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A local Borg-Marchenko theorem for difference equations with complex coefficients

Rudi Weikard

Abstract. We investigate the asymptotic behavior of the Titchmarsh-Weyl m-function for a difference equation with complex coefficients and prove a local Borg-Marchenko theorem. The proofs are based on a nesting circles analysis.

1. Introduction

The goal of this paper is to establish the asymptotic behavior of the Titchmarsh-Weyl m-function and a local Borg-Marchenko theorem for a Jacobi difference expression in the case of complex coefficients. Note that the associated Jacobi operator will not be selfadjoint in this case.

Let $\mathbb{C}^{\mathbb{N}_0}$ and $\mathbb{C}^{\mathbb{N}}$ be the sets of complex-valued sequences defined on $\mathbb{N}_0$ and $\mathbb{N}$, respectively, and denote the first order forward difference operator by a $'$, i.e., let $f'(n) = f(n+1) - f(n)$. The (symmetric) Jacobi difference expression $L : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}}$ is defined by[1]

$$(Ly)(n) = a_{n-1}y(n-1) + b_n y(n) + a_n y(n+1) = (ay')'(n-1) + (a_{n-1} + b_n + a_n)y(n)$$

where

(1) $a_0 = 1$,
(2) $a_n \neq 0$ for all $n \in \mathbb{N}$,
(3) $\sum_{n=1}^{\infty} 1/|a_n| = \infty$, and
(4) $Q(L)$, the closed convex hull of the set $\{a_{n-1}+b_n+a_n-ra_n : n \in \mathbb{N}, r \geq 0\}$, is a proper subset of the complex plane (and hence a subset of a closed half plane).

If $y \in \mathbb{C}^{\mathbb{N}_0}$ we denote the vector in $\mathbb{C}^{\mathbb{N}}$ obtained by chopping off the first component by $\hat{y}$. Then we may write $Ly = \mu\hat{y}$ to represent all equations $(Ly)(n) = \mu y(n)$ for $n \in \mathbb{N}$.

The (Dirichlet) m-function associated with L is defined by

$$m : \Omega \to \mathbb{C}_\infty : \mu \mapsto \frac{\psi'(\mu, 0)}{\psi(\mu, 0)}$$

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[1] We will mostly write a_n for $a(n)$.

where Ω is the subset of $\mathbb{C}$ for which there is a square summable solution of $Ly = \mu\hat{y}$ which is then denoted by $\psi(\mu, \cdot)$. It will be shown in the next section that this is a valid definition since, under the given circumstances, there can not be two linearly independent square summable solutions of $Ly = \mu\hat{y}$.

In this paper the following three statements will proved:

(1) The set Ω contains at least a half plane (Theorem 3.2). This was proved by Wilson [**12**] by a nesting circles analysis. As it provides useful estimates needed later and to make this paper self-contained, a (much condensed) version of that proof is provided in Section 3.
(2) $m(\mu) = -1 + 1/\mu + O(1/\mu^2)$ as μ tends to infinity (Theorem 4.2).
(3) Two Dirichlet m-functions, m and $\tilde{m}$, respectively associated with difference expressions L and $\tilde{L}$, satisfy $m(\mu) - \tilde{m}(\mu) = O(\mu^{-2N-1})$ on a suitable ray if and only if the coefficients of L and $\tilde{L}$ satisfy $b_n = \tilde{b}_n$ and $a_{n-1}^2 = \tilde{a}_{n-1}^2$ whenever $1 \le n \le N$ (Theorem 5.1, check there for a precise statement).

To any difference expression L there is associated in a natural way a Jacobi operator J from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ (possibly defined only on a dense subset of $\ell^2(\mathbb{N})$). If U is a bounded multiplication operator with a bounded inverse, then $J' = U^{-1}JU$ is also a Jacobi operator with the same m-function as J. The associated difference expression will be of the form

$$\tilde{a}_{n-1}y(n-1) + b_n y(n) + \tilde{c}_n y(n+1)$$

where $\tilde{a}_n\tilde{c}_n = a_n^2$ but, generally, $\tilde{a}_n \neq \tilde{c}_n$. Therefore our results apply also to certain nonsymmetric Jacobi expressions.

Selfadjoint Jacobi operators have, of course, a long history. We mention here the monumental work by Berezans'kiĭ [**2**], a recent monograph by Teschl [**11**], and the paper [**6**] by Gesztesy and Simon. The classical Titchmarsh-Weyl nesting circles analysis was first applied to a one-dimensional Schrödinger equation with a complex potential by Sims [**10**] in 1957. That work was later generalized to nonselfadjoint Sturm-Liouville equations by Brown et al. [**3**] and carried over to difference equations by Wilson [**12**]. A local version of the celebrated Borg-Marchenko theorem for a onedimensional Schrödinger operator with real potential was first given by Simon [**9**]. Simplified versions are due to Gesztesy and Simon [**7**] and Bennewitz [**1**]. In [**4**] this result was extended to complex-valued potentials. Borg-Marchenko theorems for selfadjoint Jacobi operators were given by Gesztesy and Simon [**6**] and by Gesztesy, Kiselev, and Makarov [**5**]. Closely related to the present work is Guseĭnov [**8**], who considers the inverse problem for a nonselfadjoint Jacobi operator after introducing a generalized spectral function.

2. Basic notation and some basic facts

We denote by $S(L)$ the set of all open half planes which do not intersect $Q(L)$. Whenever Λ is in $S(L)$ then there is a Möbius transformation of the form $\lambda \mapsto \mathrm{e}^{-i\eta}(\lambda - K)$ which maps Λ to the upper half of the complex plane. Thus, if $\mu \in \Lambda$ and $q \in \Lambda^c$ (the complement of a set $\Omega \in \mathbb{C}$ is denoted by Ω^c), then there are $\eta \in [0, 2\pi)$ and $K \in \mathbb{C}$ such that $\Im(\mathrm{e}^{-i\eta}(\lambda - K)) > 0$ and $\Im(\mathrm{e}^{-i\eta}(K - q)) \ge 0$.

The equation $(Ly)(n) = \mu y(n)$ is equivalent to

$$\begin{pmatrix} y(n) \\ a_n y(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1/a_{n-1} \\ -a_{n-1} & (\mu - b_n)/a_{n-1} \end{pmatrix} \begin{pmatrix} y(n-1) \\ a_{n-1} y(n) \end{pmatrix}.$$

Since the matrix figuring here has determinant one we see that the (modified) Wronskian

$$[f,g](n) = a_n(f(n)g'(n) - g(n)f'(n)) = a_n(f(n)g(n+1) - g(n)f(n+1))$$

of two solutions f and g of $Ly = \mu\hat{y}$ is independent of n.

For any $\mu \in \mathbb{C}$ let $c(\mu,\cdot)$ and $s(\mu,\cdot)$ be the unique linearly independent solutions of $Ly = \mu\hat{y}$ satisfying $c(\mu,0) = s'(\mu,0) = 1$ and $c'(\mu,0) = s(\mu,0) = 0$. Note that $c(\cdot,n)$ and $s(\cdot,n)$ are polynomials for every $n \in \mathbb{N}_0$. We have $[c(\mu,\cdot), s(\mu,\cdot)](n) = 1$ for all $n \in \mathbb{N}_0$. This equation and Schwarz's inequality give that

$$\left(\sum_{n=0}^{\infty} \frac{1}{|a_n|}\right)^2 \le 4\sum_{n=0}^{\infty} |c(\mu,n)|^2 \sum_{n=0}^{\infty} |s(\mu,n)|^2.$$

Since, by our assumptions, the left hand side of this inequality is infinity, at least one of $c(\mu,\cdot)$ and $s(\mu,\cdot)$ is not square summable. In particular, the space of square summable solutions of $Ly = \mu\hat{y}$ is either trivial or one-dimensional.

Lemma 2.1. *If $(Ly)(n) = \mu y(n)$, $1 \le n \le N$, then*

$$a_N y'(N)\overline{y(N)} = a_0 y'(0)\overline{y(0)} + \sum_{n=1}^{N} \{a_{n-1}|y'(n-1)|^2 + (\mu - q_n)|y(n)|^2\}$$

where $q_n = a_{n-1} + b_n + a_n$.

Proof. Using the summation by parts formula

$$\sum_{n=1}^{N} g(n)f'(n-1) = g(N)f(N) - g(0)f(0) - \sum_{n=1}^{N} g'(n-1)f(n-1)$$

and $(Ly)(n) = \mu y(n)$ yields

$$\sum_{n=1}^{N} \mu|y(n)|^2 = a_N y'(N)\overline{y(N)} - a_0\overline{y(0)}y'(0) + \sum_{n=1}^{N} (q_n|y(n)|^2 - a_{n-1}|y'(n-1)|^2).$$

A rearrangement of the terms completes the proof. □

Lemma 2.2. *Fix $n \in \mathbb{N}$. The function $s(\cdot,n)$ is a polynomial of order $n-1$ with leading coefficient $(a_0 \ldots a_{n-1})^{-1}$. In particular, $s(\mu,n)/s(\mu,n+1) = a_n\mu^{-1} + O(\mu^{-2})$ as μ tends to infinity and $C_n|\mu|^{n-1} \le |s(\mu,n)| \le D_n|\mu|^{n-1}$ for a suitable constants C_n and D_n and sufficiently large μ.*

Proof. The first statement follows from induction on n. The remaining statements are immediate corollaries of the first. □

3. Nesting circles analysis

Let N be a natural number and β a complex number. For every μ there is a one-dimensional space of finite sequences y defined on $\{0, \ldots, N+1\}$ satisfying $(Ly)(n) = \mu y(n)$ for $n = 1, \ldots, N$ and the boundary condition

$$\frac{y(N)}{a_N y'(N)} = \beta.$$

Except for a countable set of μ any such sequence y is a multiple of $\psi_m(\mu,\cdot) = c(\mu,\cdot) + m s(\mu,\cdot)$ for some appropriate number m which depends on μ, β and N. In

fact, the connection between β and m (for fixed μ and N) is given by the Möbius transformation
$$\beta = B_{\mu,N}(m) = \frac{\psi_m(\mu,N)}{a_N\psi'_m(\mu,N)} = \frac{c(\mu,N)+ms(\mu,N)}{a_N[c'(\mu,N)+ms'(\mu,N)]}.$$
The inverse of this transformation is
$$m = B^{-1}_{\mu,N}(\beta) = -\frac{a_N c'(\mu,N)\beta - c(\mu,N)}{a_N s'(\mu,N)\beta - s(\mu,N)}.$$
We are looking for a condition on μ such that the half plane given by $\Im(\beta\mathrm{e}^{i\eta}) \geq 0$, for a suitably chosen $\eta \in [0,2\pi)$, is mapped to a disk by $B^{-1}_{\mu,N}$.

Recall that the Möbius transformation
$$\alpha \mapsto -\frac{A\alpha+B}{C\alpha+D}$$
maps the upper half plane onto a disk of diameter $|AD-BC|/\Im(D\overline{C})$ provided that the point $\alpha_s = -D/C$, which is mapped to infinity, lies in the lower half plane. Also note that $\alpha_s = -D\overline{C}/|C|^2$ is in the lower half plane if and only if $\Im(D\overline{C}) = \Im(-\overline{D}C) > 0$.

According to this the half plane $\Im(\beta\mathrm{e}^{i\eta}) \geq 0$ is mapped to a disk $D_N(\mu,\eta)$ if and only if
$$d_N(\mu,\eta)^{-1} = \Im(a_N s'(\mu,N)\overline{s(\mu,N)}\mathrm{e}^{-i\eta}) > 0. \tag{1}$$
The number $d_N(\mu,\eta)$ is then the diameter of the disk.

By Lemma 2.1 we have
$$a_N s'(\mu,N)\overline{s(\mu,N)} = \sum_{n=1}^{N}(\mu-\gamma_\infty(n))|s(\mu,n)|^2$$
where
$$\gamma_\infty(n) = q_n - a_{n-1}\left|\frac{s'(\mu,n-1)}{s(\mu,n)}\right|^2 \in Q(L).$$
Hence
$$d_N(\mu,\eta)^{-1} = \Im[\mathrm{e}^{-i\eta}(\mu-K)]\sum_{n=1}^{N}|s(\mu,n)|^2 + \sum_{n=1}^{N}\Im[\mathrm{e}^{-i\eta}(K-\gamma_\infty(n))]|s(\mu,n)|^2.$$
Suppose $\mu \in \Lambda \in S(L)$. If η and K are chosen such that Λ is mapped to the upper half plane by the Möbius transformation $\lambda \mapsto \mathrm{e}^{-i\eta}(\lambda-K)$ then the first summand on the right of this equation is positive while the second is nonnegative.

Hence we proved the following lemma.

Lemma 3.1. *Suppose $\mu \in \Lambda \in S(L)$. Then there is an $\eta \in [0,2\pi)$ such that the Möbius transformation $B^{-1}_{\mu,N}$ maps the closed half plane $\Im(\beta\mathrm{e}^{i\eta}) \geq 0$ onto a closed disk $D_N(\mu,\eta)$ of diameter $d_N(\mu,\eta)$, given through equation* (1).

We now investigate the behavior of the disks $D_N(\mu,\eta)$ when N tends to infinity. First note that $m \in D_N(\mu,\eta)$ if and only if
$$\beta\mathrm{e}^{i\eta} = \mathrm{e}^{i\eta}\frac{\psi_m(\mu,N)}{a_N\psi'_m(\mu,N)}$$
is in the upper half plane which happens if and only if
$$\Im(a_N\psi'_m(\mu,N)\overline{\psi_m(\mu,N)}\mathrm{e}^{-i\eta}) < 0.$$

Applying once more Lemma 2.1 we get

$$a_N \psi'_m(\mu, N)\overline{\psi_m(\mu, N)} = m + \sum_{n=1}^{N} (\mu - \gamma_m(n))|\psi_m(n)|^2$$

where

$$\gamma_m(n) = q_n - a_{n-1} \left| \frac{\psi'_m(\mu, n-1)}{\psi_m(n)} \right|^2 \in Q(L).$$

Hence $m \in D_N(\mu, \eta)$ if and only if

$$\sum_{n=1}^{N} \Im[\mathrm{e}^{-i\eta}(\mu - \gamma_m(n))]|\psi_m(\mu, n)|^2 < \Im(-\mathrm{e}^{-i\eta} m). \tag{2}$$

Since $\gamma_m(n) \in Q(L) \subset \Lambda^c$ and $\mu \in \Lambda$ each summand on the left hand side is positive and therefore the sum may only increase when N increases. The right hand side, however, is independent of N. Therefore, if $M > N$ and $m \in D_M(\mu, \eta)$ then also $m \in D_N(\mu, \eta)$, i.e., the discs are nested.

It follows immediately from inequality (2) that

$$\Im[\mathrm{e}^{-i\eta}(\mu - K)] \sum_{n=1}^{N} |\psi_m(\mu, n)|^2 < \Im(-\mathrm{e}^{-i\eta} m)$$

which shows that $\psi_m(\mu, \cdot) \in \ell^2(\mathbb{N}_0)$ provided that $m \in \bigcap_{N=1}^{\infty} D_N(\mu, \eta)$. Since we know that under our assumptions there is at most one square summable solution, this lemma shows also that $\bigcap_{N=1}^{\infty} D_N(\mu, \eta)$ consists of just a single point, the value of the m-function at the point μ. In particular, $D_\infty(\mu) = \bigcap_{N=1}^{\infty} D_N(\mu, \eta)$ is independent of η (if there ever was choice for η).

We have therefore the following theorem.

Theorem 3.2. *Suppose $\mu \in \Lambda \in S(L)$. If $m \in D_\infty(\mu)$ then $\psi_m(\mu, \cdot) = c(\mu, \cdot) + ms(\mu, \cdot)$ is in $\ell^2(\mathbb{N}_0)$, i.e., there exists a square summable solution of $Ly = \mu\hat{y}$.*

4. Asymptotic behavior of the m function

A ray $\mathcal{R}(t) = bt$, $t \geq 0$ is called admissible for L if it eventually lies in some $\Lambda \in S(L)$ and if it is not parallel to the boundary of that Λ.

Lemma 4.1. *Fix $N \in \mathbb{N}$. Let μ be sufficiently large and on an admissible ray. Then the diameter $d_N(\mu, \eta)$ of the disk $D_N(\mu, \eta)$ satisfies $d_N(\mu, \eta) = O(\mu^{1-2N})$ as μ tends to infinity on the ray.*

Proof. Note that

$$\frac{a_N s'(\mu, N)}{s(\mu, N)} = \mu - q_N + a_{N-1} \left(1 - \frac{s(\mu, N-1)}{s(\mu, N)}\right).$$

Since $\Im(\mathrm{e}^{-i\eta}(\mu - q_N)) \geq 2\varepsilon|\mu|$ for sufficiently large μ on the ray and since, by Lemma 2.2, $a_{N-1}(1 - s(\mu, N-1)/s(\mu, N))$ is bounded we find that eventually

$$\Im(\mathrm{e}^{-i\eta} a_N s'(\mu, N)/s(\mu, N)) \geq \varepsilon|\mu|.$$

Also, again by Lemma 2.2, $|s(\mu, N)| \geq C_N|\mu|^{N-1}$. These estimates complete the proof because

$$d_N(\mu, \eta)^{-1} = |s(\mu, N)|^2 \Im(\mathrm{e}^{-i\eta} a_N s'(\mu, N)/s(\mu, N)).$$

□

THEOREM 4.2. *Suppose μ is on an admissible ray. Then*

$$m(\mu) = -1 + \frac{1}{\mu - b_1} + O(\mu^{-3}) = -1 + \frac{1}{\mu} + \frac{b_1}{\mu^2} + O(\mu^{-3})$$

as μ tends to infinity on the ray.

PROOF. The point $B_{\mu,2}^{-1}(0) = -c(\mu,2)/s(\mu,2)$ lies in the disk $D_2(\mu,\eta)$ whose diameter is of order μ^{-3} at most as μ tends to infinity. Hence $m + c(\mu,2)/s(\mu,2) = O(\mu^{-3})$. Computing $c(\mu,2)$ and $s(\mu,2)$ explicitly completes the proof. $\square$

5. A local Borg-Marchenko theorem

If $m = m(\mu)$ define

$$m_n(\mu) = -1 + \frac{\psi_m(\mu,n+1)}{a_n\psi_m(\mu,n)} = \frac{\psi_m(\mu,n+1) - a_n\psi_m(\mu,n)}{a_n\psi_m(\mu,n)}.$$

Note that $m_n(\mu)$ is the value of the m-function for the Jacobi problem where all coefficients have been shifted n times to the left because the restriction of $\psi_m(\mu,\cdot)$ to the set $\{n, n+1, ...\}$ is still square summable but where one has to take account of the fact that a_n might be different from one. In particular, $m(\mu) = m_0(\mu)$. The difference equation yields that the m_n satisfy the recurrence relation

$$a_n^2(m_n(\mu) + 1) = \mu - b_n - \frac{1}{m_{n-1}(\mu) + 1}. \tag{3}$$

Let Σ denote a fixed open sector of the complex plane whose vertex is at the origin and let $\mathcal{L}_\Sigma$ denote the set of those Jacobi expressions satisfying the conditions in Section 1 for which there is $\Lambda \in S(L)$ such that $\Lambda^c \cap \Sigma$ is bounded.

THEOREM 5.1. *Let L and $\tilde{L}$ be two Jacobi expressions in $\mathcal{L}_\Sigma$ and let m and $\tilde{m}$ be the associated m-functions. Let $\mathcal{R}$ be a ray in Σ. Then the following statement holds: $m(\mu) - \tilde{m}(\mu) = O(\mu^{-2N-1})$ on $\mathcal{R}$ if and only if $b_n = \tilde{b}_n$ and $a_{n-1}^2 = \tilde{a}_{n-1}^2$ for $n \in \{1, ..., N\}$.*

PROOF. First note that $\mathcal{R}$ is admissible for both L and $\tilde{L}$.

Assume $b_n = \tilde{b}_n$ and $a_{n-1}^2 = \tilde{a}_{n-1}^2$ for $n \in \{1, ..., N\}$. Using this information one may show inductively that $a_1c(\mu,2) = \tilde{a}_1\tilde{c}(\mu,2)$, ..., $a_1...a_Nc(\mu,N+1) = \tilde{a}_1...\tilde{a}_N\tilde{c}(\mu,N+1)$. Similarly, $a_1...a_Ns(\mu,N+1) = \tilde{a}_1...\tilde{a}_N\tilde{s}(\mu,N+1)$. Hence

$$-\frac{c(\mu,N+1)}{s(\mu,N+1)} = -\frac{\tilde{c}(\mu,N+1)}{\tilde{s}(\mu,N+1)} = B_{\mu,N+1}^{-1}(0)$$

is a point on the boundary of both $D_{N+1}(\mu,\eta)$ and $\tilde{D}_{N+1}(\mu,\tilde{\eta})$. If μ is as described then m and $\tilde{m}$ are in these disks, respectively. Therefore their distance cannot be any larger than the sum of the diameters of those disks. Because of Lemma 4.1 the first part of the theorem is therefore proven.

Now assume that $m(\mu) - \tilde{m}(\mu) = O(\mu^{-2N-1})$ on $\mathcal{R}$. Firstly, Theorem 4.2 and our assumption give

$$\frac{1}{\mu - b_1} - \frac{1}{\mu - \tilde{b}_1} = O(\mu^{-3}).$$

This yields $b_1 = \tilde{b}_1$.

We will now prove by induction on n that $a_{n-1}^2 = \tilde{a}_{n-1}^2$, $b_n = \tilde{b}_n$, and $m_{n-1}(\mu) - \tilde{m}_{n-1}(\mu) = O(\mu^{-2(N-n)-3})$ for all $n \in \{1, ..., N\}$. This is true for $n = 1$. Assume now that these statements are true for some $n \in \{1, ..., N-1\}$. Then, using equation (3),

$$a_n^2(m_n(\mu)+1) - \tilde{a}_n^2(\tilde{m}_n(\mu)+1) = \frac{m_{n-1}(\mu) - \tilde{m}_{n-1}(\mu)}{(m_{n-1}(\mu)+1)(\tilde{m}_{n-1}(\mu)+1)} = O(\mu^{-2(N-n)-1}). \tag{4}$$

Since m_n and $\tilde{m}_n$ are m-functions, Theorem 4.2 gives their asymptotic behavior so that (4) implies

$$\frac{a_n^2}{\mu - b_{n+1}} - \frac{\tilde{a}_n^2}{\mu - \tilde{b}_{n+1}} = O(\mu^{-3}).$$

Thus $a_n^2 = \tilde{a}_n^2$ and $b_{n+1} = \tilde{b}_{n+1}$. Equation (4) yields now also that $m_n(\mu) - \tilde{m}_n(\mu) = O(\mu^{-2(N-n)-1})$. This completes the induction. $\square$

We give now another proof for the "only if" part of Theorem 5.1, following Bennewitz [**1**]. So, assume again that μ on $\mathcal{R}$ and $m(\mu) - \tilde{m}(\mu) = O(\mu^{-2N-1})$ as μ tends to infinity. Fix $n \in \{1, ..., N+1\}$. By Lemma 2.2, $\tilde{s}(\mu, n)/s(\mu, n)$ is bounded as μ tends to infinity. Moreover,

$$\frac{1}{s(\mu, n)\psi_m(\mu, n)} = \frac{a_n s(\mu, n+1)}{s(\mu, n)} - a_n^2(m_n(\mu)+1)$$

tends to infinity because the first term on the right does while the second tends to zero. Therefore

$$\begin{aligned}\tilde{s}(\mu, n)\psi_m(\mu, n) &- s(\mu, n)\tilde{\psi}_m(\mu, n) \\ &= \tilde{s}(\mu, n)c(\mu, n) - s(\mu, n)\tilde{c}(\mu, n) + (m(\mu) - \tilde{m}(\mu))\tilde{s}(\mu, n)s(\mu, n)\end{aligned}$$

tends to zero as μ tends to infinity. By Lemma 2.2 and our assumption on $m - \tilde{m}$ the last term on the right hand side of this equation tends to zero so that the polynomial

$$\tilde{s}(\mu, n)c(\mu, n) - s(\mu, n)\tilde{c}(\mu, n)$$

also tends to zero as μ tends to infinity. Hence it must be identically equal to zero, i.e.,

$$\frac{c(\mu, n)}{s(\mu, n)} = \frac{\tilde{c}(\mu, n)}{\tilde{s}(\mu, n)}$$

for any $n \in \{1, ..., N+1\}$ and all $\mu \in \mathbb{C}$. Since

$$\frac{1}{a_{n-1}s(\mu, n)s(\mu, n-1)} = \frac{c(\mu, n-1)}{s(\mu, n-1)} - \frac{c(\mu, n)}{s(\mu, n)}$$

we obtain

$$a_{n-1}s(\mu, n)s(\mu, n-1) = \tilde{a}_{n-1}\tilde{s}(\mu, n)\tilde{s}(\mu, n-1) \tag{5}$$

and, using this and the difference equation,

$$(\mu - b_{n-1})s(\mu, n-1)^2 = (\mu - \tilde{b}_{n-1})\tilde{s}(\mu, n-1)^2. \tag{6}$$

In particular, for $n = 2 \le N+1$ we find $b_1 = \tilde{b}_1$.

We now prove by induction on k that $a_{k-1}^2 = \tilde{a}_{k-1}^2$, $b_k = \tilde{b}_k$, and $s(\mu, k)^2 = \tilde{s}(\mu, k)^2$ for $k = 1, ..., N$. These statements are true for $k = 1$. Assume now that they are true for some $k \in \{1, ..., N-1\}$. We may then use the square of equation

(5) for $n = k+1$ and equation (6) for $n = k+2$ to obtain the homogeneous linear system

$$\begin{pmatrix} a_k^2 & -\tilde{a}_k^2 \\ \mu - b_{k+1} & -(\mu - \tilde{b}_{k+1}) \end{pmatrix} \begin{pmatrix} s(\mu, k+1)^2 \\ \tilde{s}(\mu, k+1)^2 \end{pmatrix} = 0.$$

The determinant of the matrix must be zero for almost all μ and this proves $a_k^2 = \tilde{a}_k^2$ and $b_{k+1} = \tilde{b}_{k+1}$. Squaring the difference equation and using (5) for $n = k$ shows also that $s(\mu, k+1)^2 = \tilde{s}(\mu, k+1)^2$. This completes the proof.

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Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294–1170, USA

E-mail address: rudi@math.uab.edu