COMPARISON AND OSCILLATION THEOREMS FOR FIRST ORDER SYSTEMS WITH DISTRIBUTIONAL COEFFICIENTS

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Abstract. We establish a comparison theorem and an oscillation theorem for a $2 \times 2$-system of ordinary differential equations when the coefficients are (possibly singular) continuous, real, and finite measures.

1. Introduction

The goal of the present paper is to establish a comparison theorem and an oscillation theorem for a $2 \times 2$-system of ordinary differential equations when the coefficients are (possibly singular) continuous, real, and finite measures. This is a well studied subject for a Sturm-Liouville equation $-(py')' + vy = \lambda ry$ posed on a finite interval $(a, b)$ when $\lambda$ is a parameter and $1/p$, $v$, and $r$ are real and locally integrable functions. Hinton [12] gives an excellent survey on this subject. We also refer the reader to the books by Coddington and Levinson [6], Swanson [16], Teschl [17], and Zettl [21], but there are, of course, many others. Most of the extant results are for the definite case where $p, r > 0$. It appears that the so called indefinite case, where $p$ or $r$ may change sign, was first addressed by Richardson [15]. A (much more) recent contribution in this regard – with references to further literature – is by Binding and Volkmer [3].

Writing the Sturm-Liouville equation as a first order system yields

$$Ju' + qu = \lambda wu$$

where $u = (y, py')^\top$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 1/p \\ 0 & -1/p \end{pmatrix}$, and $w = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$. Comparison and oscillation theorems for more general systems with integrable coefficients have also been studied. We refer, for instance, to Coppel [7], Reid [14], and Brown, Eastham, and Schmidt [4].

Our goal here, as we said above, is to allow the entries of the matrices $q$ and $w$ to be finite, continuous measures. We ask the measures to be finite as that leads to regular endpoints of the interval $(a, b)$. We are restricting ourselves to continuous measures for reasons which we shall explain in Section 3. Schrödinger equations with distributional potentials (worse than measures) have been studied by Shkalikov and Ben Amara [1] and by Homa and Hryniv [13]. However, these equations can be cast as a system with integrable coefficients as was pointed out by Eckhardt et al. [8]. A comparison theorem for such cases was obtained by Ghatasheh and Weikard [9].

We end this introduction by briefly describing the content of each section. In Section 2 we discuss distributions of order 0 and their connection to measures. In Section 3 we define Prüfer angles and prove two basic results about comparison of Prüfer angles for two equations $Ju' = pju$ where $p_2 \geq p_1$. Moreover, we establish
a condition under which the zeros of the first component of a solution of $Ju' = pu$
are isolated since the solution can then cross the vertical axis only in a clockwise
direction. These fundamental facts are then used in Section 4 to prove a comparison
theorem, i.e., a theorem comparing the number of zeros of the first components $u_j$ of
solutions of $Ju' = p_j u$ when $p_2 \geq p_1$, and a separation theorem, i.e., a theorem
showing that between two consecutive zeros of $u_1$ there is always a zero of $v_1$ if $u$
and $v$ are linearly independent solution of $Ju' = pu$. Finally, in Section 5 we replace
$p$ by $\lambda w - q$ with a spectral parameter $\lambda$ and appropriate measure coefficients
$q$ and $w$. In this case the equation $Ju' + qu = \lambda w u$ gives rise to a self-adjoint
relation and the object of the oscillation theorems proved there is to relate the eigenvalue
count with the number of zeros of the first components of the eigenfunctions.

2. Some basic facts about distributions

Let $(a, b)$ be an open interval in the real line. The compactly supported infinitely
differentiable functions from $(a, b)$ to $\mathbb{C}$ are called test functions and a linear
functional $p$ defined on the set of test functions is called a distribution on $(a, b)$, if
it satisfies the following condition: for every compact subset $K$ of $(a, b)$ there are
numbers $C \geq 0$ and $k \in \mathbb{N}_0$ such that

\[ |p(\phi)| \leq C \sum_{j=0}^{k} \|\phi^{(j)}\|_{\infty} \]

whenever $\phi$ is a test function with support in $K$. The distribution $p$ is called a
distribution of order 0, if one may choose $k = 0$ for any compact set $K$. The space
distributions of order 0 on $(a, b)$ is denoted $D'(a, b)$.

If $U$ is an open set in $(a, b)$ and $p(\phi) = 0$ whenever the support of $\phi$ is contained
in $U$ one says that the distribution $p$ vanishes on $U$. The support of $p$ is the
complement of the largest open set in $(a, b)$ on which $p$ vanishes.

The most important fact about distributions is that they all have derivatives. If $p$ is a distribution its derivative $p'$ is defined by $p'(\phi) = -p(\phi')$ and is again a
distribution. Distributions also have antiderivatives and we have the following
important lemma.

**Lemma 2.1** (Du Bois-Reymond). Suppose the derivative of the distribution $p$ is
zero. Then $p$ is the constant distribution, i.e., there is a complex number $C$ such
that $p(\phi) = C \int \phi dx$ for every test function $\phi$.

If $P$ is a function of locally bounded variation on $(a, b)$ and $dP$ is the associated
local Lebesgue-Stieltjes measure one may assign the number $p(\phi) = \int \phi dP$ to the
test function $\phi$ to obtain a distribution of order 0. It is, in fact, a consequence
of Riesz’s representation theorem that every distribution of order 0 on $(a, b)$ is
represented in this way by a (local) Lebesgue-Stieltjes measure. See, for instance,
[10] for more details.

The most famous example of a distribution of order 0 is the Dirac distribution
$\delta_0(\phi) = \phi(0)$ (assuming $0 \in (a, b)$). It may be represented by the integral of $\phi$
with respect to Dirac measure which is the Lebesgue-Stieltjes measure generated
by the Heaviside step function. The support of $\delta_0$ is $\{0\}$. Perhaps the most familiar
example of a distribution is $f(\phi) = \int \phi f dx$ when $f$ is a locally integrable function.

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1Only if $P$ is actually of bounded variation or if it is non-decreasing does it generate a measure
on $(a, b)$. Otherwise it generates a measure on any compact subset of $(a, b)$. 

Now suppose that $P \in \text{BV}_{\text{loc}}((a, b))$, the space of complex-valued functions of locally bounded variation on $(a, b)$. We then have the distributions $p(\phi) = \int \phi dP$. Since $P$ is also locally integrable we also have the distribution $P(\phi) = \int P \phi \, dx$ (we shall henceforth identify a function $P \in L_{\text{loc}}^1((a, b))$ with the distribution $\phi \mapsto \int P \phi \, dx$; context will serve to distinguish the two meanings if necessary). Integration by parts shows that

$$P'(\phi) = -P(\phi') = -\int P \phi' \, dx = -\int P d\phi = \int \phi dP = p(\phi)$$

since the boundary terms vanish. Thus $P' = p$. This suggests to use the notation $p(\phi) = \int \phi dP = \int p\phi$ even if $p$ is not a function.

It is perhaps useful to recall the product rule for functions of locally bounded variation underlying the integration by parts formula. If $F$ and $G$ are in $\text{BV}_{\text{loc}}((a, b))$, then so is their product and

$$d(FG) = (FG)' = F^+G' + F'G^- = F^+ dG + dF G^-,$$

see, e.g., Hewitt and Stromberg [11], Theorem 21.67 and Remark 21.68. Note that, if one of $F$ and $G$ is continuous, one may ignore the superscripts $\pm$ on $F$ and $G$.

Not so simple is the chain rule which appears to have first been established by Vol’pert [18], see also Vol’pert and Hudjaev [19].

**Theorem 2.2.** Suppose $u \in \text{BV}_{\text{loc}}((a, b))$, $V$ is an open set in $\mathbb{C}$ containing the range of $u$, and $f \in C^1(V)$. Then $f \circ u \in \text{BV}_{\text{loc}}((a, b))$ and

$$(f \circ u)' = \left[ \int_0^1 f'(su^+ + (1-s)u^-) \, ds \right] u'.$$

**Corollary 2.3.** If $u \in \text{BV}_{\text{loc}}((a, b))$ is continuous and $f \in C^1(V)$, then $(f \circ u)' = (f' \circ u)u'$.

The conjugate of a distribution $p$, denoted by $\overline{p}$, is defined by $\overline{p}(\phi) = p(\overline{\phi})$. A distribution is called real if $p = \overline{p}$. Equivalently, $p$ is real if $p(\phi) \in \mathbb{R}$ whenever $\phi$ is real-valued. Similarly, $p$ is called non-negative, written as $p \geq 0$, if $p(\phi) \geq 0$ for all $\phi \geq 0$. If $p$ is real or non-negative, then so is the associated measure.

If $p = dP$ is a distribution of order 0 and $f \in L_{\text{loc}}^1(|dP|)$, we may define the product $pf = fp$ as follows: $(pf)(\phi) = \int f \phi \, dP$, again a distribution of order 0.

Finally, a word about matrix-valued distributions. As usual the entry in row $j$ and column $k$ of a matrix $p$ is denoted by $p_{jk}$. Sometimes we will consider matrices $p_1$ and $p_2$. Then $p_{\ell,jk}$ denotes the entry in row $j$ and column $k$ of $p_{\ell}$. Similarly, $u_{k}$ and $u_{\ell,k}$ denote respectively the $k$-th component of the vectors $u$ and $u_{\ell}$. To avoid cumbersome notation we may use the symbol $u_1$ for the vector $u_1$, and for the first component of the vector $u$. Which meaning is appropriate will be clear from the context. We are interested in the case of $2 \times 2$-matrices whose entries are distributions of order 0 on $(a, b)$, a space which we denote by $\mathcal{D}^0((a, b))^{2 \times 2}$. A matrix $r \in \mathcal{D}^0((a, b))^{2 \times 2}$ is called hermitian if $r_{jk} = r_{kj}$ for $j, k = 1, 2$. This is equivalent to the requirement that $z^* r z$ is real for all $z \in \mathbb{C}^2$. One calls $r$ non-negative, if $z^* r z$ is non-negative for all $z \in \mathbb{C}^2$. If $r_1, r_2 \in \mathcal{D}^0((a, b))^{2 \times 2}$, we write $r_1 \geq r_2$ if $r_1 - r_2$ is non-negative. We say that $r$ vanishes on an open set $U \subset (a, b)$ if this is true for each of its entries.
3. The Pr"ufer transform

In this section we consider matrix-valued distributions in $\mathcal{D}^n((a,b))^{2 \times 2}$ whose entries are real and correspond to finite and continuous measures. The set of such distributions will be denoted by $\mathcal{C}$.

For $p \in \mathcal{C}$ and $J = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ we consider the equation

$$Ju' = pu$$

on the interval $(a, b)$. Under our assumptions $\mathbb{C}^2$-valued solutions $u$ of (1) satisfying arbitrary initials conditions are guaranteed to exist, are continuous, and have limits at $a$ and $b$. Moreover, since $\pi$ is a solution of (1) if $u$ is, it follows that the real and imaginary parts of $u$ are also solutions. Consequently, the space of solutions is spanned by real (i.e., $\mathbb{R}^2$-valued) solutions.

We now define the Pr"ufer variables $r$ and $\theta$ for a non-trivial real solution $u$ of (1) as the unique continuous functions on $[a, b]$ satisfying the following conditions

$$u(x) = r(x) \begin{pmatrix} \sin \theta(x) \\ \cos \theta(x) \end{pmatrix}, \quad r(x) > 0 \text{ and } \theta(a) \in [0, 2\pi).$$

Note that $r(x) = |u(x)|$ when $| \cdot |$ denotes the euclidean norm in $\mathbb{R}^2$. Using Theorem 2.2 and Corollary 2.3 gives that both $r$ and $\theta$ are functions of bounded variation and that

$$0 = Ju' - pu = r' J \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} + r \theta' J \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} - rp \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}.$$

Multiplying this from the left by the row vectors $(\sin \theta, \cos \theta)$ and $(\cos \theta, -\sin \theta)$ gives, respectively,

$$\theta' = (\sin \theta, \cos \theta)p \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}$$

and

$$\frac{r'}{r} = - (\cos \theta, -\sin \theta)p \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}.$$  (2)

If we were to allow for the presence of discrete measures, solutions might have jump discontinuities. In such a case the tangent of the Pr"ufer angle would still be well defined but to define the angle itself would necessitate choosing a branch of arctangent. We are yet unclear how to do this in a consistent manner. In this context, it may be useful to point out that the variable $r$ has dropped out of equation (2). This would not be the case, if $r$ and $\theta$ had jump discontinuities, rendering, perhaps, Pr"ufer’s approach less powerful.

We now return to the case of continuous measures. If the coefficients $p_1$ and $p_2$ of two different equations of the form (1) are comparable, then the Pr"ufer angles of their solutions are also comparable. The following results and their proofs are adopted from [2] and [20].

**Theorem 3.1** (Angles comparison I). Let $p_1, p_2 \in \mathcal{C}$ and suppose that $p_2 \geq p_1$. If $u_j, \theta_j \ (j = 1, 2)$ are non-trivial real solutions of $Ju' = p_j u$ and their Pr"ufer angles, respectively, the following statements hold.

(a) If $\theta_2(c) \geq \theta_1(c)$ for some $c \in (a, b)$, then $\theta_2(x) \geq \theta_1(x)$ for all $x \in [c, b]$.

(b) If $\theta_2(c) \leq \theta_1(c)$ for some $c \in (a, b)$, then $\theta_2(x) \leq \theta_1(x)$ for all $x \in (a, c]$. 

Proof. Define the continuous function \( \delta = \theta_2 - \theta_1 \) on \((a, b)\) and the distributions \( h \) and \( f \) in \( D^0((a, b)) \) by
\[
 h = (\sin \theta_1, \cos \theta_1)(p_2 - p_1) \left( \frac{\sin \theta_1}{\cos \theta_1} \right) \\
= (\sin \theta_1)^2(p_{2;11} - p_{1;11}) + (\cos \theta_1)^2(p_{2;22} - p_{1;22}) \\
+ (\sin 2\theta_1)(\frac{p_{2;12} + p_{2;21}}{2} - \frac{p_{1;12} + p_{1;21}}{2})
\]
and
\[
 f = (p_{2;22} - p_{2;11}) \frac{\sin \theta_2 - \sin \theta_1}{\theta_2 - \theta_1} (\sin \theta_2 + \sin \theta_1) - \frac{p_{2;12} + p_{2;21}}{2} \frac{\sin 2\theta_2 - \sin 2\theta_1}{\theta_2 - \theta_1}
\]
where we replace the quotients by their limits when they become undefined. Then we have
\[
 \delta' = h - \delta f.
\]
Let \( F \in BV((a, b)) \) be an antiderivative of \( f \) and define on \((a, b)\) the continuous function \( k(x) = \exp(F(x)) \). Since \( F \) is continuous the chain rule shows that \( k' = kf \).
Then the product rule gives
\[
 (k\delta)' = \delta k' + k \delta' = \delta kf + k \delta' = k(\delta f + \delta') = kh.
\]
Since \( k \) is a positive function and \( h \) is a non-negative distribution, \( (k\delta)' \) is a non-negative distribution on \((a, b)\).

Suppose \( \theta_2(c) \geq \theta_1(c) \), i.e., \( \delta(c) \geq 0 \), for some \( c \in (a, b) \). Then we get, for any \( x \in (c, b) \), that
\[
 (k\delta)(x) = (k\delta)(c) + \int_{(c,x)} (k\delta)' \geq 0.
\]
Thus \( \theta_2(x) \geq \theta_1(x) \) for all \( x \in [c, b) \). This finishes the proof of (a); that of (b) is similar. \( \square \)

Theorem 3.2 (Angles comparison II). Let \( p_1, p_2 \in \mathcal{C} \) and suppose that \( p_2 \geq p_1 \). Let \( u_j, \theta_j \) (\( j = 1, 2 \)) be non-trivial real solutions of \( Ju' = p_j u \) and their Prüfer angles, respectively. If \( (c, d) \) is a non-empty subinterval of \((a, b)\) such that \( (p_2 - p_1) \) does not vanish on \((c, d)\), the following statements hold.

(a) If \( \theta_2(c) \geq \theta_1(c) \), then \( \theta_2(x) > \theta_1(x) \) for all \( x \in [d, b) \).
(b) If \( \theta_2(d) \leq \theta_1(d) \), then \( \theta_2(x) < \theta_1(x) \) for all \( x \in [a, c) \).

Proof. By part (a) of Theorem 3.1, \( \theta_2(x) \geq \theta_1(x) \) in \([c, b)\). Suppose there exists \( x_0 \) in \([d, b)\) such that \( \theta_2(x_0) = \theta_1(x_0) \), then \( \theta_2(x) \leq \theta_1(x) \) in \((a, x_0)\) by part (b) of Theorem 3.1. Thus \( \theta_2(x) = \theta_1(x) \) on \([c, x_0]\). This means \( \delta = \delta' = 0 \), and hence \( h = 0 \) in \((c, x_0)\) using the notation of the proof of Theorem 3.1. Since \( |u_1|^2 h = u_1^2 (p_2 - p_1) u_1 \) this implies that \( (p_2 - p_1) u_1 \) is the zero distribution on \((c, x_0)\) and, in particular, on \((c, d)\). This finishes the proof of (a); that of (b) is similar. \( \square \)

Lemma 3.3. Suppose \( p \in \mathcal{C} \), \( x_0 \in [a, b) \), and \( \theta(x_0) \) is an integer multiple of \( \pi \), i.e., the first component of \( u(x_0) \) is 0. If
\[
 \int_{(x_0,x)} \frac{p_{22}}{2} \geq \frac{1}{2} \max \left\{ \left| \int_{(x_0,t)} \frac{p_{22}}{2} \right| : x_0 \leq t \leq x \right\}
\]
for all \( x \) in some non-empty interval \((x_0, x_1)\), then there is a \( \delta > 0 \) such that \( \theta(x) > \theta(x_0) \) for all \( x \in (x_0, x_0 + \delta) \).
Proof. If we define $\theta_0(t) = \int_{(x_0,t)} p_{22}$ equation (2) gives

$$\theta(t) - \theta(x_0) - \theta_0(t) = \int_{(x_0,t)} \left[ (p_{11} - p_{22})(\sin \theta)^2 + (p_{12} + p_{21})(\sin \theta)(\cos \theta) \right].$$

The mean value theorem implies that

$$\frac{\sin \theta(t)}{\theta(t) - \theta(x_0)} = \frac{\sin \theta(t) - \sin \theta(x_0)}{\theta(t) - \theta(x_0)} = \cos(t')$$

for some $t' \in (x_0, t)$. Hence $| \sin \theta(t) | \leq | \theta(t) - \theta(x_0) |$. Now we define $M(t) = \max\{|\theta(s) - \theta(x_0)| : x_0 \leq s \leq t\}$ to obtain

$$|\theta(t) - \theta(x_0) - \theta_0(t)| \leq M(t) \left( M(t) \int_{(x_0,t)} |p_{11} - p_{22}| + \int_{(x_0,t)} |p_{12} + p_{21}| \right).$$

The latter factor on the right of this inequality, which we call $F(t)$, is continuous as a function of $t$ and vanishes at $x_0$. Therefore there is a number $\delta \in (0, x_1 - x_0)$ such that $F(t)$ is less than $1/4$ as long as $x_0 \leq t \leq x < x_0 + \delta$. For such $t$ we have now

$$|\theta(t) - \theta(x_0)| - \theta_0(t) \leq |\theta(t) - \theta(x_0) - \theta_0(t)| \leq \frac{1}{4} M(t)$$

noting that $\theta_0(t) \geq 0$. Using our hypothesis we find $|\theta(t) - \theta(x_0)| \leq \theta_0(t) + \frac{1}{4} M(t) \leq 2\theta_0(x) + \frac{1}{4} M(x)$ so that $\frac{3}{4} M(x) \leq 2\theta_0(x)$. The latter inequality in (5) becomes then

$$|\theta(x) - \theta(x_0) - \theta_0(x)| \leq \frac{1}{4} M(x) \leq \frac{3}{4} \theta_0(x)$$

which proves our claim. \(\square\)

If $p_{22}$ is a positive measure with support $(a, b)$, one may choose $x_1 = b$ for every $x_0 \in (a, b)$ to satisfy condition (4). The positivity of $p_{22}$ is, however, not a necessary condition as the following example shows. The function $P_{22}$ defined by $P_{22}(x) = x + 2x^2 \sin(1/x)$ for $x > 0$ and $P_{22}(0) = 0$ is absolutely continuous on $[0, 1]$. Therefore its derivative gives rise to a measure $p_{22}$ which is absolutely continuous with respect to Lebesgue measure. It is easy to see that $p_{22}$ is not a positive measure. However, condition (4) holds for all $x \in (0, 1)$.

Considering instead $P_{22}(x) = 2x^2 \sin(1/x)$, it turns out that condition (4) is violated. The system $Ju' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u$ is solved by the function $u = (-2x^2 \sin(1/x), 1)^\top$. This shows that the conclusion of Lemma 3.3 is violated and hence that some condition on $p_{22}$ is necessary.

If condition (4) is replaced by

$$\int_{(x,x_0)} p_{22} > \frac{1}{2} \max \left\{ \left| \int_{(t,x_0)} p_{22} \right| : x_0 \leq t \leq x \right\}$$

for all $x$ in some interval $(x_1, x_0)$ we may similarly conclude that $\theta(x) < \theta(x_0)$ for all $x$ in some interval $(x_0 - \delta, x_0)$. We have therefore the following theorem.

**Theorem 3.4.** Suppose $p \in C$ and $u$ and $\theta$ are non-trivial real solution of $Ju' = pu$ and its Prüfer angle, respectively. If conditions (4) and (6) are satisfied near $x_0$ and if $u_1(x_0) = 0$, then $x_0$ is an isolated zero of $u_1$. Moreover, the Prüfer angle $\theta$ equals an integer multiple of $\pi$ at $x_0$ and is strictly increasing there. In other words, the point $u$ crosses the vertical axis clockwise.

If, instead, conditions (4) and (6) hold for $-p_{22}$ instead of $p_{22}$, then the Prüfer angle $\theta$ is strictly decreasing at $x_0$ and $u$ crosses the vertical axis counterclockwise.
**Theorem 3.5** (Isolated zeros). Suppose \( p \in C \) and \( p_{22} \) is a positive measure with support \((a, b)\) and let \( u \) be a non-trivial real solution of (1). Then the zeros of \( u_1 \), the first component of \( u \), are isolated and the number of these zeros in \((a, b)\) is bounded above by

\[
1 + \frac{1}{\pi} \int_{(a,b)} \left( |p_{11}| + p_{22} + \frac{|p_{12} + p_{21}|}{2} \right),
\]

where \( |p_{jk}| \) is the total variation measure of \( p_{jk} \).

**Proof.** We mentioned earlier that condition (4) is satisfied for any \( x_0 \in (a, b) \) if \( p_{22} \) is a positive measure. Thus Theorem 3.5 shows that the zeros of \( u_1 \) are isolated.

Theorem 3.5 also shows that the trajectory of \( u \) transverses the vertical axis in the clockwise direction at \( x_0 \). Thus \( u_1 \) picks up one more zero whenever \( \theta \) increases an amount of \( \pi \). The number of zeros of \( u_1 \) is therefore bounded above by

\[
1 + \frac{1}{\pi} |\theta(b) - \theta(a)| = 1 + \frac{1}{\pi} \int_{(a,b)} \left( (\sin \theta)^2 p_{11} + (\cos \theta)^2 p_{22} + (\sin 2\theta)p_{12} + p_{21} \right) \leq 1 + \frac{1}{\pi} \int_{(a,b)} \left( |p_{11}| + p_{22} + \frac{|p_{12} + p_{21}|}{2} \right).
\]

\( \square \)

Let \( x_1, \ldots, x_N \) be finitely many points in \((a, b)\) and suppose \( (x_{n-1}, x_n) \) is either a positive or negative measure for \( n = 1, \ldots, N + 1 \) and if \( \text{supp} p_{22} = (a, b) \), then the zeros of \( u_1 \) will still be isolated. However, the upper bound of these zeros mentioned in Theorem 3.5 may not hold anymore, since each interval \((x_{n-1}, x_n)\) could contribute a zero without \( \theta \), the Prüfer angle of \( u \), changing very much.

4. Comparison and separation theorems

**Theorem 4.1** (Comparison theorem). Let \( p_1 \) and \( p_2 \) be elements of \( C \) and suppose that \( p_2 \geq p_1 \). Furthermore, assume that \( p_{1;22} \) and \( p_{2;22} \) are positive measures with support equal to \((a, b)\). For \( j = 1, 2 \) let \( u_j \) be a non-trivial real solution of \( Ju' = p_j u \). If \( x_0 < x_1 \) are two consecutive zeros of \( u_{1;1} \) and if \( u_1 \) and \( u_2 \) are linearly independent, then \( u_{2;1} \) has at least one zero in \((x_0, x_1)\).

**Proof.** First we note, in view of Theorem 3.5, that the zeros of \( u_{1;1} \) and \( u_{2;1} \) are isolated. Without loss of generality we can assume that \( \theta_1(x_0) = 0 \) and \( 0 \leq \theta_2(x_0) < \pi \). Since \( x_0 \) and \( x_1 \) are two consecutive zeros of \( u_{1;1} \), we have \( \theta_1(x_1) = \pi \).

If \( (p_2 - p_1)u_1 \neq 0 \) we apply Theorem 3.2 with \( c = x_0 \) and \( d = x_1 \) and conclude that \( \theta_2(x_1) > \theta_1(x_1) = \pi \). The continuity of \( \theta_2 \) shows that there is a point \( y \in (x_0, x_1) \) such that \( \theta_2(y) = \pi \) so that \( u_{2;1}(y) = 0 \). If, on the other hand, \( (p_2 - p_1)u_1 = 0 \), then both \( u_1 \) and \( u_2 \) satisfy the equation \( Ju' = p_{22} u \). Since \( u_1 \) and \( u_2 \) are linearly independent, we must have \( u_{2;1}(x_1) \neq 0 \) and hence \( \theta_2(x_1) \neq \pi \). Since, using Theorem 3.1, we also have \( \theta_2(x_1) \geq \theta_1(x_1) = \pi \) we conclude again that \( \theta_2(x_1) > \pi \) and we may now argue as before for a zero of \( u_{2;1} \) in \((x_0, x_1)\). \( \square \)

**Theorem 4.2** (Separation theorem). Suppose \( p \in C \) and \( p_{22} \) is a positive measure whose support equals \((a, b)\). Let \( u_j \), \( j = 1, 2 \), be linearly independent (thus non-trivial) solutions of \( Ju' = p u \). If \( x_0 < x_1 \) are two consecutive zeros of \( u_{1;1} \), then \( u_{2;1} \)
has precisely one zero in \((x_0, x_1)\). In other words, the zeros of the first component of linearly independent solutions are interlacing.

**Proof.** Since \(u_1\) and \(u_2\) are linearly independent, it follows that \(u_{2,1}(x_0) \neq 0\). Choosing \(p_2 = p_1 = p\), Theorem 4.1 gives therefore that \(u_{2,1}\) has at least one zero \(y\) in \((x_0, x_1)\). By way of contradiction, let \(y_0\) and \(y_1\) be the first and second zero of \(u_{2,1}\) in \((x_0, x_1)\). Interchanging the role of \(u_1\) and \(u_2\) in Theorem 4.1, \(u_{1,1}\) has a zero in \((y_0, y_1)\). This contradicts the hypothesis on \(x_0\) and \(x_1\). \(\square\)

5. **The oscillation theorem**

In this section we let the coefficient \(p\) in equation (1) depend on another real parameter \(\lambda\). Specifically, we set

\[ p = \lambda w - q \]

where \(q\) is hermitian and \(w\) is non-negative. Since we still require that the entries of \(q\) and \(w\) are real, this entails, in fact, that they are symmetric. Thus the differential equation we consider reads now

\[ Ju' + qu = \lambda wu. \] (7)

Next we describe briefly how to define linear relations associated with the differential equation (7); more details may be found in [10]. Since \(w \geq 0\) its trace is a positive scalar measure. Therefore we may introduce the vector space \(L^2(w)\) consisting of \(C^2\)-valued functions \(f\) where both components are measurable with respect to \(\text{tr} w\) and which satisfy \(\int f^*w f < \infty\). The vector space \(L^2(w)\) is a semi-inner product space with semi-inner product \(\langle f, g \rangle = \int f^* w g\) and semi-norm \(\|f\| = (\langle f, f \rangle)^{1/2}\). The corresponding Hilbert space, i.e., the quotient of \(L^2(w)\) by the kernel of \(\| \cdot \|\), will be denoted by \(L^2(w)\). We now introduce the linear relations

\[ T_{\text{max}} = \{(u, f) \in L^2(w) \times L^2(w) \colon u \in \text{BV}_{\text{loc}}((a, b))^2, Ju' + qu = w f\} \]

and

\[ T_{\text{min}} = \{(u, f) \in T_{\text{max}} : \text{supp} u \text{ is compact in } (a, b)\}. \]

Then, in the Hilbert space setting, we represent our differential equation by the relations

\[ T_{\text{max}} = \{([u], [f]) \in L^2(w) \times L^2(w) : (u, f) \in T_{\text{max}}\} \]

and

\[ T_{\text{min}} = \{([u], [f]) \in T_{\text{max}} : (u, f) \in T_{\text{min}}\}. \]

It was shown in [10] (see also [5] for a proof under more general circumstances) that \(T_{\text{min}}^* = T_{\text{max}}\) which shows that \(T_{\text{min}}\) is a symmetric relation. Since \(q\) and \(w\) are finite measures \(T_{\text{min}}\) has equal deficiency indices so that we have self-adjoint extensions. The deficiency indices are equal to 1 if and only if there is a solution of \(Ju' + qu = 0\) of norm 0. We will exclude this case, i.e., we assume subsequently that the deficiency indices \(n_{\pm}\) are equal to 2. For easy reference we collect our conditions in the following hypothesis.

**Hypothesis 5.1.** Suppose \(q, w \in D^0((a, b))^2\times2\) have real entries whose associated measures are finite and continuous. Moreover, \(q\) is symmetric and \(w\) non-negative. Finally, if \(Ju' + qu = 0\) and \(wu = 0\), then \(u\) is identically equal to 0.
Given numbers $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$ we consider now the relation
\[ T_{\alpha,\beta} = \{(u,f) \in T_{\max} : (\cos \alpha - \sin \alpha)u(a) = (\cos \beta - \sin \beta)u(b) = 0\} \]
and the corresponding relation
\[ T_{\alpha,\beta} = \{([u], [f]) \in T_{\max} : (u, f) \in T_{\alpha,\beta}\}. \]
This is a self-adjoint relation, see Section 7.2 in [10].

Since $\pi$ is an eigenfunction associated with $\lambda$ precisely when $u$ is, it follows that $\Re u$ and $\Im u$ are also eigenfunctions. In fact, $\Re u$ and $\Im u$ must be multiples of each other since this is true for their initial values at $a$. It follows that the eigenspace of $\lambda$ is one-dimensional and is spanned by a real solution of $Ju' + qu = \lambda w u$.

**Theorem 5.2 (Oscillation theorem).** Suppose $q$ and $w$ satisfy Hypothesis 5.1, $-q_{22} \geq 0$, and $\text{supp } q_{22} = (a, b)$. Then the eigenvalues of $T_{\alpha,\beta}$ are isolated and can accumulate only at $\infty$ and $-\infty$.

Let $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ be the positive eigenvalues and $\lambda(\cdot, \lambda_n)$ be the corresponding real eigenfunctions. Let $\theta(\cdot, 0)$ be the Prüfer angle for a real solution of $Ju' + qu = 0$ satisfying the boundary condition at $a$ and $k_r$ the unique non-negative integer satisfying $\theta(b, 0) = k_r \pi + \beta$ where $\beta \in (0, \pi]$. If $\gamma < \beta$, then $u_1(\cdot, \lambda_n)$ has precisely $k_r + n - 1$ zeros in $(a, b)$. If $\gamma \geq \beta$, then $u_1(\cdot, \lambda_n)$ has precisely $k_r + n$ zeros in $(a, b)$.

If there is a number $\mu_0 < 0$ such that $q_{22} - \mu_0 w_{22} \geq 0$ let $\theta(\cdot, \mu_0)$ be the Prüfer angle for a real solution of $Ju' + (q - \mu_0 w)u = 0$ satisfying the boundary condition at $a$ and $k_\delta$ the unique non-negative integer satisfying $\theta(b, \mu_0) = \delta - k_\delta \pi$ where $\delta \in (0, \pi]$. Let $\mu_1 > \mu_2 > \mu_3 > \ldots$ be the eigenvalues of $T_{\alpha,\beta}$ strictly below $\mu_0$ and $u(\cdot, \mu_n)$ the corresponding real eigenfunctions. If $\alpha > 0$ then $u_1(\cdot, \mu_n)$ has precisely $k_\delta + n$ or $k_\delta + n - 1$ zeros in $(a, b)$, depending on whether $\delta \leq \beta$ or $\delta > \beta$. If $\alpha = 0$ the count of zeros is smaller by 1.

**Proof.** Let $u(\cdot, \lambda)$ be a real solution of $Ju' + qu = \lambda w u$ satisfying the initial condition $u(a, \lambda) = (\sin \alpha, \cos \alpha)^T$. Then $u(\cdot, \lambda)$ satisfies the boundary condition at $a$. Consider the function $F$ defined by
\[ F(\lambda) = (\cos \beta - \sin \beta)u(b, \lambda). \]
It follows that $\lambda$ is an eigenvalue of $T_{\alpha,\beta}$ precisely if it is a zero of $F$. It was shown in [10] that the entries of $u(b, \cdot)$ are entire. Hence $F$ is also entire and this implies that the eigenvalues are isolated and that they cannot accumulate at any finite point.

Because of Theorem 3.4 and since $-q_{22} \geq 0$, the function $\theta(\cdot, 0)$ can pass (or reach) an integer multiple of $\pi$ only by increasing strictly. In particular, $\theta(b, 0) > 0$. When $\theta(\cdot, 0)$ passes an integer multiple of $\pi$, then $u_1$ has a zero and any zero of $u_1$ is of this kind. It follows that $u_1(\cdot, 0)$ has precisely $k_r$ zeros in the open interval $(a, b)$. For $\lambda > \lambda' > 0$ we have that $p_2 = \lambda w - q \geq \lambda' w - q = p_1$. Now Theorem 3.2 shows that $\theta(b, \lambda) > \theta(b, \lambda')$, i.e., the function $\lambda \mapsto \theta(b, \lambda)$ is strictly increasing and continuous. If $\gamma < \beta$, then $\theta(b, \cdot)$ will not pass a multiple of $\pi$ before it reaches $k_r \pi + \beta$ at $\lambda_1$ the first positive eigenvalue. If $\gamma \geq \beta$, on the other hand, $\theta(b, \cdot)$ will pass a multiple of $\pi$ before it reaches $(k_r + 1)\pi + \beta$ at $\lambda_1$. The Prüfer angles at $b$ for the eigenfunctions of two consecutive eigenvalues will differ precisely by $\pi$ and will therefore cross a multiple of $\pi$ exactly once.

The proof for the last statement is almost the same, except that the Prüfer angle $\theta(\cdot, \mu)$ is now strictly decreasing as it passes a multiple of $\pi$. \qed
There may of course be no $\mu_0$ such that $\mu_0 w_{22} - q_{22} \leq 0$, for instance when $q_{22}(x) = -1$ and $w_{22}(x) = x$ on $(0, 1)$. In such a case the last part of Theorem 5.2 becomes void.

**Theorem 5.3.** Suppose $q$ and $w$ satisfy Hypothesis 5.1, $w_{22} = 0$, $-q_{22} \geq 0$, and supp $q_{22} = (a, b)$. Then the eigenvalues of $T_{\alpha, \beta}$ are bounded below. The eigenfunction associated with the smallest eigenvalue has no zeros in $(a, b)$ provided supp $w_{11} = (a, b)$.

**Proof.** A slight modification of the previous proof (whose notation and ideas we use freely) shows that the number of zeros of the eigenfunctions associated with consecutive non-positive eigenvalues will differ by exactly 1 with the eigenfunction associated with the smaller eigenvalue having fewer zeros. It follows that there can be at most $k_\gamma + 1$ non-positive eigenvalues if $\theta(b, 0) = k_\gamma \pi + \gamma$ with $\gamma \in (0, \pi]$. Hence there is a smallest eigenvalue. Denote the number of zeros in $(a, b)$ of the eigenfunction associated with the smallest eigenvalue by $k_\alpha$.

Since $\theta(\cdot, \lambda)$ can reach an integer multiple of $\pi$ only from below, it follows that $\theta(x, \lambda) \geq 0$ for all $x \in (a, b)$ and all $\lambda \in \mathbb{R}$. We also know from Theorem 3.2 that $\theta(x, \cdot)$ is strictly increasing. Hence the function $\theta_0(x) = \lim_{\lambda \to -\infty} \theta(x, \lambda)$ exists and is non-negative for every $x \in (a, b)$.

The Prüfer equation of $Ju' + qu = \lambda w u$ reads

$$\theta' = (\sin \theta, \cos \theta)(\lambda w - q) \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}.$$ 

Since $w_{22} = 0$ and $w \geq 0$ we have, in fact, $w = w_{11}(1 \ 0 \ 0)$. Thus

$$(\sin \theta)^2 w_{11} = \frac{1}{\lambda} \left[ \theta' + (\sin \theta, \cos \theta)q \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \right]$$

and, integrating over $(a, b)$,

$$\int_{(a, b)} (\sin \theta)^2 w_{11} = \frac{\theta(b, \lambda) - \theta(a, \lambda)}{\lambda} + \frac{1}{\lambda} \int_{(a, b)} (\sin \theta, \cos \theta)q \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}$$

$$\leq \frac{\alpha}{|\lambda|} + \frac{1}{|\lambda|} \int_{(a, b)} (|q_{11}| + |q_{22}| + |q_{12}|).$$

Applying the dominated convergence theorem to pass to the limit as $\lambda$ approaches $-\infty$ shows that $\int_{(a, b)} (\sin \theta_0)^2 w_{11} = 0$. Since supp $w_{11} = (a, b)$, we get that $\sin \theta_0 = 0$ in a dense subset $S$ of $(a, b)$. Thus $\theta_0(x)$ is an integer multiple of $\pi$ if $x \in S$.

If $\lambda < 0$ and $a < s < t < b$, we get from (8)

$$\theta(t, \lambda) - \theta(s, \lambda) \leq \int_{(s, t)} (\sin \theta)^2 - q_{22}(\cos \theta)^2 - q_{12} \sin 2\theta).$$

Using again the dominated convergence theorem gives $\theta_0(t) - \theta_0(s) \leq \mu((s, t))$ where $\mu = |q_{11}| + |q_{22}| + |q_{12}|$ is a finite, continuous measure on $(a, b)$. Hence there is a point $x_1 \in (a, b)$ such that $0 \leq \theta_0(x) \leq \theta_0(a) + \mu((a, x_1)) < \pi$ for $x \in (a, x_1)$.

Therefore $\theta_0 = 0$ on $S \cap (a, x_1)$. If $x_2$ satisfies $\mu((x_1, x_2)) < \pi$ we obtain similarly that $\theta_0 = 0$ on $S \cap (x_1, x_2)$. After finitely many steps we may conclude that $\theta_0 = 0$ on $S$. Now let $x$ be an any point in $(a, b)$ and $n \mapsto s_n$ a strictly increasing sequence in $S$ converging to $x$. Then $0 \leq \theta_0(x) \leq \theta_0(s_n) + \mu((s_n, x))$. But $\theta_0(s_n) = 0$ and $\mu((s_n, x))$ converges to 0 as $n$ tends to infinity showing that $\theta_0(x) = 0$. Therefore, finally, $\theta_0 = 0$ on $(a, b)$. 

It follows now that $\theta(b, \lambda_0) = \beta \leq \pi$ if $\lambda_0$ is the smallest eigenvalue of $T_{\alpha,\beta}$ and this means that the corresponding eigenfunction $u(\cdot, \lambda_0)$ has no zero in $(a, b)$. □

References


