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A local Borg–Marchenko theorem for complex potentials ☆

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Abstract

The Borg–Marchenko theorem states that the Weyl–Titchmarsh m-function of the differential expression $-\mathrm{d}^2/\mathrm{d}x^2 + q$ with a real-valued potential q determines this potential uniquely. We investigate the validity of the Borg–Marchenko theorem (and its local version) for complex-valued potentials.

Keywords: Complex potentials; Sturm-Liouville expressions; Borg–Marchenko theroem; m-function

1. Introduction

The celebrated Borg–Marchenko theorem states that the Titchmarsh–Weyl m-function of a selfadjoint operator associated with the differential expression $L = -\mathrm{d}^2/\mathrm{d}x^2 + q$ determines the potential q uniquely (cf. [3,10–12]). Recently, Simon [14] formulated a local version of this theorem stating that q is uniquely determined on some interval $[0, a]$ if the m-function is given along some non-real ray within an error term of order $\exp(-2a\Im(\sqrt{\lambda}))$ (where $\Im(\sqrt{\lambda}) > 0$). Simplified proofs of Simon's result have since been provided by Gesztesy and Simon [7] and Bennewitz [1]. In this paper, we will explore the corresponding result for complex potentials and complex boundary conditions.

In Section 2, we will present our main results together with some basic notation, a review of the nesting circle analysis, the definition of the m-function, and an application for complex potentials on a compact interval. The proofs of some technical details have been delayed to Section 3.

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2. Main results

2.1. Basic notation

Throughout this paper Σ denotes a fixed open sector of the complex plane whose vertex is at the origin. Let q be a complex-valued, locally integrable function on some interval $[0,b)$ where b is either a positive real number or perhaps infinity. (b may be different for different potentials and we will often speak about the pair (q,b) if we want to emphasize the domain of q.) We will consider only such potentials q for which there is an open half-plane Λ satisfying the following two requirements: [1]

(1) $\Lambda^c \cap \Sigma$ is bounded.
(2) The set $Q(q) = \overline{\mathrm{co}}(\{q(x) + r : x \in [0,b), 0 < r < \infty\})$ does not intersect Λ.

Note that $Q(q)$ changes even if q changes only on a set of measure zero. For definiteness we assume therefore that

$$q(x) = \lim_{n\to\infty} n \int_x^{x+1/n} \tilde{q}(t)\,\mathrm{d}t,$$

where $\tilde{q}$ is any representative in the class of potentials which are equal to each other almost everywhere. Of course, q is then also one such representative and does not depend on the choice of $\tilde{q}$.

Conditions of this type have first been introduced by Brown et al. [4]. The set of all such potentials will be denoted by $\mathcal{Q}_\Sigma$.

Note that Σ will not contain the positive real axis if $\mathcal{Q}_\Sigma$ is not empty.

When one is interested in real-valued potentials only (so that the sets $Q(q)$ are subsets of the real line) one may choose for Σ any sector (with vertex zero) contained in the upper or lower half-plane (the half-plane containing Σ would be Λ). When q is real and bounded below Σ could be any sector (with vertex zero) not containing the positive real axis.

Now let q be a potential in $\mathcal{Q}_\Sigma$ and let L be the differential expression $L = -\mathrm{d}^2/\mathrm{d}x^2 + q$. The same proof as in the self-adjoint case (see, e.g., [5] or [6]) shows that all solutions of $Ly = \lambda y$ are square integrable on $[0,b)$ for all complex numbers λ if this is the case for just one particular complex number λ, i.e., the distinction of whether or not all solutions of $Ly = \lambda y$ are square integrable on $[0,b)$ is independent of λ. This gives rise to the following definition.

Definition 2.1. If at most one (up to constant multiples) solution of $Ly = \lambda y$ is square integrable on $[0,b)$ we say that the pair (q,b) is of Class I. Otherwise, if all solutions of $Ly = \lambda y$ are square integrable on $[0,b)$, we say that the pair (q,b) is of Class II.

Now choose complex numbers h_1, h_2, H_1, and H_2 such that $h_1H_2 - h_2H_1 \neq 0$. For any $\lambda \in \mathbb{C}$ let $\theta(\cdot,\lambda)$ and $\phi(\cdot,\lambda)$ be the unique linearly independent solutions of $Ly = \lambda y$ satisfying

$$\phi(0,\lambda) = h_1, \quad \theta(0,\lambda) = h_2,$$

$$\phi'(0,\lambda) = H_1, \quad \theta'(0,\lambda) = H_2.$$

[1] If S is a subset of the complex plane we denote its complement by S^c and its closed convex hull by $\overline{\mathrm{co}}(S)$.

The maximal operator T associated with q is the operator defined by $Ty = Ly$ on the set

$$D(T) = \{f \in L^2([0,b)) : f, f' \in \mathrm{AC}_{\mathrm{loc}}([0,b)), Lf \in L^2([0,b))\}.$$

Note that a square integrable solution of $Ly = \lambda y$ is an element of $D(T)$.

For once differentiable functions f and g we use the notation $[f,g](x) = f(x)g'(x) - f'(x)g(x)$. It is well known that $\lim_{x\to b}[f,g](x)$, which we will denote by $[f,g](b)$, exists whenever $f, g \in D(T)$.

Given $\eta \in [-\pi/2, \pi/2]$ and $K \in \mathbb{C}$ define

$$\Lambda_{\eta,K} = \{\lambda \in \mathbb{C} : \Re[\mathrm{e}^{\mathrm{i}\eta}(\lambda - K)] < 0\},$$

i.e., $\Lambda_{\eta,K}$ is the preimage of the left half-plane under the Möbius transformation $\lambda \mapsto \mathrm{e}^{\mathrm{i}\eta}(\lambda - K)$.

Given a set $Q(q) = \overline{\mathrm{co}}\{q(x) + r : x \in [0,b), 0 < r < \infty\}$ we define

$$S(q) = \{(\eta, K) \in [-\pi/2, \pi/2] \times \mathbb{C} : \Lambda_{\eta,k} \cap Q(q) = \emptyset\}$$

which is not empty by our assumptions on q. We emphasize that $(\eta, K) \in S$ if and only if

$$\forall \lambda \in Q(q) : \Re[\mathrm{e}^{\mathrm{i}\eta}(\lambda - K)] \geqslant 0$$

and that $(\eta, K) \in S(q)$ and $\Re[\mathrm{e}^{\mathrm{i}\eta}K] \geqslant \Re[\mathrm{e}^{\mathrm{i}\eta}K']$ implies that $(\eta, K') \in S(q)$.

Throughout the paper, we will use the notation $k = \sqrt{-\lambda}$ (using the principal branch of the square root so that $\Re(k) \geqslant 0$).

2.2. Some remarks about Class II problems

If (q,b) is of Class II we need to specify a boundary condition at b. This is done in exactly the same way as in the selfadjoint case but we still record the basic facts here.

Lemma 2.2. *Suppose* (q,b) *is of Class* II *and* $u \in D(T)$. *If for some* $\lambda \in \mathbb{C}$ *it is true that* $[u, \theta(\cdot,\lambda)](b) = [u, \phi(\cdot,\lambda)](b) = 0$, *then this is true for all* $\lambda \in \mathbb{C}$.

Proof. Fix $\lambda_0 \in \mathbb{C}$ and $f \in D(T)$. One computes directly that

$$[u,f](x)[\theta(\cdot,\lambda_0),\phi(\cdot,\lambda_0)](x) = [f,\phi(\cdot,\lambda_0)](x)[u,\theta(\cdot,\lambda_0)](x) - [f,\theta(\cdot,\lambda_0)](x)[u,\phi(\cdot,\lambda_0)](x)$$

for all $x \in [0,b)$ and hence also for $x = b$. Since $[\theta(\cdot,\lambda_0),\phi(\cdot,\lambda_0)](x) \neq 0$ replacing f by $\phi(\cdot,\lambda)$ and $\theta(\cdot,\lambda)$ proves the claim. □

This allows us to make the following definition.

Definition 2.3. If (q,b) is of Class II and $u \in D(T)$ satisfies $[\phi(\cdot,\lambda), u](b) \neq 0$ or $[\theta(\cdot,\lambda), u](b) \neq 0$ for some $\lambda \in \mathbb{C}$ then u is called a boundary condition for b.

Lemma 2.4. *Fix* $\lambda_0 \in \mathbb{C}$. *If* (q,b) *is of Class* II *and* u *is in* $D(T)$ *then there exist complex numbers* α *and* β *such that*

$$[f,u](b) = [f, \alpha\phi(\cdot,\lambda_0) + \beta\theta(\cdot,\lambda_0)](b)$$

for all $f \in D(T)$.

Proof. Choose $\alpha = -[u, \theta(\cdot, \lambda_0)](b)/[\theta(\cdot, \lambda_0), \phi(\cdot, \lambda_0)](b)$ and $\beta = [u, \phi(\cdot, \lambda_0)](b)/[\theta(\cdot, \lambda_0), \phi(\cdot, \lambda_0)](b)$. □

Note that every nontrivial solution of $Ly = \lambda_0 y$ for some fixed λ_0 is a boundary condition for b. Indeed, the previous lemma shows that these give rise to all boundary conditions for b. Hence, without loss of generality, we may restrict ourselves to such boundary conditions u for which it is never the case that $u(x) = u'(x) = 0$.

2.3. Nesting circles analysis

Weyl's nesting circles analysis for the expression $-y'' + qy$ was first extended to nonselfadjoint problems by Sims [15]. Recently, Brown et al. [4] extended this to cover the more general Sturm–Liouville expressions $(-(py')' + qy)/w$. In addition, their approach allows the removal of Sims's restriction that $\Im(q)$ is bounded below. Here we present their basic ideas for easy reference in the case $w = p = 1$.

With the aid of θ and ϕ we define the Möbius transformation

$$M_X(z) = -\frac{\theta(X,\lambda)z + \theta'(X,\lambda)}{\phi(X,\lambda)z + \phi'(X,\lambda)}.$$

Let z_s be the preimage of infinity under M_X, i.e., $z_s = -\phi'(X,\lambda)/\phi(X,\lambda)$. The half-plane $\Re[z\mathrm{e}^{\mathrm{i}\eta}] \geqslant 0$ is mapped onto a closed disk $D_X(\eta,\lambda)$ if and only if z_s satisfies $\Re[z_s\mathrm{e}^{\mathrm{i}\eta}] < 0$. The diameter $d_X(\eta,\lambda)$ of that disk is then given by the equation

$$|h_1H_2 - h_2H_1| d_X(\eta,\lambda)^{-1} = -|\phi(X,\lambda)|^2 \Re[z_s\mathrm{e}^{\mathrm{i}\eta}] = \Re[\mathrm{e}^{\mathrm{i}\eta}\phi'(X,\lambda)\overline{\phi(X,\lambda)}].$$

Using integration by parts and the fact that $\phi(\cdot,\lambda)$ satisfies $Ly = \lambda y$, we have

$$\int_0^X \lambda|\phi|^2\,\mathrm{d}x = -\phi'(X,\lambda)\overline{\phi(X,\lambda)} + \overline{h_1}H_1 + \int_0^X (|\phi'|^2 + q|\phi|^2)\,\mathrm{d}x.$$

Multiplying by $\mathrm{e}^{\mathrm{i}\eta}$, taking real parts, and rearranging terms gives

$$\begin{aligned}\Re[\mathrm{e}^{\mathrm{i}\eta}\phi'(X)\overline{\phi(X)}] &= \Re[\mathrm{e}^{\mathrm{i}\eta}\overline{h_1}H_1] + \int_0^X \Re[\mathrm{e}^{\mathrm{i}\eta}((q-\lambda)|\phi|^2 + |\phi'|^2)]\,\mathrm{d}x \\ &= \int_0^X \Re\left[\mathrm{e}^{\mathrm{i}\eta}\left(q + \frac{|\phi'|^2}{|\phi|^2} - K\right)\right]|\phi|^2\,\mathrm{d}x + \Re[\mathrm{e}^{\mathrm{i}\eta}(K + K' - \lambda)]\int_0^X |\phi|^2\,\mathrm{d}x,\end{aligned}$$

where $K' = \overline{h_1}H_1/\int_0^X |\phi|^2\,\mathrm{d}x$. Since $q(x) + |\phi'(x,\lambda)|^2/|\phi(x,\lambda)|^2 \in Q(q)$ the first integral on the right-hand side is nonnegative if (η, K) is in $S(q)$. The second integral is positive if λ is in $\Lambda_{\eta,K+K'}$. Hence we proved.

Lemma 2.5. *Suppose $(\eta, K) \in S(q)$ and $\lambda \in \Lambda_{\eta,K+K'}$ where $K' = \overline{h_1}H_1/\int_0^X |\phi|^2\,\mathrm{d}x$. Then the half-plane $\Re[z\mathrm{e}^{\mathrm{i}\eta}] \geqslant 0$ is mapped by M_X onto a closed disk $D_X(\eta,\lambda)$.*

Definition 2.6. A pair $(\eta, K) \in S(q)$ is called X-admissible if, for every $\lambda \in \Lambda_{\eta,K}$, the Möbius transformation M_X maps the half-plane $\Re[z\mathrm{e}^{\mathrm{i}\eta}] \geqslant 0$ onto a closed disk $D_X(\eta, \lambda)$.

If $h_1 = 0$ or $H_1 = 0$ then every element of $S(q)$ is X-admissible for every $X \in [0, b)$.

Lemma 2.7. *Let $X \in (0, b)$, $h_1 = H_2 = 0$, and $h_2 = H_1 = 1$. Let $\tau \in (-\pi/2, \pi/2)$ and let $\mathscr{R}$ be the ray defined by $\mathscr{R}(t) = -t^2\mathrm{e}^{2\mathrm{i}\tau}$ where $t \geqslant 0$. Assume $\mathscr{R}$ stays eventually in some X-admissible half-plane $\Lambda_{\eta,K}$ and that $\cos(\eta+\tau) > 0$. Then, given any positive ε, there is an $R(\varepsilon)$ such that*

$$d_X(\eta, \lambda) \leqslant \frac{8|k|\mathrm{e}^{-2\Re(k)X}}{\cos(\eta+\tau)} \leqslant \mathrm{e}^{-2(X-\varepsilon)\Re(k)}$$

whenever $\lambda \in \mathscr{R}$ satisfies $|\lambda| > R(\varepsilon)$.

Proof. First note that

$$\left|\phi'\bar{\phi} - \frac{\mathrm{e}^{(k+\bar{k})x}}{4\bar{k}}\right| \leqslant |(\phi' - \phi_0')\bar{\phi}| + |\phi_0'(\bar{\phi} - \overline{\phi_0})| + \left|\left(\phi_0' - \frac{1}{2}\mathrm{e}^{kx}\right)\overline{\phi_0})\right| + \left|\frac{1}{2}\mathrm{e}^{kx}\left(\overline{\phi_0} - \frac{1}{2\bar{k}}\mathrm{e}^{\bar{k}x}\right)\right|.$$

Here, we choose $\phi_0(x) = \sinh(kx)/k$ so that the relationship between ϕ and ϕ_0 is the same as that between u and u_0 in Lemma 3.1. Therefore, we may apply that lemma to obtain

$$|\bar{\phi} - \overline{\phi_0}| \leqslant \frac{|\mathrm{e}^{kx}|}{|k|}\left(\mathrm{e}^{\int_0^x |q/k|\,\mathrm{d}t} - 1\right),$$

$$|\phi' - \phi_0'| \leqslant |\mathrm{e}^{kx}|\left(\mathrm{e}^{\int_0^x |q/k|\,\mathrm{d}t} - 1\right).$$

We also obtain

$$|\phi| \leqslant |\phi_0| + |\phi - \phi_0| \leqslant \frac{|\mathrm{e}^{kx}|}{|k|}\mathrm{e}^{\int_0^x |q/k|\,\mathrm{d}t}.$$

These estimates together with appropriate estimates on $\cosh(kx) - \mathrm{e}^{kx}/2$, etc., show that

$$\left|\phi'\bar{\phi} - \frac{\mathrm{e}^{(k+\bar{k})x}}{4\bar{k}}\right| \leqslant \frac{\mathrm{e}^{2\Re(k)x}}{|k|}\left(\mathrm{e}^{\int_0^x |q/k|\,\mathrm{d}t} + 1\right)\left(\mathrm{e}^{\int_0^x |q/k|\,\mathrm{d}t} - 1\right) + \frac{3}{4|k|}$$

$$\leqslant \frac{\mathrm{e}^{2\Re(k)x}}{|k|}\left(\frac{C(x)}{|k|} + \frac{3}{4}\mathrm{e}^{-2\Re(k)x}\right)$$

for some appropriate function $C(x)$ which does not depend on λ. Hence, for sufficiently large λ on the ray $\mathscr{R}$,

$$\Re\left[\mathrm{e}^{\mathrm{i}\eta}\phi'\bar{\phi}\right] \geqslant \frac{\mathrm{e}^{2\Re(k)x}}{4|k|}\left(\Re[\mathrm{e}^{\mathrm{i}(\eta+\tau)}] - \frac{4C(x)}{|k|} - 3\mathrm{e}^{-2\Re(k)x}\right) \geqslant \frac{\mathrm{e}^{2\Re(k)x}}{8|k|}\cos(\eta+\tau). \qquad \square$$

We now investigate the behavior of the disks $D_X(\eta,\lambda)$ when X tends to b. First note that K' depends on X. Given K choose $X_0\in(0,b)$ and define

$$\hat{K}=\begin{cases} K & \text{if } \Re[e^{i\eta}\overline{h_1}H_1]\geqslant 0,\\ K+K'(X_0) & \text{if } \Re[e^{i\eta}\overline{h_1}H_1]<0. \end{cases} \tag{1}$$

In the following, we assume that $(\eta,K)\in S(q)$ and that $\lambda\in\Lambda_{\eta,\hat{K}}$. Then $(\eta,\hat{K})$ is X-admissible for all $X\in[X_0,b)$.

Since

$$M_X^{-1}(m)=-\frac{\psi_m'(X,\lambda)}{\psi_m(X,\lambda)},$$

where $\psi_m=\theta+m\phi$ we have the following statement: m is contained in $D_X(\eta,\lambda)$ if and only if

$$\Re[e^{i\eta}\psi_m'(X,\lambda)\overline{\psi_m(X,\lambda)}]=-|\psi_m|^2\Re[e^{i\eta}M_X^{-1}(m)]\leqslant 0.$$

Treating $\int_0^X\lambda|\psi_m|^2\,dx$ in the same manner as $\int_0^X\lambda|\phi|^2\,dx$ before gives

$$\Re[e^{i\eta}\psi_m'(X,\lambda)\overline{\psi_m(X,\lambda)}]-\Re[e^{i\eta}\overline{(h_2+mh_1)}(H_2+mH_1)]=\int_0^X\Re[e^{i\eta}(|\psi_m'|^2+(q-\lambda)|\psi_m|^2)]\,dx.$$

Since $\lambda\in\Lambda_{\eta,\hat{K}}$ the integrand on the right-hand side is positive for all $x\in[X_0,b)$. If $m\in D_X(\eta,\lambda)$ the left-hand side is bounded above by $-\Re[e^{i\eta}\overline{(h_2+mh_1)}(H_2+mH_1)]$ which is independent of X. This implies that $D_Y(\eta,\lambda)\subset D_X(\eta,\lambda)$ if $Y\geqslant X$ and hence that $D_b(\eta,\lambda)=\bigcap_{X_0\leqslant X<b}D_X(\eta,\lambda)$ is not empty.

Moreover, since

$$\int_0^X\Re[e^{i\eta}(|\psi_m'|^2+(q-\lambda)|\psi_m|^2)]\,dx=\Re[e^{i\eta}(\hat{K}-\lambda)]\int_0^X|\psi_m|^2\,dx$$
$$+\int_0^X\Re[e^{i\eta}(|\psi_m'|^2+(q-\hat{K})|\psi_m|^2)]\,dx$$

we find that

$$\int_0^X|\psi_m|^2\,dx\leqslant\frac{-\Re[e^{i\eta}\overline{(h_2+mh_1)}(H_2+mH_1)]}{\Re[e^{i\eta}(\hat{K}-\lambda)]}$$

if $\lambda\in\Lambda_{\eta,\hat{K}}$ and $m\in D_X(\eta,\lambda)$.

We have therefore the following lemma.

Lemma 2.8. *Suppose $(\eta,K)\in S(q)$, that $\hat{K}$ is defined by (1), and that $\lambda\in\Lambda_{\eta,\hat{K}}$. Then the set $D_b(\eta,\lambda)=\bigcap_{X_0\leqslant X<b}D_X(\eta,\lambda)$ is not empty. Furthermore, if $m\in D_b(\eta,\lambda)$ then ψ_m is in $L^2([0,b))$, i.e., there exists at least one square integrable solution of $-y''+qy=\lambda y$.*

This lemma shows also that $D_b(\eta,\lambda)$ consists of just a single point if (q,b) is of Class I, i.e., a problem of Class I is always in the limit-point case. On the other hand, if $D_b(\eta,\lambda)$ does not consist of just a single point (the limit-circle case) then (q,b) is necessarily of Class II. For selfadjoint problems it is well known that the converses of these statements also hold, i.e., that the classification of problems into Classes I and II is equivalent to the classification of problems into limit-point and limit-circle case. This is not the case for nonselfadjoint problems where there are problems of Class II for which $D_b(\eta,\lambda)$ does consist of a single point. Sims [15] introduced therefore a threefold classification. But, if (q,b) is of Class II, it appears (cf. Remark 2.4 and Remark 4.11 of Brown et al. [4]) that neither the classification into limit-point and limit-circle case nor the definition of an m-function via the limit point is (η,K)-independent. It seems therefore that for complex potentials the notions of limit-point case and limit-circle case are less helpful.

2.4. Definition of the m-function

For Class I problems and a given λ there is at most one (up to constant multiples) solution of $Ly=\lambda y$ which is square integrable. For a Class II problem every solution of $Ly=\lambda y$ is square integrable but there is only one (up to constant multiples) which satisfies a given boundary condition u at b. Therefore, we define

$$\mathscr{D}=\begin{cases} D(T) & \text{if } (q,b) \text{ is of Class I},\\ \{f\in D(T):[f,u](b)=0\} & \text{if } (q,b) \text{ is of Class II}\end{cases}$$

and

$$\mathscr{M}=\{\lambda\in\mathbb{C}:\exists y\in\mathscr{D}:Ly=\lambda y\}.$$

The set $\mathscr{M}$ will serve as the domain of the m-functions.

We will define an m-function for every choice of the tuple (h_1,h_2,H_1,H_2) satisfying $h_1H_2-h_2H_1\neq 0$ and every choice of $\mathscr{D}$ (only for Class II problems is a choice to be made). The tuple of initial conditions determines uniquely the functions ϕ and θ. Since there is a unique element (up to constant multiples) in $\mathscr{D}$ which solves the equation $Ly=\lambda y$ as long as λ is in $\mathscr{M}$, there exists a unique complex number m such that $\theta(\cdot,\lambda)+m\phi(\cdot,\lambda)$ is in $\mathscr{D}$, unless $\phi(\cdot,\lambda)$ itself is in $\mathscr{D}$ in which case we define $m=\infty$. Assigning this value of m to $\lambda\in\mathscr{M}$ defines then the m-function.

Thus, m is a function from $\mathscr{M}$ to the Riemann sphere $\mathbb{C}\cup\{\infty\}$ (or, equivalently, to the complex projective line). Note that for (q,b) in Class II we have that $\mathscr{M}=\mathbb{C}$. However, if (q,b) is in Class I the set $\mathscr{M}$ may well be smaller than the whole plane. It was shown in Lemma 2.8 that in any case $\mathscr{M}$ contains at least a half-plane. It should also be mentioned that

$$m(\lambda)=-\frac{[\theta(\cdot,\lambda),u](b)}{[\phi(\cdot,\lambda),u](b)}$$

in the Class II case with boundary condition u.

We will now discuss the dependence of the m-function on the choice of the tuple (h_1,h_2,H_1,H_2) which gives rise to initial conditions for ϕ and θ. Consider two such tuples (h_1,h_2,H_1,H_2) and $(\tilde h_1,\tilde h_2,\tilde H_1,\tilde H_2)$ and denote the two m-functions associated with those pairs (and otherwise equal data) by m and $\tilde m$, respectively. One finds after some algebra that

$$m(\lambda)=\frac{\tilde h_2H_2-h_2\tilde H_2+(\tilde h_1H_2-h_2\tilde H_1)\tilde m(\lambda)}{h_1\tilde H_2-\tilde h_2H_1+(h_1\tilde H_1-\tilde h_1H_1)\tilde m(\lambda)}$$

which is a Möbius transform since its "determinant" is

$$(h_1H_2 - h_2H_1)(\tilde{h}_1\tilde{H}_2 - \tilde{h}_2\tilde{H}_1)$$

which does not equal zero.

Thus, $\tilde{m}$ encodes precisely the same information about q (and u for a Class II problem) as m does. In other words, switching from one pair (θ, ϕ) to another is nothing but a change of coordinates. We will make the Dirichlet choice, i.e., $h_1 = H_2 = 0$ and $h_2 = H_1 = 1$. All m-functions in the remainder of the paper will be Dirichlet m-functions unless an explicit statement to the contrary is made.

2.5. Main results

We will be concerned with the behavior of m-functions along certain rays. When we say ray we mean a ray emanating from zero.

Definition 2.9. A ray $\mathscr{R} \in \Sigma$ is called admissible for a Dirichlet m-function m (defined by (q, b) and perhaps a boundary condition u for b) if there exists a pair $(\eta, K) \in S(q)$ with the following properties:

(1) $\mathscr{R}(t)$ stays eventually in $\Lambda_{\eta,K}$.
(2) $\cos(\eta + \tau) > 0$ when $\tau = \arg(-\mathscr{R}(t))/2 \in (\pi/2, \pi/2)$.
(3) If (q, b) is of Class II and b is not regular then $\Re[e^{i\eta}u'(x)/u(x)] \leqslant 0$.

Remarks. (1) Since $\mathscr{R}$ is in Σ it can not coincide with the positive real axis. Thus, if $\mathscr{R}(t) = -t^2\exp(2i\tau)$ where $t > 0$, then, without loss of generality $\tau \in (-\pi/2, \pi/2)$.

(2) $\mathscr{R}(t) = -t^2\exp(2i\tau) \in \Lambda_{\eta,K}$ for all sufficiently large t implies that $-t^2\cos(\eta + 2\tau) < \Re[e^{i\eta}K]$ and hence that $\eta + 2\tau \in [-\pi/2, \pi/2]$. Since we also have that $\eta \in [-\pi/2, \pi/2]$ this implies

$$-\frac{\pi}{2} \leqslant \eta + \tau \leqslant \frac{\pi}{2}$$

with equality if and only if $\tau = 0$. Hence, if $\tau \neq 0$ then the requirement that $\mathscr{R}$ stays eventually in $\Lambda_{\eta,K}$ implies already that $\cos(\eta + \tau) > 0$. If $\tau = 0$, however, then $\cos(\eta) > 0$ is an additional requirement which forbids that the boundary of $\Lambda_{\eta,K}$ is parallel to the real axis.

(3) The third condition in Definition 2.9 is empty when (q, b) is of Class I and when b is regular. Otherwise it amounts to a restriction of allowed boundary conditions u. We do not know whether this is a technical problem only or whether it is possible for our results to be false when this condition is violated.

Theorem 2.10. *Suppose that two potentials (q, b) and $(\tilde{q}, \tilde{b})$ are in $\mathscr{Q}_\Sigma$. Let m and $\tilde{m}$ be the associated Dirichlet m-functions (defined after specifying boundary conditions u and/or $\tilde{u}$, if necessary) and let $a \in (0, \min\{b, \tilde{b}\}]$. If $q = \tilde{q}$ on $[0, a]$, then for any $\varepsilon > 0$ and for any ray $\mathscr{R}$, admissible for both m and $\tilde{m}$, we have*

$$m(\lambda) - \tilde{m}(\lambda) = \mathrm{O}(e^{-2(a-\varepsilon)\Re(\sqrt{-\lambda})})$$

as $\lambda \to \infty$ on $\mathscr{R}$.

Proof. Assume that $q = \tilde{q}$ on $[0, a]$. Let $\varepsilon > 0$ and $\mathscr{R}$ be a ray as stated in the hypothesis of the theorem. Let $\hat{m}$ be the m-function for a regular problem on $[0, a]$ generated by (q, a) and the boundary condition $\hat{u}(x) = x - a$ for a. We will show below that

$$m(\lambda) - \hat{m}(\lambda) = \mathrm{O}(\mathrm{e}^{-2(a-\varepsilon)\Re(k)}) \tag{2}$$

as $\lambda \to \infty$ along $\mathscr{R}$. Since $q = \tilde{q}$ on $[0, a]$ we obtain also that

$$\tilde{m}(\lambda) - \hat{m}(\lambda) = \mathrm{O}(\mathrm{e}^{-2(a-\varepsilon)\Re(k)})$$

as $\lambda \to \infty$ along $\mathscr{R}$. The triangle inequality gives then the desired result.

It remains to show the validity of (2). Note that

$$\hat{m}(\lambda) = -\frac{\theta(a, \lambda)}{\phi(a, \lambda)}$$

lies on the boundary of the disk $D_a(\eta, \lambda)$. If b is not regular then our assumptions on $\mathscr{R}$ and Lemma 2.5 show that $m(\lambda)$ is in $D_a(\eta, \lambda)$ and hence that

$$|m(\lambda) - \hat{m}(\lambda)| \leqslant d_a(\eta, \lambda).$$

Lemma 2.7 says that $d_a(\eta, \lambda) = \mathrm{O}(\mathrm{e}^{-2(a-\varepsilon)\Re(k)})$ which is (2).

For a regular problem we first compare $m(\lambda)$ with $m_0(\lambda)$ which is obtained by putting a Dirichlet boundary condition at b, i.e., $m_0(\lambda) = -\theta(b, \lambda)/\phi(b, \lambda)$. When $u(b) = 0$ then $m = m_0$. Otherwise

$$\begin{aligned} |m(\lambda) - m_0(\lambda)| &= \left| M_b(-u'(b)/u(b)) + \frac{\theta(b, \lambda)}{\phi(b, \lambda)} \right| = \frac{1}{|\phi(b, \lambda)[\phi'(b, \lambda) + (-u'(b)/u(b))\phi(b, \lambda)]|} \\ &\leqslant 16|k||\mathrm{e}^{-2kb}| \leqslant |\mathrm{e}^{-2k(a-\varepsilon)}| \end{aligned}$$

using $a \leqslant b$ and Corollary 3.2.

Lemma 2.5 gives now that $m_0(\lambda) \in \partial D_b(\eta, \lambda)$ and that $\hat{m}(\lambda) \in \partial D_a(\eta, \lambda)$. By Lemma 2.8 $D_b(\eta, \lambda) \subset D_a(\eta, \lambda)$ and by Lemma 2.7 $d_a(\eta, \lambda) = \mathrm{O}(\mathrm{e}^{-2(a-\varepsilon)\Re(k)})$. The triangle inequality finishes the proof. □

We now present a converse of Theorem 2.10.

Theorem 2.11. *Suppose that $q, \tilde{q} \in \mathscr{Q}_\Sigma$ and let m and m be associated Dirichlet m-functions. If there are two distinct rays $\mathscr{R}_1$ and $\mathscr{R}_2$, admissible for both m and $\tilde{m}$ such that, given any $\varepsilon > 0$,*

$$m(\lambda) - \tilde{m}(\lambda) = \mathrm{O}(\mathrm{e}^{-2(a-\varepsilon)\Re(\sqrt{-\lambda})})$$

as $\lambda \to \infty$ along each ray, then $q = \tilde{q}$ almost everywhere on $[0, a]$.

Proof. Fix $x \in (0, a)$. Define $\psi = \theta + m\phi$ and $\tilde{\psi} = \tilde{\theta} + \tilde{m}\tilde{\phi}$. Using Lemma 3.1 one shows that $\phi(x, \lambda)/\tilde{\phi}(x, \lambda)$ tends to one as λ tends to infinity on $\mathscr{R}_1$ or $\mathscr{R}_2$. This fact and Lemma 3.4 yield

$$\tilde{\phi}(x, \lambda)\psi(x, \lambda) = \frac{\tilde{\phi}(x, \lambda)}{\phi(x, \lambda)}(\phi(x, \lambda)\psi(x, \lambda)) \to 0$$

and

$$\phi(x,\lambda)\tilde{\psi}(x,\lambda) = \frac{\phi(x,\lambda)}{\tilde{\phi}(x,\lambda)}(\tilde{\phi}(x,\lambda)\tilde{\psi}(x,\lambda)) \to 0$$

as $\lambda \to \infty$ on $\mathscr{R}_1$ or $\mathscr{R}_2$. Thus, their difference

$$\tilde{\phi}(x,\lambda)\theta(x,\lambda) - \phi(x,\lambda)\tilde{\theta}(x,\lambda) + (m(\lambda) - \tilde{m}(\lambda))\phi(x,\lambda)\tilde{\phi}(x,\lambda)$$

converges to zero along $\mathscr{R}_1$ and $\mathscr{R}_2$. Choose ε such that $0 < \varepsilon < a - x$. Then our hypothesis on $m - \tilde{m}$ and the fact that $\phi(x,\lambda)\tilde{\phi}(x,\lambda) = \mathrm{O}(\mathrm{e}^{2kx})$ shows that

$$|(m(\lambda) - \tilde{m}(\lambda))\phi(x,\lambda)\tilde{\phi}(x,\lambda)| \leqslant C\mathrm{e}^{-2(a-\varepsilon-x)\Re(k)} \to 0$$

as λ tends to infinity on $\mathscr{R}_1$ or $\mathscr{R}_2$. Therefore, the entire function

$$g(\lambda) = \tilde{\phi}(x,\lambda)\theta(x,\lambda) - \phi(x,\lambda)\tilde{\theta}(x,\lambda)$$

converges on the rays and hence is bounded there. The Phragmén–Lindelöf theorem implies that g is bounded in $\mathbb{C}$, so it is constant by Liouville's theorem and, in fact, identically equal to zero, i.e.,

$$\tilde{\phi}(x,\lambda)\theta(x,\lambda) = \phi(x,\lambda)\tilde{\theta}(x,\lambda).$$

Since $x \in (0,a)$ was arbitrary this equation holds for all $x \in (0,a)$ and for all $\lambda \in \mathbb{C}$.

Let now $\lambda \in \mathbb{C}$ be fixed and suppress both arguments for the rest of this proof. We rewrite the previous equation as $\theta/\phi = \tilde{\theta}/\tilde{\phi}$ and differentiate both sides with respect to x to get

$$\frac{\phi\theta' - \phi'\theta}{\phi^2} = \frac{\tilde{\phi}\tilde{\theta}' - \tilde{\phi}'\tilde{\theta}}{\tilde{\phi}^2}.$$

Since the numerators on both sides are one, we have $\phi^2 = \tilde{\phi}^2$. Differentiating once more gives $2\phi\phi' = 2\tilde{\phi}\tilde{\phi}'$ and hence $\phi'/\phi = \tilde{\phi}'/\tilde{\phi}$. Differentiating a third time gives finally

$$q - \lambda = \frac{\phi''}{\phi} = \frac{\tilde{\phi}''}{\tilde{\phi}} = \tilde{q} - \lambda$$

and these equations are valid almost everywhere on $(0,a)$. $\square$

2.6. An application

Let b be finite and $q \in L^1([0,b]) \cap \mathscr{Q}_\Sigma$. The goal of this subsection is to describe various pieces of information each of which determines q uniquely.

It is here advantageous to consider the Neumann m-function, i.e., the m-function for the choice $h_1 = H_2 = 1$ and $h_2 = H_1 = 0$, which we denote by m_N. Since

$$m_N(\lambda) = -\frac{\theta(b,\lambda)u'(b) - \theta'(b,\lambda)u(b)}{\phi(b,\lambda)u'(b) - \phi'(b,\lambda)u(b)}.$$

Lemma 3.1 implies that there is a constant C such that $|m_N(\lambda)| \leqslant C/\sqrt{|\lambda|}$ for any λ on a certain sequence of circles whose radii tend to infinity (if $u(b) \neq 0$ those radii can eventually be chosen as $(2n+1)^2\pi^2/(4b^2)$, if $u(b) = 0$ they can eventually be chosen as $n^2\pi^2/b^2$).

Note that m_N is a meromorphic function (recall that $\theta(b,\cdot)$, $\theta'(b,\cdot)$, $\phi(b,\cdot)$, and $\phi'(b,\cdot)$ are entire). Denote the distinct poles of m_N and their multiplicities by λ_n and j_n, respectively. Using the asymptotic behavior just established one can apply the residue theorem to prove that

$$m_N(\lambda) = \sum_{n=1}^{\infty}\sum_{p=1}^{j_n} \frac{a_{n,-p}}{(\lambda-\lambda_n)^p},$$

where the $a_{n,-p}$ are the coefficients in the Laurent expansion of m_N about λ_n. More precisely, they are defined by

$$m_N(\mu) = \sum_{k=-j_n}^{\infty} a_{n,k}(\mu-\lambda_n)^k.$$

Now define the operators T_1 and T_2 as restrictions of the maximal operator T to the respective domains

$$D(T_j) = \{y \in D(T) : y'(0) = 0 \text{ and } y(b)u_j'(b) - y'(b)u_j(b) = 0\},$$

where u_1 and u_2 are linearly independent boundary conditions for b. The spectra of the operators T_j consist only of eigenvalues, which are precisely the poles of the associated m-function while the algebraic multiplicity of such an eigenvalue equals the order of the pole of the m-function (all eigenvalues have geometric multiplicity one). Furthermore, the functions $\phi(\cdot,\lambda)$ is an eigenfunction of T_j if λ is an eigenvalue of T_j and $\phi^{(0,p)}(\cdot,\lambda)$, $p=1,\dots,j-1$ are the generalized eigenfunctions of λ if its multiplicity is j. In particular, they satisfy the boundary condition at b. We define the sets

$$S_j = \{(\lambda_n, j_n) : \lambda_n \text{ is an eigenvalue of } T_j \text{ of multiplicity } j_n\}$$

and, for $p \in \{0,\dots,j_n-1\}$, the quantities [2]

$$N_{n,p} = \int_0^b (j_n\phi^{(0,p)}(x,\lambda_n)\phi^{(0,j_n-1)}(x,\lambda_n) - p\phi^{(0,j_n)}(x,\lambda_n)\phi^{(0,p-1)}(x,\lambda_n))\,\mathrm{d}x$$

which could be called "generalized norming constants". Note that for $j_n = 1$ one has the usual norming constants.

Theorem 2.12. *Suppose b is a positive real number, $q \in L^1([0,b]) \cap \mathcal{Q}_\Sigma$, and u_1 and u_2 are linearly independent boundary conditions for b. Then q is uniquely determined by each of the following pieces of information*:

(1) *The sets S_1, S_2,*
(2) *The sets S_1 and $\{(\phi^{(0,p)}(b,\lambda_n), \phi^{(1,p)}(b,\lambda_n)) : (\lambda_n, j_n) \in S_1, p = 0,\dots,j_n-1\}$,*
(3) *The sets S_1 and $\{N_{n,p} : (\lambda_n, j_n) \in S_1, p = 0,\dots,j_n-1\}$,*
(4) *The sets S_1 and $\{a_{n,-p} : (\lambda_n, j_n) \in S_1, p = 1,\dots,j_n\}$.*

[2] If f is a function of two variables we use the notation $f^{(j,k)} = \partial^{j+k} f/(\partial x^j \partial y^k)$.

Remarks.

- In the self-adjoint case (1) is of course Borg's [2] original result. Levinson [9] gave a shorter proof of it. The nonselfadjoint case was first proven by Marchenko [11]. A proof is also given in [8].
- Yurko [16] has established previously that the m-function determines q uniquely when all of its poles are simple. However, Yurko allows q to have inverse square singularities.
- In (2), we really need to know only one of the two numbers $\phi^{(0,p)}(b,\lambda_n)$ and $\phi^{(1,p)}(b,\lambda_n)$ in every pair since the other one is then determined by the boundary condition. If $u_1(b)$ and $u_1'(b)$ are both nonzero either of the numbers will do. If $u_1(b)=0$ then, necessarily, $\phi^{(0,p)}(b,\lambda_n)=0$ and we need to know the value of $\phi^{(1,p)}(b,\lambda_n)$. Similarly, if $u_1'(b)=0$ then, necessarily, $\phi^{(1,p)}(b,\lambda_n)=0$ and we need to know the value of $\phi^{(0,p)}(b,\lambda_n)$.
- Obviously, given q all the information in (1)–(4) is uniquely determined.

Proof. The information in (4) determines a Neumann m-function m_N. The Dirichlet m-function is then given as $1/m_N$. This in turn determines q according to Theorem 2.11. We will briefly sketch how to obtain the information in (4) from that in (1), (2), or (3). See [13] for more details.

(1) $\to$ (2): Let $g_j=\phi(b,\cdot)u_j'(b)-\phi'(b,\cdot)u_j(b)$. With the aid of Hadamard's factorization theorem and the knowledge of the asymptotic behavior of $\phi(b,\cdot)$ and $\phi'(b,\cdot)$ one can recover the function g_1 from S_1 and g_2 from S_2. The pair $(\phi^{(0,p)}(b,\lambda_n),\phi^{(1,p)}(b,\lambda_n))$ is then given as the solution of the system

$$\phi^{(0,p)}(b,\lambda_n)u_1'(b)-\phi^{(1,p)}(b,\lambda_n)u_1(b)=0,$$

$$\phi^{(0,p)}(b,\lambda_n)u_2'(b)-\phi^{(1,p)}(b,\lambda_n)u_2(b)=g_2^{(p)}(\lambda_n)$$

which is satisfied when λ_n is an eigenvalue of T_1 (and hence a zero of g_1).

(3) $\to$ (2): Since the x-derivative of $\phi^{(0,j_n)}\phi^{(1,p)}-\phi^{(0,p)}\phi^{(1,j_n)}$ equals the integrand in the definition of $N_{n,p}$ we have that

$$\begin{aligned}u_1(b)N_{n,p}&=\phi^{(0,j_n)}(b,\lambda_n)\phi^{(1,p)}(b,\lambda_n)u_1(b)-\phi^{(0,p)}(b,\lambda_n)\phi^{(1,j_n)}(b,\lambda_n)u_1(b)\\&=\phi^{(0,p)}(b,\lambda_n)g_1^{(j_n)}(\lambda_n).\end{aligned}$$

Since $g_1^{(j_n)}(\lambda_n)\neq 0$ we have that $N_{n,p}$ determines $\phi^{(0,p)}(b,\lambda_n)$ provided that $u_1(b)\neq 0$. If $u_1(b)=0$ one looks at $u_1'(b)N_{n,p}$ instead.

(2) $\to$ (4): Now define also $f_1=\theta(b,\cdot)u_1'(b)-\theta'(b,\cdot)u_1(b)$ so that $m_N=-f_1/g_1$. It is clear that the first j_n coefficients in the Taylor series expansion of f_1 about λ_n determine the numbers $a_{n,-p}$, $p=1,\ldots,j_n$ so we will set out to determine these. The equation $[\theta(\cdot,\mu),\phi(\cdot,\mu)](b)=1$ is equivalent to $\phi(b,\mu)f_1(\mu)=u_1(b)$. Differentiating this latter equation successively with respect to μ up to j_n-1 times and evaluating the resulting expressions at λ_n gives then the values $f_1^{(p)}(\lambda_n)$ assuming again that $u_1(b)\neq 0$ (with obvious modifications if $u_1(b)=0$). $\square$

3. Technical details

3.1. Estimates for solutions of initial value problems

Lemma 3.1. *Suppose that $u(\cdot,\lambda)$ solves the equation $-y''+qy=\lambda y$ and $u(0,\lambda), u'(0,\lambda)$ are independent of λ. Let $k=\sqrt{-\lambda}$, choosing the branch of the root such that $\Re(k)\geqslant 0$,*

$$u_0(x,\lambda)=u(0,\lambda)\cosh(kx)+u'(0,\lambda)\frac{\sinh(kx)}{k},$$

and $c(\lambda)=|u(0)|+|u'(0)|/|k|$. Then

$$|u(x)-u_0(x)|\leqslant c(\lambda)|\mathrm{e}^{kx}|(\mathrm{e}^{\int_0^x |q/k|\,\mathrm{d}t}-1) \tag{3}$$

and

$$|u'(x)-u_0'(x)|\leqslant c(\lambda)|k||\mathrm{e}^{kx}|(\mathrm{e}^{\int_0^x |q/k|\,\mathrm{d}t}-1). \tag{4}$$

These estimates hold for all $x\in[0,b)$ and all $\lambda\in\mathbf{C}$.

Proof. The variation of constants formula gives

$$u(x)=u_0(x)+\int_0^x \frac{\sinh[k(x-t)]}{k}q(t)u(t)\,\mathrm{d}t.$$

Let

$$g(x)=|\mathrm{e}^{-kx}|\left|\int_0^x \frac{\sinh[k(x-t)]}{k}q(t)u(t)\,\mathrm{d}t\right|=|\mathrm{e}^{-kx}|\,|u(x)-u_0(x)|.$$

Replacing u by $u_0+(u-u_0)$ and estimating the exponential functions appearing here yields

$$g(x)\leqslant\frac{1}{|k|}\int_0^x |q|g\,\mathrm{d}t+\frac{c(\lambda)}{|k|}\int_0^x |q|\,\mathrm{d}t. \tag{5}$$

We now multiply both sides of this inequality by $|q(x)|\mathrm{e}^{-\int_0^x |q/k|\,\mathrm{d}t}$, bring the first term of the right-hand side to the left-hand side, and integrate both sides from 0 to x. This gives

$$\mathrm{e}^{-\int_0^x |q/k|\,\mathrm{d}t}\int_0^x |q|g\,\mathrm{d}t\leqslant c(\lambda)|k|-c(\lambda)|k|\mathrm{e}^{-\int_0^x |q/k|\,\mathrm{d}t}\left(\frac{1}{|k|}\int_0^x |q|\,\mathrm{d}t+1\right).$$

Thus, with the aid of (5),

$$g(x)\leqslant\frac{1}{|k|}\int_0^x |q|g\,\mathrm{d}t+\frac{c(\lambda)}{|k|}\int_0^x |q|\,\mathrm{d}t\leqslant c(\lambda)\left(\mathrm{e}^{\int_0^x |q/k|\,\mathrm{d}t}-1\right)$$

which is equivalent to (3).

Differentiating u we get

$$u'(x) = u_0'(x) + \int_0^x \cosh[k(x-t)]q(t)u(t)\,\mathrm{d}t.$$

Replacing again u by $u_0 + (u - u_0)$ we now estimate

$$\begin{aligned}|u'(x) - u_0'(x)| &\leqslant |\mathrm{e}^{kx}| \int_0^x c(\lambda)|q(t)|(\mathrm{e}^{\int_0^t |q/k|\,\mathrm{d}s} - 1)\,\mathrm{d}t + |\mathrm{e}^{kx}| \int_0^x c(\lambda)|q(t)|\,\mathrm{d}t \\ &= c(\lambda)|k||\mathrm{e}^{kx}| \left(\mathrm{e}^{\int_0^x |\frac{q}{k}|\,\mathrm{d}t} - 1\right),\end{aligned}$$

to obtain (4). □

Corollary 3.2. *If x and z are fixed and $\Re(k)$ is sufficiently large then*

$$|\phi(x,\lambda)| \geqslant \frac{|\mathrm{e}^{kx}|}{4|k|},$$

$$|\phi'(x,\lambda) + z\phi(x,\lambda)| \geqslant \frac{|\mathrm{e}^{kx}|}{4}.$$

Proof. Since $\phi_0(x,\lambda) = \sinh(kx)/k$ and $\phi = \phi_0 + (\phi - \phi_0)$ Lemma 3.1 and the triangle inequality give the first statement. The second statement follows similarly. □

3.2. Asymptotic behavior of the m-function

Lemma 3.3. *Let $\mathscr{R}$ be an admissible ray for a Dirichlet m-function m. Then m satisfies*

$$m(\lambda) = -\sqrt{-\lambda} + \mathrm{O}(1)$$

as $\lambda \to \infty$ along $\mathscr{R}$.

Proof. Fix $X_0 \in [0,b)$. If (q,b) is of Class I then $\bigcap_{X\in[X_0,b)} D_X(\eta,\lambda) = \{m(\lambda)\}$. If (q,b) is of Class II, then

$$m(\lambda) = -\frac{[\theta(\cdot,\lambda),u](b)}{[\phi(\cdot,\lambda),u](b)} = \lim_{X\to b} M_X(-u'(X)/u(X)).$$

If b is not regular our assumption that $\mathscr{R}$ is admissible forces $M_X(-u'(X)/u(X)) \in D_X(\eta,\lambda) \subset D_{X_0}(\eta,\lambda)$.

In either case we have therefore $m(\lambda) \in D_{X_0}(\eta,\lambda)$. Furthermore, the point $-\theta(X_0,\lambda)/\phi(X_0,\lambda)$ is the image of infinity under the Möbius transformation M_{X_0} and is therefore on the boundary of $D_{X_0}(\eta,\lambda)$. Thus,

$$\left| m(\lambda) + \frac{\theta(X_0,\lambda)}{\phi(X_0,\lambda)} \right| \leqslant d_{X_0}(\eta,\lambda).$$

Since $\phi_0(x,\lambda)=\sinh(kx)/k$ and $\theta_0(x,\lambda)=\cosh(kx)$ Lemma 3.1 allows us to estimate

$$|\phi(X_0,\lambda)| \geqslant \frac{|e^{kX_0}|}{4|k|}$$

and

$$|\theta(X_0,\lambda)-k\phi(X_0,\lambda)| \leqslant |e^{-kX_0}| + 2|e^{kX_0}|\left(e^{\int_0^{X_0}|q/k|\,dt}-1\right).$$

Hence, there is a constant $C(X_0)$, independent of λ, such that

$$\left|\frac{\theta(X_0,\lambda)}{\phi(X_0,\lambda)}-k\right| \leqslant C(X_0).$$

The triangle inequality and Lemma 2.7 give now the desired result.

Finally, considering the regular case, we have

$$m(\lambda)=M_b(-u'(b)/u(b)).$$

If $u(b)\neq 0$ then the asymptotic behavior of m is given by the one of $-\theta'(b,\cdot)/\phi'(b,\cdot)$. If $u(b)=0$ then $m(\lambda)=-\theta(b,\cdot)/\phi(b,\cdot)$. In either case Lemma 3.1 gives the desired result. □

3.3. Asymptotic behavior of Green's function

We finish this section with a result that states a manner in which the diagonal Green's function for a given problem converges to zero.

Lemma 3.4. *Let $\mathscr{R}$ be an admissible ray for an m-function m. Let $\psi(x,\lambda)=\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$ and fix $a\in(0,b)$. Then $\phi(a,\lambda)\psi(a,\lambda)\to 0$ as $\lambda\to\infty$ along $\mathscr{R}$.*

Proof. Let m_a be the Dirichlet m-function for a problem on the interval $[a,b)$ (if the problem on $[0,b)$ is of Class II then so is the problem on $[a,b)$ and we use the same boundary condition for b). More precisely, if $q_a(x)=q(x+a)$ (and $u_a(x)=u(x+a)$ if a boundary condition for b is present) then m_a is the Dirichlet m-function for the problem $(q_a, b-a)$ (and the boundary condition u_a for $b-a$ if necessary).

Let θ_a,ϕ_a be solutions of $-y''+qy=\lambda y$ satisfying the initial conditions

$$\phi_a(a,\lambda)=0,\quad \theta_a(a,\lambda)=1,$$

$$\phi_a'(a,\lambda)=1,\quad \theta_a'(a,\lambda)=0.$$

Then define

$$\psi_a(\cdot,\lambda)=\theta_a(\cdot,\lambda)+m_a(\lambda)\phi_a(\cdot,\lambda).$$

As usual we also have

$$\psi(\cdot,\lambda)=\theta(\cdot,\lambda)+m(\lambda)\phi(\cdot,\lambda).$$

Both ψ and ψ_a are square integrable and, in the case of a Class II problem, satisfy the same boundary condition at b. They are therefore multiples of each other. Note that $\psi_a(a,\lambda)=1$ and $\psi_a'(a,\lambda)=m_a(\lambda)$. Thus,

$$m_a(\lambda)=\frac{\psi_a'(a,\lambda)}{\psi_a(a,\lambda)}=\frac{\psi'(a,\lambda)}{\psi(a,\lambda)}.$$

From Lemma 3.3 we have therefore that

$$\frac{\psi'(a,\lambda)}{\psi(a,\lambda)}=-k+\mathrm{O}(1)$$

as λ tends to infinity along $\mathscr{R}$.

Next, by Lemma 3.1,

$$-\frac{\phi'(a,\lambda)}{\phi(a,\lambda)}=-k+\mathrm{O}(1)$$

as λ tends to infinity along $\mathscr{R}$. To complete our proof, note that

$$\frac{1}{\phi(a,\lambda)\psi(a,\lambda)}=\frac{\phi(a,\lambda)\psi'(a,\lambda)-\phi'(a,\lambda)\psi(a,\lambda)}{\phi(a,\lambda)\psi(a,\lambda)}=\frac{\psi'(a,\lambda)}{\psi(a,\lambda)}+\left(-\frac{\phi'(a,\lambda)}{\phi(a,\lambda)}\right).$$

Since both terms in the last expression are asymptotic to $-k$ as $\lambda\to\infty$ along $\mathscr{R}$, we have that the reciprocal of the Green's function tends to infinity like $-2k$, so the Green's function $\phi(a,\lambda)\psi(a,\lambda)$ tends to 0 as $\lambda\to\infty$ along $\mathscr{R}$. $\square$

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