{q}_4(z)=-20\wp(z-\omega_j)-6\wp(z-\omega_k)-6\wp(z-\omega_\ell), \tag{4.30}$$

$$q_5(z)=-30\wp(z-\omega_j)-2\wp(z-\omega_k), \tag{4.31}$$

$$\hat{q}_5(z)=-12\wp(z-\omega_j)-12\wp(a-\omega_k)-6\wp(z-\omega_\ell)-2\wp(z-\omega_m), \tag{4.32}$$

where $j,k,\ell,m\in\{1,2,3,4\}$ ($\omega_2=\omega_1+\omega_3$, $\omega_4=0$) are mutually distinct. Then $q_4$ and $\hat{q}_4$ correspond to (arithmetic) genus $g=4$ while $q_5$ and $\hat{q}_5$ correspond to $g=5$. However, we emphasize that all four potentials contain precisely 16 summands of the type $-2\wp(x-b_n)$ (cf. the discussion following (1.10)). $q_5$ and $\hat{q}_5$ are isospectral (i.e., correspond to the same curve $K_5$) while $q_4$ and $\hat{q}_4$ are not.

## 5. Picard's Theorem for First-Order Systems

Having characterized all elliptic algebro-geometric solutions of the KdV hierarchy which are related to the second-order expression $L_2=d^2/dz^2+q$, it is natural to try to extend Theorem 4.7 to $n$-th order expressions $L_n$ (connected with the Gel'fand-Dickey hierarchy). Actually, a more general extension to integrable systems related to general first-order $n \times n$ matrix-valued differential expressions seems very desirable in order to include AKNS systems (see, e.g., [**51**]) and the matrix hierarchies of integrable equations described in detail, for instance, in [**26**], Sects. 9, 13–16, [**31**], [**33**]. Picard's Theorem 4.1 generalizes in a straightforward manner to first-order systems, that is, pairwise distinct Floquet multipliers in one of the fundamental directions and a meromorphic fundamental system of solutions guarantee the existence of a fundamental system of solutions which are elliptic of the second kind (see (5.3)). Moreover, it is possible to obtain the explicit Floquet-type structure of these solutions (cf. Theorem 5.2).

Denote by $M(n)$ the set of $n \times n$ matrices with entries in $\mathbb{C}$ and consider the linear homogeneous system

$$\Psi'(z) = Q(z)\Psi(z), \quad z \in \mathbb{C}, \tag{5.1}$$

where $Q(z) \in M(n)$ and where the entries of $Q(z)$ are elliptic functions with a common period lattice $\Lambda$ spanned by $2\omega_1$ and $2\omega_3$ which satisfy the same conditions as before.

Assuming without loss of generality that no pole of $Q(z)$ lies on the line containing the segments $[0, 2\omega_j]$, Floquet theory with respect to these directions yields the existence of fundamental matrices $\Phi_j(z)$ of the type

$$\Phi_j(z) = P_j(z) \exp\left(zK_j\right), \tag{5.2}$$

where $P_j(z)$ is a periodic matrix with period $2\omega_j$ and $K_j$ is a constant matrix. The monodromy matrix is given by $M_j = \exp(2\omega_j K_j)$. We want to establish the existence of a Floquet representation *simultaneously* for both directions $2\omega_1$ and $2\omega_3$. More precisely, we intend to find solutions $\underline{\phi}$ of $\Psi'(z) = Q(z)\Psi(z)$ satisfying

$$\underline{\phi}(z + 2\omega_j) = \rho_j \underline{\phi}, \quad j = 1, 3, \tag{5.3}$$

where $\rho_j \in \mathbb{C}\backslash\{0\}$. Solutions $\underline{\phi}(z)$ of (5.1) satisfying (5.3) are again called elliptic of the second kind.

Even though Picard did mention certain extensions of his result to first-order systems (see, e.g., [**60**], p. 248–249), apparently he did not seek a Floquet representation for systems in the elliptic case. The first to study such a representation seems to have been Fedoryuk who proved the following result.

THEOREM 5.1. *([**41**]) Let $Q(z)$ be an $n \times n$ matrix whose entries are elliptic functions with fundamental periods $2\omega_1$ and $2\omega_3$ and suppose that (5.1) has a single-valued fundamental matrix of solutions. Then (5.1) admits a fundamental matrix $\Phi(z)$ of the type*

$$\Phi(z) = D(z) \exp\left(zS + \zeta(z)T\right), \quad z \in \mathbb{C}, \tag{5.4}$$

*where $S, T \in M(n)$, $D(z)$ is invertible and doubly periodic and*

$$S = \frac{1}{\pi i}\left[2\omega_3 \zeta\left(\omega_1\right) K_3 - 2\omega_1 \zeta\left(\omega_3\right) K_1\right], \quad T = -\frac{2\omega_1\omega_3}{\pi i}\left(K_3 - K_1\right). \tag{5.5}$$

*Moreover, $K_1$ and $K_3$, and hence $S$ and $T$ commute.*

Fedoryuk's representation (5.4) has the peculiar feature that it seems to stress an apparent essential singularity structure of solutions at $z = 0$. Indeed, since $\zeta(z)$

has a first-order pole at $z = 0$, the term $\exp(\zeta(z)T)$ in (5.4) exhibits an essential singularity unless $T$ is nilpotent. Hence, the doubly periodic matrix $D(z)$, in general, will cancel the essential singularity of $\exp(\zeta(z)T)$ and therefore cannot be meromorphic and hence not elliptic. Thus Fedoryuk's result cannot be considered the natural extension of Picard's Theorem 4.1.

In the remainder of this section we shall focus on an alternative to Theorem 5.1 and describe a generalization of Picard's theorem in the context of first-order systems with elliptic coefficients.

THEOREM 5.2. *([**52**]) Let $Q(z)$ be an elliptic $n \times n$ matrix with fundamental periods $2\omega_1$ and $2\omega_3$ and suppose that (5.1) has a meromorphic fundamental matrix $\Psi(z)$ of solutions. Then (5.1) admits a fundamental matrix of the type*

$$\begin{aligned} \Phi(z) =& E(z)\sigma(z)^{-1}\sigma\left(zI_n - \frac{2\omega_1\omega_3}{\pi i}(K_3 - K_1)\right) \\ &\times \exp\left\{\frac{z}{\pi i}\left[2\omega_3\zeta(\omega_1)K_3 - 2\omega_1\zeta(\omega_3)K_1\right]\right\}, \quad z \in \mathbb{C}, \end{aligned} \tag{5.6}$$

*where $E(z)$ is an elliptic matrix with periods $2\omega_j$ and $K_1$, $K_3$ (and hence $M_j = \exp(2\omega_j K_j)$, $j = 1, 3$) are commuting matrices. Moreover, linearly independent solutions $\underline{\phi}_m(z) \in \mathbb{C}^n$, $1 \le m \le n$ of (5.1), that is, column vectors of (5.6), are of the type*

$$\underline{\phi}_m(z) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \underline{e}_{m,k_1,k_2}(z)\exp(z\mu_{m,k_1,k_2})\, z^{k_1}\zeta(z)^{k_2}, \tag{5.7}$$

*where the vectors $\underline{e}_{m,k_1,k_2}(z)$ are elliptic, the numbers $\mu_{m,k_1,k_2}$ denote the (not necessarily distinct) eigenvalues of*

$$(1/\pi i)\left[2\omega_3\zeta(2\omega_1/2)K_3 - 2\omega_1\zeta(2\omega_3/2)K_1\right], \tag{5.8}$$

*and, most notably, the upper limits of the sums in (5.7) satisfy*

$$n_1 + n_2 \le n - 1. \tag{5.9}$$

*In particular, there exists at least one solution $\underline{\phi}_{m_0}(z)$ of (5.1) which is elliptic of the second kind, that is, $\underline{\phi}_{m_0}(z)$ is meromorphic on $\mathbb{C}$ and*

$$\underline{\phi}_{m_0}(z + 2\omega_j) = \rho_{m_0,j}\underline{\phi}_{m_0}(z), \quad j = 1, 3, \quad z \in \mathbb{C} \tag{5.10}$$

*for some $\rho_{m_0,j} = \exp(2\omega_j\mu_{m_0,0,0}) \in \mathbb{C}\backslash\{0\}$, $j = 1, 3$. In addition, if all eigenvalues of $M_1$ or $M_3$ are distinct, then there exists a fundamental system of solutions $\{\underline{\phi}_m(z)\}_{1\le m\le n}$ of (5.1) with all $\underline{\phi}_m(z)$ elliptic of the second kind.*

For the proof one considers a meromorphic fundamental matrix $\tilde{\Psi}(z)$ of 5.1 and defines

$$\begin{aligned} E(z) =& \tilde{\Psi}(z)\exp\left(-\frac{z}{\pi i}\left[2\omega_3\zeta(\omega_1)K_3 - 2\omega_1\zeta(\omega_3)K_1\right]\right) \\ &\times \sigma\left(zI_n - \frac{2\omega_1\omega_3}{\pi i}(K_3 - K_1)\right)^{-1}\sigma(z). \end{aligned} \tag{5.11}$$

By hypothesis, $E(z)$ is meromorphic and, applying the addition theorem

$$\sigma(z + 2\omega_j) = -\sigma(z)\exp\left\{2\zeta(\omega_j)\left[z + (\omega_j)\right]\right\}, \quad 1 \le j \le 3 \tag{5.12}$$

(more precisely, a matrix-valued generalization thereof), one verifies that

$$E(z+2\omega_j)=E(z), \quad j=1,3, \tag{5.13}$$

that is, $E(z)$ is elliptic. The remaining assertions in Theorem 5.2 follow by transforming $(K_3-K_1)$ and, say, $K_1$ (separately) into their Jordan normal forms (cf. [**52**]).

Remark 5.3. If $M_1$ or $M_3$ has distinct eigenvalues, one can show that equation (5.1) has a fundamental system of solutions $\underline{\phi}_m(z)$ of the form,

$$\begin{aligned}\underline{\phi}_m(z) =& \underline{e}_m(z)\sigma(z)^{-1}\sigma\left(z-\frac{2\omega_1\omega_3}{\pi i}k_{3,1,m}\right)\\ &\times\exp\left(\frac{2\omega_3 z}{\pi i}\zeta\left(\omega_1\right)k_{3,1,m}\right)\exp\left(zk_{1,m}\right), \quad 1\le m\le n, \quad z\in\mathbb{C},\end{aligned} \tag{5.14}$$

where $\underline{e}_m(z)$ are elliptic with period lattice $\Lambda$ and $\{k_{3,1,m}\}_{1\le m\le n}$ and $\{k_{1,m}\}_{1\le m\le n}$ are the eigenvalues of $(K_3-K_1)$ and $K_1$, respectively.

Remark 5.4. In the special scalar case (4.1), the bound (5.9) was proved in [**66**], p. 377–378 for $n=2$ and stated (without proof) for $n\ge 3$.

## 6. The Higher-Order Scalar Case

In our final section we consider a differential expression $L_n$ of the form

$$L_n y = y^{(n)} + q_{n-2}y^{(n-2)} + \ldots + q_0 y, \tag{6.1}$$

where initially the coefficients $q_0,\ldots,q_{n-2}$ are continuous complex-valued functions of a real variable periodic with period $\Omega>0$. Again we denote the ($n$-dimensional) vector space of solutions of the differential equation $L_n y = Ey$ by $\mathcal{L}(E)$ and the operator which shifts the argument of a function in $\mathcal{L}(E)$ by a period $\Omega$ by $T(E)$. As before $T(E)$ and $L_n$ commute which implies that $T(E)$ maps $\mathcal{L}(E)$ to itself. Floquet multipliers, that is, eigenvalues of $T(E)$ are given as zeros of the polynomial

$$\mathcal{F}(E,\rho)=(-1)^n\rho^n+(-1)^{n-1}a_1(E)\rho^{n-1}+\ldots-a_{n-1}(E)\rho+1=0, \tag{6.2}$$

where the functions $a_1,\ldots,a_{n-1}$ are entire. This is obvious after choosing the basis $\phi_1(E,x),\ldots,\phi_n(E,x)$ of $\mathcal{L}(E)$ satisfying the initial conditions $\phi_j^{k-1}(E,x_0)=\delta_{j,k}$.

Note that $\mathcal{F}(E,\cdot)$ has $n$ distinct zeros unless the discriminant of $\mathcal{F}(E,\cdot)$, which is an entire function of $E$, is equal to zero. Thus, this happens at most at countably many points. Denote by $m_g(E,\rho)$ and $m_f(E,\rho)$ the geometric and algebraic multiplicity, respectively, of the eigenvalue $\rho$ of $T(E)$. Then the number $m_f(E,\rho)-m_g(E,\rho)\in\{0,1,\ldots,n-1\}$ counts the "missing" Floquet solutions of $L_n y = Ey$ with multiplier $\rho$. We will be interested in the case where this number is positive only for finitely many points $E$.

For $\theta\in\mathbb{C}$, consider the operator $H(\theta)$ associated with the differential expression $L_n$ in $L^2([x_0,x_0+\Omega])$ with domain

$$D(H(\theta))=\{g\in H^{2,n}([x_0,x_0+\Omega]) : g^{(k)}(x_0+\Omega)=e^{i\theta}g^{(k)}(x_0),\ k=0,\ldots,n-1\}, \tag{6.3}$$

where $H^{p,r}(\cdot)$ denote the usual Sobolev spaces with $r$ distributional derivatives in $L^p(\cdot)$.

$H(\theta)$ has discrete spectrum. In fact, its eigenvalues, which will be called Floquet eigenvalues, are given as the zeros of $\mathcal{F}(\cdot,\rho)$. Moreover, the algebraic multiplicity

$m_a(E,\rho)$ of $E$ as an eigenvalue of $H(\theta)$ is given as the order of $E$ as a zero of $\mathcal{F}(\cdot,\rho)$ (see, e.g., [**57**]).

With any differential expression $L_n$ given by (6.1) with continuous complex-valued periodic coefficients $q_0,...,q_{n-2}$ of a real variable we associate the corresponding closed operator $H : H^{2,n} \to L^2(\mathbb{R})$, $Hy = L_n y$.

Rofe-Beketov's result [**99**], referred to in Section 3, originally was proved for an $n$-th order operator. Hence the spectrum $\sigma(H)$ of $H$ equals the conditional stability set $\mathcal{S}(L_n)$ of $L_n$, that is, the set of all complex numbers $E$ for which the differential equation $L_n y = Ey$ has a nontrivial bounded solution. For $E$ to be in $\mathcal{S}(L_n)$ it is necessary and sufficient that $L_n y = Ey$ has a Floquet multiplier of modulus one. Hence

$$\mathcal{S}(L_n) = \{E \in \mathbb{C} : \mathcal{F}(E, e^{it}) = 0 \text{ for some } t \in \mathbb{R}\}, \tag{6.4}$$

where $\mathcal{F}$ is given by (6.2). Since $\mathcal{F}$ is entire in both its variables, it follows that $\sigma(H) = \mathcal{S}(L_n)$ consists of (generally) infinitely many regular analytic arcs (i.e., spectral bands). They end at a point where the arc fails to be regular analytic or extends to infinity. Finite endpoints of the spectral bands are called band edges.

Definition 6.1. (i) $H$ is called a **finite-band operator** if and only if $\sigma(H)$ consists of a finite number of regular analytic arcs.
(ii) $L_n$ is called a **Picard differential expression** if and only if all its coefficients are elliptic functions associated with a common period lattice and if $L_n y = Ey$ has a meromorphic fundamental system (with respect to the independent variable) for any value of the spectral parameter $E \in \mathbb{C}$.

Next, let $\Phi(E,x)$ be the fundamental matrix of $L_n y = Ey$ satisfying the initial condition $\Phi(E,x_0) = I_n$ where $I_n$ is the $n \times n$ identity matrix. The Floquet multipliers of the differential equation $L_n y = Ey$ are then the eigenvalues of the monodromy matrix $\Phi(E, x_0 + \Omega)$.

Our aim is to determine multiplicities of Floquet eigenvalues and multipliers for large values of the spectral parameter $E$. For large values of $E$ the equation $L_n y = Ey$ can be treated as a perturbation of $y^{(n)} = Ey$. In this case there exist $n$ linearly independent Floquet solutions $\exp(\lambda\sigma_k x)$ with associated Floquet multipliers $\exp(\lambda\sigma_k)$, where $\lambda$ is such that $\lambda^n = -E$ and the $\sigma_k$ are the different $n$-th roots of $-1$. The characteristic polynomial of the associated monodromy matrix is therefore given by

$$\mathcal{F}_0(E,\rho) = (-1)^n\rho^n + (-1)^{n-1}a_{F,1}(E)\rho^{n-1} + ... - a_{F,n-1}(E)\rho + 1, \tag{6.5}$$

where the $a_{F,j}$ are the elementary symmetric polynomials in the variables $\exp(\lambda\sigma_1)$, ..., $\exp(\lambda\sigma_n)$. Perturbation theory now yields that the coefficients $a_j$ in (6.2) are related to the coefficients $a_{F,j}$ in (6.5) by

$$a_j(E) = a_{F,j}(E) + b_j(E), \quad j = 1, ...n-1, \tag{6.6}$$

where, for some suitable positive constant $M$,

$$|b_j| \leq \frac{M}{|\lambda|} |\exp(\lambda\sigma_{n+1-j}) ... \exp(\lambda\sigma_n)|, \quad j = 1, ..., n-1 \tag{6.7}$$

having ordered the roots in such a way that $|\exp(\lambda\sigma_j)| \leq |\exp(\lambda\sigma_{j+1})|$ for $j = 1, ..., n-1$. This allows one to show that asymptotically, in a certain small disk about

$\exp(\lambda\sigma_k)$, there are either one or two Floquet multpliers of $L_n y = Ey$ depending on whether or not the inequality

$$|\exp(\lambda\sigma_k) - \exp(\lambda\sigma_j)| \geq \gamma \max\{|\exp(\lambda\sigma_k)|, |\exp(\lambda\sigma_j)|\} \text{ for all } j \neq k \tag{6.8}$$

holds. From this one may prove the following theorem concerning algebraic multplicities of Floquet multipliers.

THEOREM 6.2. *([**121**]) Let $L_n$ be defined as in (6.1). For every $\varepsilon > 0$ there exists a disk $B(\varepsilon) \subset \mathbb{C}$ with the following two properties.*
*1. All values of $E$, where at least two Floquet multipliers of the differential equation $L_n y = Ey$ coincide, lie in $B(\varepsilon)$ or in the cone $\{E : |\operatorname{Im}(E)|/|\operatorname{Re}(E)| \leq \varepsilon\}$.*
*2. Every degenerate Floquet multiplier outside $B(\varepsilon)$ has multiplicity two.*

This result has been obtained earlier by McKean [**78**] for $n = 3$ and by da Silva Menezes [**25**] for general $n$.

Moreover the above observations may be used to obtain information concerning algebraic multiplicities of Floquet eigenvalues.

THEOREM 6.3. *([**121**]) Let $\theta_0 \in \mathbb{C}$. Then there exists an $R > 0$ such that every eigenvalue $E$ of the Floquet operator $H(\theta_0)$, which satisfies $|E| > R$, has at most algebraic multiplicity two.*

Now suppose that $\mathcal{F}(E,\rho) = 0$ and that $E \in \mathcal{S}(L_n)$ is suitably large. Since $1 \leq m_f(E,\rho), m_a(E,\rho) \leq 2$ we have to distinguish four cases and Weierstrass's preparation theorem provides us with the following information:
1. If $m_f(E,\rho) = m_a(E,\rho) = 1$ then one spectral band passes through $E$.
2. If $m_f(E,\rho) = 2$ and $m_a(E,\rho) = 1$ then two (possibly coinciding) spectral bands end in $E$.
3. If $m_f(E,\rho) = 1$ and $m_a(E,\rho) = 2$ then two spectral bands intersect in $E$ forming a right angle.
4. If $m_f(E,\rho) = m_a(E,\rho) = 2$ then two (possibly coinciding) spectral bands pass through $E$.

This shows that a necessary condition for a suitably large $E$ to be a band edge is that $m_a(E,\rho) = 1$ and $m_f(E,\rho) = 2$ for some $\rho$. In particular, such an $E$ is necessarily a point where strictly less than $n$ linearly independent Floquet solutions exist. In addition, when $n$ is odd then $\sigma(H)$ is ultimately in a cone with the imaginary axis as symmetry axis while the possible band edges (where $m_f(E,\rho) = 2$) are in a cone whose axis is the real axis. Therefore we have the following result.

THEOREM 6.4. *([**121**]) The operator $H$ associated with the differential expression $L_n$ introduced after (6.3) is a finite-band operator whenever $n$, the order of $L_n$, is odd.*

Finally we turn to the case where $L_n$ is a Picard differential expression. The principal result of this section, Theorem 6.5 below, then shows that algebraic and geometric multiplicities of Floquet multipliers of $L_n y = Ey$ can be different only when $E$ is one of finitely many points.

THEOREM 6.5. *([**121**]) Suppose the differential expression $L_n$ is Picard. Then there exist $n$ linearly independent solutions of $L_n y = Ey$ which are elliptic of the second kind for all but finitely many values of the spectral parameter $E$.*

Sketch of proof. The proof is modeled closely after the one of Theorem 4.7. Again, inside a compact set there can be only a finite number of values of $E$ where Floquet multipliers associated with a fundamental period of the coefficients of $L_n$ are degenerate. On the other hand when $|E|$ becomes large we only have to prove that for one of the fundamental periods of the coefficients of $L_n$ all Floquet multipliers of $L_n y = Ey$ are distinct according to Picard's Theorem 4.1.

Assume that the fundamental periods $2\omega_1$ and $2\omega_3$ are such that the angle $\phi$ between them is less than $\pi/n$ and assume that $z_0$ is such that no singularity of $q_0, ..., q_{n-2}$ lies on the line through $z_0$ and $z_0 + 2\omega_1$ or on the line through $z_0$ and $z_0 + 2\omega_3$.

Substituting $w(x) = y(2\omega_1 x + z_0)$ and defining $p_k(x) = (2\omega_1)^{n-k} q_k(2\omega_1 x + z_0)$ transforms $L_n y = Ey$ into

$$w^{(n)} + p_{n-2}(x) w^{(n-2)} + ... + p_0(x) y = (2\omega_1)^n E w. \tag{6.9}$$

Therefore, Theorem 6.2 implies that all Floquet multipliers associated with the periods $2\omega_1$ $(2\omega_3)$ are pairwise distinct provided the spectral parameter $(2\omega_j)^n E$ lies outside the set $S_j$, $j = 1, 3$, where

$$S_j = \left\{ z : \left| \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \right| \le \frac{\phi}{3} \right\} \cup \{ z : |z| \le R_j \}, \tag{6.10}$$

with $R_j$, $j = 1, 3$ being suitable positive constants.

The two sets $S_1$ and $S_3$ do not intersect outside a sufficiently large disk $D$. Hence, for each value of $E$ outside $D$, Picard's theorem guarantees the existence of $n$ linearly independent solutions of $L_n y = Ey$ which are elliptic functions of the second kind. □

In particular, when $|E|$ is large and $L_n$ is Picard, we infer that $m_f(E,\rho) = m_g(E,\rho) \le m_a(E,\rho)$. Moreover, we have shown earlier that necessarily $m_a(E,\rho) = 1$ and $m_f(E,\rho) = 2$ for band edges $E$ with $|E|$ sufficiently large. Thus there are no band edges with sufficiently large absolute values for Picard expressions. One may also show that at most two bands extend to infinity. Hence we have the following final theorem which, in view of Theorem 6.4, has significance only when $n$ is even.

Theorem 6.6. *([**121**]) Let $L_n$ be a Picard differential expression and $H$ the associated operator. Then, if $\sigma(H)$ does not contain closed regular analytic arcs, $\sigma(H)$ consists of finitely many analytic arcs which are regular in their interior.*

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