THE INVERSE RESONANCE PROBLEM FOR HERMITE OPERATORS

B. MALCOLM BROWN, SERGUEI NABOKO, AND RUDI WEIKARD

Abstract. In this paper the inverse resonance problem for the Hermite operator is investigated. The Hermite operator $H = \mathfrak{a} + \mathfrak{a}^* + b$ with the creation operator $\mathfrak{a}$, the annihilation operator $\mathfrak{a}^*$, and a finitely supported multiplication operator b, is an unbounded operator on $\ell^2(\mathbb{N}_0)$ having finitely many eigenvalues and infinitely many resonances (except for $b = 0$ when there are no eigenvalues or resonances). It is shown that knowing the location of eigenvalues and resonances determines the potential b uniquely.

1. Introduction

In this paper we prove an inverse resonance theorem for the Hermite operator $H = \mathfrak{a} + \mathfrak{a}^* + b$, where $\mathfrak{a}$ is the creation operator acting on $\ell^2(\mathbb{N}_0)$, i.e., $(\mathfrak{a}y)(n) = \sqrt{n+1}y(n+1)$, $\mathfrak{a}^*$ is its adjoint (the annihilation operator), and b is a multiplication operator. As the simplest model in quantum optics the study of this Jacobi matrix operator is interesting in its own right. It is also highly interesting as an operator for which the difficulties it poses in regard to inverse spectral theory are in between those of the discrete and the continuous Schrödinger operator.

Classical results due to Gelfand and Levitan, Krein, and Marchenko (see, e.g., the monographs by Levitan [21], Marchenko [22], or Naimark [24]) show that, under certain circumstances, the potential q of the Schrödinger equation $-y''+qy = \lambda y$ on a half line is determined from either the spectral function or else from the scattering phase, the eigenvalues, and the norming constants of the associated eigenfunctions. In a practical sense such information is difficult to come by while the location of the eigenvalues and small resonances, central objects in quantum physics, are attainable in the laboratory. Therefore a fundamental question arises: what may be said about a potential, known to be contained in a particular class, when the location of the associated eigenvalues and small resonances is known. Analogous questions arise for difference equations and the associated Jacobi operators including the Hermite operator investigated here. This paper is a contribution to this circle of ideas.

For integrable potentials q one may show (see e.g., Coddington and Levinson [8] Theorem 8.1) that there is a unique solution of $-y'' + qy = z^2 y$ which behaves asymptotically like e^{izx} as long as z is in the upper half plane. This solution is called the Jost solution and we denote it by $\psi(\cdot, z)$. At least when q is superexponentially decaying at infinity the function $\psi(x, \cdot)$ may be extended to the complex plane as an entire function of growth order one for any fixed $x \in [0, \infty)$. The eigenvalues of

---

Date: 19. June 2008.

2000 *Mathematics Subject Classification.* 47B36, 39A70.

Key words and phrases. Inverse problems, eigenvalues and resonances, Hermite operator, perturbation determinant.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-0304280 and RFBR grant 06-01-00249.

the Schrödinger operator associated with q and a Dirichlet boundary condition at $x = 0$ are the squares of the zeros of the Jost function, that is the function $\psi(0, \cdot)$, in the upper half plane (the physical sheet). The squares of the zeros of the Jost function in the lower half plane (the unphysical sheet) are precisely the resonances mentioned above (there cannot be any zeros on the real z-line except possibly at $z = 0$). Hadamard's factorization theorem implies now that knowing the location of all eigenvalues and resonances determines the Jost function up to a factor e^{az+b}. But the parameters a and b are determined also since it is known that the Jost function tends to one as z tends to infinity along the positive imaginary axis for any potential under consideration. The Jost function, in turn, determines directly the scattering phase, the eigenvalues and the norming constants so that, with the aid of Marchenko's inverse scattering theorem (assuming q is real-valued and has an integrable first moment), we now have the result that the location of eigenvalues and resonances determine a superexponentially decaying potential. We call this an inverse resonance result.

One can raise analogous questions in other circumstances, for instance for complex potentials, perturbations of algebro-geometric potentials, or in the situation of a difference rather than a differential equation. An important quantity which is always available, even when the spectral function or the scattering phase are not defined, is the Titchmarsh-Weyl m-function. Therefore an approach was developed in [4] by which the m-function is constructed from the Jost function (and hence from its zeros) with the aid of Cauchy's residue theorem. The method is illustrated in [4] to prove an inverse resonance result for the case of compactly supported but complex-valued potentials of the continuous Schrödinger equation, in [6] for complex-valued perturbations of certain algebro-geometric potentials, and in [5] for the discrete Schrödinger equation[1]. The present paper deals with the Hermite operator $H = \mathfrak{a} + \mathfrak{a}^* + b$ and hence with the difference equation

$$\sqrt{n}y(n-1) + b_n y(n) + \sqrt{n+1}y(n+1) = \lambda y(n).$$

In all these cases a major ingredient is the Riemann surface which serves as the domain of the Jost function (or the Jost solution at a fixed value of the spatial variable). This Riemann surface is a twofold covering of the complex plane in which the spectral parameter λ varies. In the case of the continuous Schrödinger equation the surface is defined by the relationship $\lambda = z^2$ and is itself a complex plane. In the case of the discrete Schrödinger equation the surface is defined by the relationship $\lambda = z + 1/z$ and is a complex plane punctured at zero. In the case of the Hermite operator the surface consists of two disjoint copies of the complex plane rendering this case significantly different from the other ones.

Postulating certain abstract properties of the Jost solutions one shows then, using little more than Cauchy's residue theorem and Hadamard's factorization theorem, that the location of the resonances determines the m-function, which, as is well known, determines the potential. Theorem 3.1 below gives a prototypical example of this approach. In applications one then needs to establish the postulated properties of the Jost solutions for a given class of potentials. Most of these properties are fairly easily confirmed. However, one needs a lower bound to the modulus of the Jost function on a sequence of contours. This is no problem on the physical sheet where only finitely many zeros are located. But on the unphysical sheet the Jost function may have infinitely many zeros at the resonances. For the

[1]We also like to point the reader to related work by Korotyaev [18] and [19].

discrete Schrödinger equation a classical theorem of Wiman states that the minimum modulus of an entire function of growth order less than $1/2$ is unbounded and this guarantees that the Jost solution is bounded away from zero on a sequence of circles. For the continuous Schrödinger equation and the present case of the Hermite operator the Jost functions have growth order 1 and 2 respectively. Therefore detailed information on the asymptotic behavior of the Jost function between the resonances is necessary. In the present case the distance of these resonances goes to zero as they become larger in absolute value making the analysis of the modulus of the Jost function a very sensitive issue.

For this reason we restrict our attention in this paper to the Hermite operator with a finitely supported complex-valued "potential" b, a situation which is set in between the discrete and the continuous Schrödinger equation. Like the discrete Schrödinger equation the operator under investigation here is a second order difference operator, however, the interplay between kinetic and potential energy is much more like the continuous Schrödinger equation. In support of this statement we mention here the following facts. (1) The kinetic energy $T = \mathfrak{a} + \mathfrak{a}^*$ is unbounded. (2) Adding a potential to T, however small and even if finitely supported, creates immediately infinitely many resonances, in contrast to the case of the discrete Schrödinger equation where a finitely supported potential has only finitely many resonances. (3) As shown in Lemma 4.4, the asymptotic behavior of the resonances depends only on the location of the last site occupied by the potential and its value there, reflecting strongly the situation for the continuous Schrödinger equation[2]. In particular, in both cases, the analysis of a decaying but not finitely supported potential would be completely different from the analysis needed for finite or compact support.

However, there are also differences to the case of the continuous Schrödinger equation. We mentioned already the fact, that the underlying Riemann surface consists of two disconnected sheets. Resonances and eigenvalues may occur on either sheet and the distance between resonances shrinks as they grow larger where in the case of the continuous Schrödinger equation with a compactly supported potential resonances have asymptotically constant distance. Another new feature is played by a certain entire function M_0 of growth order two which governs the asymptotic behavior of the Jost solution and hence the Jost function. In fact, the Jost function has the form $\rho(\lambda) + \sigma M_0(\sigma\lambda)\tau(\lambda)$ where ρ and τ are polynomials and where σ assumes the values ± 1 signifying the two disjoint sheets of the Riemann surface.

If N denotes the index of the last site occupied, the polynomials ρ and τ have degree $2N - 1$ and $2N$, respectively, and their coefficients are polynomials of b_0, ..., b_N. Provided that N is known one might be tempted to recover first the $4N + 1$ coefficients of ρ and τ from the location of finitely many resonances and then the b_j from these coefficients. While the first step is linear it is not clear how to establish that the problem is nonsingular. The second step is highly nonlinear and to guarantee uniqueness of a solution appears to be very difficult.

[2]More precisely, in the Hermite case the asymptotic behavior of the resonances is given by the solutions of the equation $i\sigma\sqrt{2\pi}\exp(-z^2/2) = N! z^{-2N}/b_N$ where σ assumes the values ± 1 signifying the two disjoint sheets of the Riemann surface, N is the index of the last occupied site and b_N is the value of the potential there. In the case of a compactly supported perturbation of the continuous Schrödinger equation the asymptotic behavior of the resonances is given by the solutions of $\exp(2izR) = 4z^2/q(R)$ when R denotes the right endpoint of the support of the potential q.

The major result of this paper is a uniqueness theorem for the inverse resonance problem. Recall, though, that from the physical point of view one expects large resonances to be insignificant. Therefore it would be very interesting to establish a stability result stating that two potentials having the same eigenvalues and the same small resonances should be very close in some appropriate norm. Such a result was obtained in [23] for the case of a discrete Schrödinger equation but there is no analogue for the continuous Schrödinger equation or for the present case of the Hermite equation. We expect the Hermite case to be more tractable and plan to address it in the future.

Next we will describe the result obtained in this paper in more precise terminology. As usual the symbol $\mathbb{N}$ stands for the set of natural numbers. We will also use the abbreviations $\mathbb{N}_0$ and $\mathbb{M}$ for the sets $\mathbb{N}\cup\{0\}$ and $\mathbb{N}\cup\{0,-1\}$, respectively. The set of complex-valued functions defined on a set S is denoted by $\mathbb{C}^S$. The object of investigation in this paper is the operator $h:\mathbb{C}^{\mathbb{M}}\to\mathbb{C}^{\mathbb{N}_0}$ given by

$$(hy)(n)=a_{n-1}y(n-1)+b_ny(n)+a_ny(n+1),\quad n\in\mathbb{N}_0$$

where $a_{-1}=1$, $a_n=\sqrt{n+1}$ when $n\in\mathbb{N}_0$, and where b is a complex-valued sequence with semi-bounded imaginary part. The operator h is the Hermite "difference expression". Since the set $\{y\in\ell^2(\mathbb{M}):y(-1)=0\}$ is isomorphic to $\ell^2(\mathbb{N}_0)$ and since the annihilation operator $\mathfrak{a}^*$ acts on $y\in\ell^2(\mathbb{N}_0)$ as if there were an element y_{-1} which is zero, we may think of $H=\mathfrak{a}+\mathfrak{a}^*+b$ as the operator in $\ell^2(\mathbb{N}_0)$ associated with h and a Dirichlet boundary condition on the left. Accordingly one might want to call H the Dirichlet-Hermite operator.

Due to the fact that the sequence a is not summable there is, for any $z\in\mathbb{C}$, at most one linearly independent solution $\psi(\cdot,z)$ of the difference equation

$$(hy)(n)=zy(n),\quad n\in\mathbb{N}_0 \tag{1}$$

which is square summable, see Carleman [7]. Under suitable conditions on the sequence b the functions $\psi(n,\cdot)$ may be extended to entire functions and we may assume without loss of generality that they do not have any nonconstant common factors. A zero z_0 of $\psi(-1,\cdot)$ is then, by definition, either an eigenvalue or a resonance depending on whether $\psi(\cdot,z_0)$ is square summable or not. We emphasize that there could be real resonances, embedded in the continuous spectrum. In fact, in the situation at hand where the potential is finitely supported, the following statements hold. (1) The real line is covered by absolutely continuous spectrum of local multiplicity one. This follows from the asymptotic behavior of solutions of the equation $hy=zy$ for real z, see [16]. (2) The asymptotic behavior of the solutions shows also that eigenvalues can never be real. (3) If b is real so that H is self-adjoint there are no real resonances. This is proved below in Appendix A, Corollary A.2. (4) It is easy to construct explicit examples of complex potentials for which there are real resonances. However, their number is finite since the asymptotic behavior of the Jost solution prevents the existence of zeros outside a compact interval.

Our main theorem is the following one.

Theorem 1.1. *Suppose $b\in\mathbb{C}^{\mathbb{N}_0}$ is finitely supported. Then the location of all eigenvalues and resonances of the associated Hermite operator H and their multiplicities determine b uniquely.*

This theorem is a corollary of Theorems 3.1 and 4.1 below and its proof can be found after the proof of Theorem 4.1. Although we consider perturbations of the main diagonal only a similar approach appears to work for general finitely supported three-diagonal perturbations.

We also show that the Jost function may be expressed as a Fredholm determinant replicating a fact for both the continuous and discrete Schrödinger equation. This result may be found in Appendix B.

The Hermite operator is intimately connected to the weight $w(z) = \exp(-z^2)$ which produces the coefficients a_n to grow like $n^{1/2}$. We expect that our approach will work in principle also for the more general case of weights of the form $w(z) = \exp(-|z|^\alpha)$, $0 < \alpha < 1$, whence $a_n = O(n^{1/\alpha})$. The associated Jacobi difference equations have drawn much attention recently. In particular, detailed investigations of the asymptotic behavior of their solutions was carried out by Dombrowski [9], Dombrowski and Pedersen [11], [12], and Dombrowski, Janas, Moszyński, and Pedersen [10]. Related work was also done by Beckermann [2], Beckermann and Castro Smirnova [3], and Geronimo, Bruno, and Van Assche [13]. Finally we mention the monograph [20] by Levin and Lubinsky.

The paper is organized as follows. In Section 2 we discuss the case of an unperturbed Hermite operator, where $b = 0$. In Section 3 we state and prove an abstract version of the inverse resonance theorem, starting from certain hypotheses on the Jost solutions. In Section 4 we show that finitely supported potentials b have Jost solutions with the required properties to apply the abstract theorem. Appendix A deals with the question of common zeros of the two polynomials appearing in the Jost function. In Appendix B we study the perturbation determinant. Finally, in Appendix C we list the Jost functions for the three most basic examples.

We are grateful to M. Ismail for useful information about classical and recent result on orthogonal polynomials and to an unknown referee for suggestions to improve the presentation.

2. The free case

In this section we consider equation (1) when all b_n are equal to zero, i.e., we consider the equation

$$a_{n-1}y(n-1,z) + a_n y(n+1,z) = zy(n,z) \tag{2}$$

for $n \in \mathbb{N}_0$. This we call the free case. In the following we remind the reader about some classical facts concerning this case.

Define $w(z) = \exp(-z^2/2)/\sqrt{2\pi}$ and $p_0(n,z) = H_n(z/\sqrt{2})/\sqrt{2^n n!}$ where the H_n are Hermite polynomials. Then the $p_0(n,\cdot)$ are orthonormal polynomials in $L^2(w)$ (i.e., $\int_{-\infty}^{\infty} p_0(n,z)p_0(m,z)w(z)dz = \delta_{n,m}$ holds for all $n, m \in \mathbb{N}_0$). They satisfy equations (2) for $n \in \mathbb{N}$ (but not for $n = 0$), i.e., they satisfy (1) when all $b_n = 0$ and when $n \geq 1$. Since $1/a_n$ is not summable we have the limit point case at infinity.

For $n \in \mathbb{N}_0$ we let

$$q_0(n,z) = \int_{-\infty}^{\infty} \frac{p_0(n,t) - p_0(n,z)}{t - z} w(t)dt.$$

These functions also satisfy equations (2) for $n \in \mathbb{N}$ since $\int_{-\infty}^{\infty} p_0(n,t)w(t)dt = \delta_{0,n} = 0$.

Given $F \in L^2(w)$ define $x_k = \int_{-\infty}^{\infty} F(t)p_0(k,t)w(t)dt$. Bessel's inequality states

$$\sum_{k=0}^{\infty} |x_k|^2 \leq \int_{-\infty}^{\infty} |F(t)|^2 w(t)dt.$$

When $\Im(z) \neq 0$ choose $F(t) = 1/(t-z)$ (which is in $L^2(w)$) and denote the corresponding number x_k by $\psi_0(k,z)$. Thus $k \mapsto \psi_0(k,z)$ is square summable. Moreover, defining

$$m_0(z) = \int_{-\infty}^{\infty} \frac{w(t)}{t-z} dt,$$

we obtain

$$\begin{aligned}\psi_0(k,z) &= \int_{-\infty}^{\infty} \frac{p_0(k,t)}{t-z} w(t) dt \\ &= \int_{-\infty}^{\infty} \frac{p_0(k,t) - p_0(k,z)}{t-z} w(t) dt + p_0(k,z) \int_{-\infty}^{\infty} \frac{w(t)}{t-z} dt \\ &= q_0(k,z) + m_0(z) p_0(k,z).\end{aligned}$$

Thus m_0 is the Titchmarsh-Weyl m-function associated with the pair (q_0, p_0). We also note that $m_0(-z) = -m_0(z)$ when $\Im(z) \neq 0$.

Define

$$M_0(z) = i\sqrt{\frac{\pi}{2}} \exp(-z^2/2) \operatorname{erfc}(-iz/\sqrt{2})$$

where erfc denotes the complementary error function defined by

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt.$$

It follows then from formulas 7.1.3 and 7.1.4 of Abramowitz and Stegun [1] that $m_0(z) = M_0(z)$ if $\Im(z) > 0$ and that $m_0(z) = -M_0(-z)$ if $\Im(z) < 0$. By Formula 7.1.23 of Abramowitz and Stegun we have

$$M_0(z) \sim -\frac{1}{z} \left(1 + \sum_{k=1}^{\infty} \frac{1 \times 3 \times \cdots \times (2k-1)}{z^{2k}} \right) \tag{3}$$

as long as $|\arg(-iz)| < 3\pi/4$, i.e., $-\pi/4 < \arg(z) < 5\pi/4$. Since $\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$ we also have

$$M_0(z) \sim i\sqrt{2\pi} \exp(-z^2/2) - \frac{1}{z} \left(1 + \sum_{k=1}^{\infty} \frac{1 \times 3 \times \cdots \times (2k-1)}{z^{2k}} \right) \tag{4}$$

as long as $|\arg(iz)| < 3\pi/4$, i.e., $-5\pi/4 < \arg(z) < \pi/4$. We also record for future use that

$$M_0(z) + M_0(-z) = i\sqrt{2\pi} \exp(-z^2/2)$$

never vanishes. Analytic continuation of m_0 across the real axis from both the upper and the lower half plane gives a function defined on the Riemann surface $\mathcal{R}_0 = \{(z,\sigma) : z \in \mathbb{C}, \sigma = \pm 1\}$ which consists of two completely disconnected copies of the complex plane. This function is denoted by $\hat{M}_0$ and we have explicitly

$$\hat{M}_0 : \mathcal{R}_0 \to \mathbb{C} : (z,\sigma) \mapsto \sigma M_0(\sigma z).$$

We also define

$$\Psi_0(n,z,\sigma) = q_0(n,z) + \hat{M}_0(z,\sigma) p_0(n,z)$$

and emphasize that $\Psi_0(n,z,\sigma) = \psi_0(n,z)$ provided that $\sigma \Im(z) > 0$.

We have constructed solutions for equation (2) when $n \in \mathbb{N}$. We define $p_0(-1,z)$, $q_0(-1,z)$, $\psi_0(-1,z)$, and $\Psi_0(-1,z,\sigma)$ so that equation (2) holds also for $n = 0$. This gives $p_0(-1,z) = 0$, $q_0(-1,z) = -1$ and $\psi_0(-1,z) = \Psi_0(-1,z,\sigma) = -1$. The function $\Psi_0 : \mathbb{M} \times \mathcal{R}_0 \to \mathbb{C}$ is called the Jost solution associated with the difference equation (2). Note that $\Psi_0(\cdot, z, \sigma)$ is square summable if $\sigma\Im(z) > 0$. Also, defining the (modified) Wronskian

$$[f, g](n) = a_n(f(n)g(n+1) - g(n)f(n+1)),$$

one computes immediately that $[\Psi_0(\cdot, z, +1), \Psi_0(\cdot, z, -1)](n) = M_0(z) + M_0(-z) = i\sqrt{2\pi}\exp(-z^2/2)$.

The function $\Psi_0(-1, \cdot, \cdot) = -1$ defined on $\mathcal{R}_0$ is called the Jost function of the problem. Since $\Psi_0(-1, \cdot, \cdot)$ has no zeros it follows that the Hermite operator has no eigenvalues or resonances in absence of a potential.

2.1. **Asymptotic properties of $\psi_0(n, \cdot)$ near infinity.**

Lemma 2.1. *For every $n \in \mathbb{N}_0$ there is a constant c_n such that*

$$\left|\psi_0(n, z) + \frac{\sqrt{n!}}{z^{n+1}}\right| \le \frac{c_n}{|z|^{n+2}|\Im(z)|}$$

for any complex number z with $\Im(z) \neq 0$.

Proof. Since $\psi_0(n, z) = \int_{-\infty}^{\infty} p_0(n, t)w(t)/(t - z)dt$ and

$$\frac{1}{t-z} = \frac{1}{t-z}\left(\frac{t}{z}\right)^{n+2} - \frac{1}{z}\sum_{\ell=0}^{n+1}\left(\frac{t}{z}\right)^{\ell}$$

we compute $\int_{-\infty}^{\infty} p_0(n, t)t^\ell w(t)dt$. Because of the orthogonality properties of the Hermite polynomials this is zero if $\ell < n$. Using formulas 22.13.19 and 22.4.6 of Abramowitz and Stegun we find that $\int_{-\infty}^{\infty} p_0(n, t)t^n w(t)dt = \sqrt{n!}$. Finally, we also have that $\int_{-\infty}^{\infty} p_0(n, t)t^{n+1}w(t)dt = 0$ since all Hermite polynomials are either even or odd. Hence

$$\int_{-\infty}^{\infty} \frac{p_0(n, t)}{t - z} w(t)dt = -\frac{\sqrt{n!}}{z^{n+1}} + \int_{-\infty}^{\infty} \frac{p_0(n, t)t^{n+2}}{(t - z)z^{n+2}} w(t)dt.$$

The claim follows if we define $c_n = \int_{-\infty}^{\infty} |p_0(n, t)t^{n+2}|w(t)dt$. □

3. The inverse resonance problem

Definition 1. Let $\mathcal{C}$ be the family of Hermite operators H for which the complex sequence b has semi-bounded imaginary part and for which there exists a function $\Psi : \mathbb{M} \times \mathcal{R}_0 \to \mathbb{C}$ with the following properties:

(1) For every complex number z the functions $\Psi(\cdot, z, +1)$ and $\Psi(\cdot, z, -1)$ are nontrivial solutions of the equation $hy = zy$.
(2) The (modified) Wronskian of $\Psi(\cdot, z, +1)$ and $\Psi(\cdot, z, -1)$ satisfies

$$[\Psi(\cdot, z, +1), \Psi(\cdot, z, -1)](n) = M_0(z) + M_0(-z) = i\sqrt{2\pi}\exp(-z^2/2)$$

for all $z \in \mathbb{C}$.
(3) $\Psi(\cdot, z, +1)$ is square summable for all z in some nonempty open subset of the upper half of the z-plane and $\Psi(\cdot, z, -1)$ is square summable for all z in some nonempty open subset of the lower half of the z-plane.
(4) $\Psi(-1, \cdot, \pm1)$ and $\Psi(0, \cdot, \pm1)$ are entire functions of finite growth order.
(5) $\Psi(-1, z, \sigma)$ tends to -1 as σz tends to infinity in the upper half plane.

(6) There exist two sequences of closed contours $n \mapsto \gamma_{\sigma,n} : [0, 2\pi] \to \mathcal{R}_0$, one on each sheet, with the following properties: (i) $|\gamma_{\sigma,n}(t)| \geq r_n$ where r_n is a sequence of positive numbers tending to infinity, (ii) $\int_0^{2\pi} |\gamma_{\sigma,n}(t)| dt = O(r_n)$ as n tends to infinity, and (iii) there is a positive constant A such that

$$|\Psi(-1, z, \sigma)| \geq A r_n^{-p} \begin{cases} |\exp(-z^2/2)| & \text{if } \sigma \Im(z) \leq 0, \\ 1 & \text{if } \sigma \Im(z) \geq 0. \end{cases}$$

for $z = \gamma_{\sigma,n}(t)$, $t \in [0, 2\pi]$, $\sigma = \pm 1$ and $n \in \mathbb{N}$.

Theorem 3.1. *Assume that the Hermite operator H, associated with the complex sequence $b \in \mathbb{C}^{\mathbb{N}_0}$ with semi-bounded imaginary part, is in $\mathcal{C}$ and let Ψ be the function from Definition 1 establishing that fact. Then the zeros of $\Psi(-1, \cdot, \cdot)$ and their multiplicities determine uniquely the quantities b_n for all $n \in \mathbb{N}_0$.*

Proof. It is well-known that the Titchmarsh-Weyl m-function determines a real potential b uniquely. This remains true in the complex-valued case, at least if b satisfies a certain assumption concerning the distribution of its values, see [28]. This assumption on b is satisfied when b is a bounded sequence.

The m-function is defined[3] as $-\psi(0, z)/\psi(-1, z)$ when $\psi(\cdot, z)$ is a square summable solution of the difference equation $hy = zy$. By Properties (1) and (3) we have that

$$m(z) = -\frac{\Psi(0, z, \sigma)}{\Psi(-1, z, \sigma)}$$

if (z, σ) is in one of the sets mentioned by Property (3). By Property (4) the function m may be analytically continued to a function $\hat{M}$ defined on the Riemann surface $\mathcal{R}_0$. Thus we need to show that $\hat{M}$ is uniquely determined by the zeros of $\Psi(-1, \cdot, \cdot)$. We will use the notation $M_+ = \hat{M}(\cdot, +1)$ and $M_- = \hat{M}(\cdot, -1)$.

Since, according to Property (4), the function $\Psi(-1, \cdot, +1)$ is entire and of finite growth we may apply Hadamard's factorization theorem. Let $z_1, z_2, ...$ be the pairwise distinct nonzero zeros of $\Psi(-1, \cdot, +1)$ labeled such that $|z_1| \leq |z_2| \leq ...$ and let $n_1, n_2, ...$ be their respective multiplicities. Also, let $z_0 = 0$ and denote by n_0 the multiplicity of zero as a zero of $\Psi(-1, \cdot, +1)$ (of course, n_0 may well be zero). Then

$$\Psi(-1, z, +1) = \exp(g(z)) z^{n_0} \prod_{j=1}^{\infty} E_\rho(z/z_j)^{n_j}$$

where ρ is to be chosen to be the smallest integer such that $\sum_{j=1}^{\infty} n_j |z_j|^{-\rho+1}$ is finite, where g is a polynomial, and where E_ρ is a canonical factor, i.e.,

$$E_\rho(z) = (1 - z) \exp(z + \frac{z^2}{2} + ... + \frac{z^\rho}{\rho}).$$

From the asymptotic behavior of $\Psi(-1, \cdot, +1)$, known from Property (5), we find that $g(z) = i\pi + \log(1 + o(1)) - n_0 \log z - \sum_{j=1}^{\infty} n_j \log E_\rho(z/z_j)$ as z tends to infinity in the upper half plane, i.e., that g is known up to terms which vanish at infinity. Since g is a polynomial this information fixes it uniquely. Thus the function $\Psi(-1, \cdot, +1)$ is completely determined by its zeros. In the same way we can determine $\Psi(-1, \cdot, -1)$ from its zeros.

---

[3] This definition is different from the one in [28]. However, the two notions are easily translated into one another.

Next, for any $j \in \mathbb{N}_0$, we take $r \le n_j - 1$ derivatives of the Wronskian relationship given by Property (2) and evaluate at z_j to obtain

$$\sum_{\ell=0}^{r} \binom{r}{\ell} \Psi^{(\ell)}(0, z_j, +1)\Psi^{(r-\ell)}(-1, z_j, -1) = -W^{(r)}(z_j)$$

where $W(z) = i\sqrt{2\pi}\exp(-z^2/2)$. This allows to determine recursively the numbers $\Psi^{(r)}(0, z_j, +1)$ for $r = 0, ..., n_j - 1$ since $\Psi(-1, z_j, -1)$ is necessarily different from zero. Analogously we find the appropriate number of derivatives of $\Psi(0, \cdot, -1)$ at the zeros of $\Psi(-1, \cdot, -1)$.

We now compute the residues of $h_z M_+$ at the zeros of $\Psi(-1, \cdot, +1)$ where $M_+(\mu) = -\Psi(0, \mu, +1)/\Psi(-1, \mu, +1)$ and $h_z(\mu) = (z/\mu)^{p+1}/(z - \mu)$ where p will be determined shortly. To simplify notation we introduce

$$f_{z,j}(\mu) = h_z(\mu)\frac{(\mu - z_j)^{n_j}}{\Psi(-1, \mu, +1)}$$

for $j \in \mathbb{N}$ and note that these function are known except for the common multiplicative constant $1/C_+$. Thus we may compute the numbers

$$\begin{aligned}\operatorname{res}_{z_j}(h_z M_+) &= -\frac{1}{(n_j - 1)!}(\Psi(0, \cdot, +1)f_{z,j})^{(n_j-1)}(z_j)\\ &= -\frac{1}{(n_j - 1)!}\sum_{r=0}^{n_j-1}\binom{n_j - 1}{r}\Psi^{(r)}(0, z_j, +1)f_{z,j}^{(n_j-1-r)}(z_j).\end{aligned}$$

Also

$$\operatorname{res}_0(h_z M_+) = -\sum_{k=0}^{p+n_0}\frac{(f_{z,0}\Psi(0, \cdot, +1))^{(k)}(0)}{k!}z^{k-n_0}$$

where $f_{z,0}(\mu) = \mu^{n_0}/\Psi(-1, \cdot, +1)$.

Let B_n be the open domain enclosed by the contour $\gamma_{+1,n}$. By the residue theorem,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\gamma_n} h_z(\mu)M_+(\mu)d\mu\\ &\quad = -M_+(z) + \sum_{k=0}^{p+n_0}\frac{(f_{z,0}\Psi(0, \cdot, +1))^{(k)}(0)}{k!}z^{k-n_0} + \sum_{z_j\in B_n}\operatorname{res}_{z_j}(h_z M_+) \qquad (5)\end{aligned}$$

if $z \in B_n$, and if z is none of the poles of M_+.

In the upper half plane M_+ equals the Titchmarsh-Weyl m-function. Its asymptotic behavior was studied in [28] showing that $M_+(z)$ is of order $1/z$ as z tends to infinity there. We now want to investigate the asymptotic behavior of M_+ in the lower half plane. Property (2) implies

$$M_+(z) = M_-(z) + \frac{i\sqrt{2\pi}e^{-z^2/2}}{\Psi(-1, z, +1)\Psi(-1, z, -1)}.$$

Using Property (5) and Property (6) we may estimate $|\Psi(-1, z, -1)| \ge 1/2$, and $|\Psi(-1, z, +1)| \ge A|\exp(-z^2/2)|r_n^{-p}$ as long as z is located on the contour $\gamma_{+1,n}$ given by Property (6) (recall that z is in the lower half plane). Also $M_-(z) = O(1/z)$ as it equals the Titchmarsh-Weyl m-function in the lower half plane. Thus M_+ grows at most like r_n^p on those contours and the integral on the left hand side of equation (5) tends to zero as n tends to infinity, proving firstly the convergence of the series and secondly that

$$M_+(z) = \sum_{k=0}^{p+n_0} \frac{(f_{z,0}\Psi(0,\cdot,+1))^{(k)}(0)}{k!} z^{k-n_0} + \sum_{j=1}^{\infty} \mathrm{res}_{z_j}(h_z M_+).$$

Our previous considerations show that we know the series on the right of this equation as well as the expressions $(f_{z,0}\Psi(0,\cdot,+1))^{(k)}(0)$ as long as $k \le n_0 - 1$. For $k \ge n_0$ these quantities are also known, since, according to Property (5) $M_+(z)$ tends to zero as z tends to infinity in the upper half plane. Thus, the function M_+ and, in an analogous fashion, the function M_-, are uniquely determined from the zeros of $\Psi(-1,\cdot,\cdot)$. □

4. Finitely supported diagonal perturbations

Now suppose we have a finite diagonal perturbation, i.e., we are looking for solutions of the equations (1) where $b_n = 0$ for $n > N \ge 0$ and $b_N \ne 0$. We emphasize that this allows for perturbations of the first $N+1$ sites, in particular, $N = 0$ does not indicate the free case.

We define a function $\Psi : \mathbb{M} \times \mathcal{R}_0 \to \mathbb{C}$. For $n \ge N$ we set $\Psi(n,z,\sigma) = q_0(n,z) + \sigma M_0(\sigma z) p_0(n,z)$. For $-1 \le n \le N-1$ we define $\Psi(n,\cdot,\cdot)$ by the recurrence relation (1) and obtain

$$\Psi(n,z,\sigma) = \rho_n(z) + \sigma M_0(\sigma z)\tau_n(z)$$

for appropriate polynomials ρ_n and τ_n. In fact, for $N > 0$, we find that ρ_n and τ_n are polynomials of degree $2N-2-n$ and $2N-1-n$, respectively, and that both have leading coefficient $-b_N A_n/N!$ where $A_{-1} = 1$ and $A_n = \sqrt{n!}$ when $n \ge 0$. This latter statement makes use of the fact that both $q_0(k,\cdot)$ and $p_0(k,\cdot)$ have leading coefficient $1/\sqrt{k!}$. If $N = 0$ then $\rho_{-1} = -1$ and $\tau_{-1} = -b_0$.

Theorem 4.1. *Suppose $b \in \mathbb{C}^{\mathbb{N}_0}$ and that there exists an $N \ge 0$ such that $b_N \ne 0$ and $b_n = 0$ for $n > N \ge 0$. Then the associated Hermite operator is in Class $\mathcal{C}$.*

Proof. By definition Ψ satisfies Property (1) of Definition 1. The sequence $\Psi(\cdot,z,\sigma)$ is the Jost solution of $hy = zy$ when $\sigma\Im(z) > 0$.

The Wronskian $[\Psi(\cdot,z,+1),\Psi(\cdot,z,-1)](n)$ is independent of n. For $n \ge N$ we are in the free case so that its value is always equal to $M_0(z) + M_0(-z) = i\sqrt{2\pi}\exp(-z^2/2)$. This is Property (2) of Definition 1.

It is also obvious that $\Psi(\cdot,z,+1)$ is square summable for any z in the upper half plane and that $\Psi(\cdot,z,-1)$ is square summable for any z in the lower half plane, i.e., Ψ satisfies Property (3) of Definition 1.

The error function is an entire function of growth order 2. The same is therefore true for the functions M_0 and $\Psi(n,\cdot,\sigma)$ for either choice of σ and all $n \in \mathbb{M}$. Thus Property (4) of Definition 1 holds.

Lemma 4.2 for $n = -1$ shows that Property (5) is satisfied, too.

Finally, we need a lower bound on certain contours $\gamma_{\sigma,n}$ which pass between the zeros of $\Psi(-1,\cdot,\sigma)$. We consider only the case $\sigma = +1$ since the other one is treated similarly. Circles would be ideal but a circle passing between two zeros near the ray with argument $-\pi/4$ may come too close to a zero on the ray with argument $-3\pi/4$. Instead we choose two semicircles of possibly different radii in the right and left half plane and join these along the imaginary axis. In the upper half plane $\Psi(-1,\cdot,+1)$ approaches the value -1 so that its modulus is eventually bounded below by 1/2. In the lower half plane Lemma 4.4 gives the desired estimate. Thus Property (6) is satisfied with $A = |b_N|/N!$ and $p = 0$. $\square$

We are now in the position to prove our main theorem.

Proof of Theorem 1.1. The potential b is identically equal to zero if and only if there are no eigenvalues or resonances. Otherwise Theorem 4.1 shows that H is in the class $\mathcal{C}$ defined by Definition 1. In that case Theorem 3.1 shows that the zeros of $\Psi(-1,\cdot,\cdot)$, i.e., the eigenvalues and resonances determine b uniquely. $\square$

4.1. **Asymptotic properties of $\Psi(n,\cdot,\sigma)$ near infinity.** We first investigate the asymptotics of $\Psi(n,\cdot,\sigma)$ when σz is in the upper half plane.

Lemma 4.2. *There exist positive constants C_N and K_N such that*

$$|\Psi(n,z,\sigma) - \Psi_0(n,z,\sigma)| \le \frac{2C_N K_N \|b\|_1}{|z|^{n+2}}$$

for all $n \in \mathbb{M}$ and all complex numbers z satisfying $\sigma\Im(z) \ge 1$ and $|z| \ge 2K_N\|b\|_1$.

Proof. For $\ell \in \mathbb{N}_0$ define recursively

$$\psi_\ell(n,z) = \sum_{k=n+1}^{N} (p_0(k,z)q_0(n,z) - q_0(k,z)p_0(n,z)) b_k \psi_{\ell-1}(k,z)$$

and note that, according to Lemma 2.1 and since $\psi_0(-1,z) = -1$,

$$|\psi_0(n,z)| \le \frac{C_N}{|z|^{n+1}}$$

for a suitable constant C_N as long as $-1 \le n \le N$ and $|\Im(z)| \ge 1$.

Next note that, since $\psi_0 = q_0 + m_0 p_0$, we have

$$p_0(k,z)q_0(n,z) - q_0(k,z)p_0(n,z) = p_0(k,z)\psi_0(n,z) - \psi_0(k,z)p_0(n,z).$$

Therefore

$$|p_0(k,z)q_0(n,z) - q_0(k,z)p_0(n,z)| \le K_N|z|^{k-n-1}$$

for a suitable constant K_N provided that $-1 \le n \le k \le N$ and $|\Im(z)| \ge 1$.

Induction shows then that

$$|\psi_\ell(n,z)| \le \frac{C_N}{|z|^{n+1}} \left(\frac{K_N\|b\|_1}{|z|}\right)^\ell.$$

Since $|z| \ge 2K_N\|b\|_1$ we find that

$$\sum_{\ell=1}^{\infty} |\psi_\ell(n,z)| \le \frac{2C_N K_N \|b\|_1}{|z|^{n+2}}.$$

From standard arguments we obtain that $\Psi(\cdot,\cdot,\sigma) = \psi_0 + \sum_{\ell=1}^\infty \psi_\ell$ if $\sigma\Im(z) > 0$. This completes the proof. $\square$

Lemma 4.3. *The asymptotic behavior of the rational function ρ_{-1}/τ_{-1} near infinity is given by*

$$\frac{\rho_{-1}(z)}{\tau_{-1}(z)} = \frac{1}{z}\left(1 + \sum_{k=1}^{N-1} \frac{1 \times 3 \times \cdots \times (2k-1)}{z^{2k}}\right) + \frac{N!}{b_N z^{2N}} + O(|z|^{-2N-1}).$$

Proof. Since
$$\frac{\rho_{-1}(z)}{\tau_{-1}(z)} = -\sigma M_0(\sigma z) + \frac{\Psi(-1,z,\sigma)}{\tau_{-1}(z)}$$
the result follows from the Lemma 4.2 and from the asymptotic behavior of M_0 given in equation (3) for z in the upper half plane. Since ρ_{-1}/τ_{-1} is rational this behavior persists in any direction. □

Lemma 4.4. *The function $\Psi(-1,\cdot,+1)$ has infinitely many zeros in the lower half plane. Denoting the principal branch of the logarithm by* log *their location is asymptotically given by*
$$z_n^2 = 4n\pi i + 2N\log(2n\pi i) - 2\log(C) + O(n^{-1/2})$$
where z_n is to be chosen in the third quadrant for positive n and in the fourth quadrant for negative n and where $C = iN!2^{-N}/(\sqrt{2\pi}b_N)$.

However, if n is sufficiently large, then
$$|\Psi(-1,z,+1)| \geq \frac{|b_N|}{N!}|z^{2N}\exp(-z^2/2)|$$
for all z located on certain quarter circles in the third and fourth quadrant passing between the z_n and z_{n+1}. This estimate holds also for z on the negative imaginary axis provided that $|z|$ is sufficiently large.

Analogous statements hold for $\Psi(-1,\cdot,-1)$.

Proof. If $\sigma\Im(z) < 0$ we get from Lemma 4.3 and equation (4)
$$\frac{\Psi(-1,z,\sigma)}{\tau_{-1}(z)} = \frac{\rho_{-1}(z)}{\tau_{-1}(z)} + \sigma M_0(\sigma z) = i\sigma\sqrt{2\pi}\exp(-z^2/2) + \frac{N!}{b_N z^{2N}} + \tilde{g}(\sigma z)$$
where $\tilde{g}(z) = O(|z|^{-2N-1})$. Define $w = z^2/2$, $C = i\sigma N!2^{-N}/(\sqrt{2\pi}b_N)$ (this is consistent with the definition made above since $\sigma = 1$ there), and
$$F(w) = \mathrm{e}^{-w} - Cw^{-N} + g(w) = \mathrm{e}^{-w} - Cw^{-N}(1 - h(w))$$
where $g(w) = \tilde{g}(\sigma\sqrt{2w})$ and $h(w) = g(w)w^N/C$. Note that g depends both on σ and on which root is chosen in computing $z = \sqrt{2w}$ but that both of these are unimportant since g is just an error term. Also, $h(w) = O(|w|^{-1/2})$.

All large zeros of $\Psi(-1,\cdot,\sigma)$ correspond to zeros of F since τ_{-1} is a polynomial (in fact, since, as we show in Appendix A, ρ_{-1} and τ_{-1} have no common zeros, all zeros of $\Psi(-1,\cdot,\sigma)$ correspond to zeros of F).

We consider here only w in the upper half plane, associated with z in the third quadrant. Analogous considerations apply to w in the lower half plane associated with z in the fourth quadrant.

If we define
$$f(w) = w - N\log(w) + \log(C) + \log(1 + h(w))$$
then w is a zero of F if and only $f(w) = 2n\pi i$ for some n. Theorem 6.1 of Olver [25] guarantees now a solution of this latter equation provided that n is sufficiently large. Moreover this solution is asymptotically equal to $2n\pi i$. Thus
$$w_n = 2n\pi i + N\log(2n\pi i) - \log(C) + O(n^{-1/2}).$$
(Note that we do not claim that n enumerates all the zeros of F in the upper half plane.) We have now proved the first statement of the lemma.

To prove the second statement define
$$K = 2\sup\{|Cw^{-N} + g(w)| : |w| \geq 1, \Im w \geq 0\},$$

and

$$R_n = \Im(w_n) + \pi$$

so that R_n is approximately π away from both $\Im(w_n)$ and $\Im(w_{n+1})$. We consider w on the semi-circle $R_n e^{i\varphi}$ and distinguish three cases:

- In the sector $0 \leq \varphi \leq \pi(1 - 2N(\ln R_n)/R_n)/2$ the term Cw^{-N} is the dominant term in $F(w)$. Since $\Re w = R_n \cos\varphi \geq 2R_n(\frac{\pi}{2} - \varphi)/\pi \geq 2N\ln(R_n) \geq N\ln(R_n) - \ln(|C|/4)$ when n is sufficiently large, we find that $|e^{-w}| \leq |Cw^{-N}|/4$. Sufficiently large n entail also the estimate $|g(w)| \leq |Cw^{-N}|/4$ so that $|F(w)| \geq |Cw^{-N}|/2$.
- In the sector $\pi(1 + (\ln K)/R_n)/2 \leq \varphi \leq \pi$ the term e^{-w} is the dominant term in $F(w)$. Now $\Re w \leq 2R_n(\frac{\pi}{2} - \varphi)/\pi \leq -\ln K$ and hence we obtain $|Cw^{-N} + g(w)| \leq K/2 \leq |e^{-w}|/2$ and hence $|F(w)| \geq |e^{-w}|/2$.
- Finally, in the sector $\pi(1 - 2N(\ln R_n)/R_n)/2 \leq \varphi \leq \pi(1 + (\ln K)/R_n)/2$ the terms e^{-w} and Cw^{-N} are possibly of the same order and may cancel each other. However, on our chosen circular segments this can be prevented. Since $C = e^{-w_n} w_n^N/(1-h(w_n))$ we may rewrite F as $F(w) = e^{-w}(1-G(w))$ where
$$G(w) = e^{w-w_n}(w_n/w)^N(1 + O(n^{-1/2})).$$
Note that, setting $w = x + iy$ and $w_n = x_n + iy_n$ with x, y, x_n, y_n real we get $y - y_n = R_n \sin\varphi - (R_n - \pi) = \pi + O((\log n)^2/n)$ and $x - x_n = O(\log n)$. Therefore $(w - w_n)/w = O(\log n/n)$ and
$$(w_n/w)^N = 1 + O(\log n/n).$$
Also
$$e^{w-w_n} = e^{x-x_n}(-1 + O((\log n)^2/n)).$$
Thus $\Re(G(w)) = e^{x-x_n}(-1 + O(n^{-1/2})) \leq 0$ which yields
$$|F(w)| = |e^{-w}||1 - G(w)| \geq |e^{-w}|\Re(1 - G(w)) \geq |e^{-w}|.$$

We also note that $|F(w)| \geq |e^{-w}|/2$ as w tends to negative infinity along the real axis.

The second statement in the Lemma follows now after recalling that $\Psi(-1, z, +1)$ equals $i\sigma\sqrt{2\pi}\tau_{-1}(z)F(z^2/2)$ and that $\tau_{-1} = -b_N z^{2N}/N! + O(z^{2N-1})$. $\square$

Appendix A. The resultant of ρ_{-1} and τ_{-1}

Theorem A.1. *Assume that $b \in \mathbb{C}^{\mathbb{N}_0}$ has its support in $\{0, ..., N\}$ with $b_N \neq 0$ and let $\Psi : \mathbb{M} \times \mathcal{R}_0 \to \mathbb{C}$ be the associated Jost solution. The resultant of the polynomials ρ_{-1} and τ_{-1} defined by the relationship*

$$\Psi(-1, z, \sigma) = \rho_{-1}(z) + \sigma M_0(\sigma z)\tau_{-1}(z)$$

is given by

$$R(\rho_{-1}, \tau_{-1}) = -\left(\frac{b_N}{N!}\right)^{2N-1}.$$

Proof. Suppose z_0 is a common zero of the polynomials ρ_{-1} and τ_{-1}. Then $\Psi(-1, z_0, +1)$ and $\Psi(-1, z_0, -1)$ are both equal to zero. This is impossible since it would imply that their Wronskian is also zero. Therefore ρ_{-1} and τ_{-1} do not have any common zeros and their resultant never vanishes. A priori, the resultant is a polynomial in the variables b_0, ..., b_N. However, since it never vanishes, it is necessarily a multiple of a power of b_N (recall that we assume $b_N \neq 0$). We can now easily compute this resultant, by assuming that $b_0 = ... = b_{N-1} = 0$. In this case one shows by induction that, for $-1 \le n \le N$,

$$\Psi(n,z,\sigma) = \Psi_0(n,z,\sigma) - \frac{b_N}{\sqrt{N!}} f_n(z)\Psi_0(N,z,\sigma)$$

where the f_n satisfy the recursion relation $a_{n-1}f_{n-1} + a_n f_{n+1} = z f_n$ and the initial conditions $f_N = 0$ and $f_{N-1} = \sqrt{(N-1)!}$. This implies that

$$f_n(z) = -\sqrt{N!}(p_0(N,z)q_0(n,z) - p_0(n,z)q_0(N,z)).$$

In particular, $f_{-1}(z) = \sqrt{N!}p_0(N,z)$ and hence

$$\rho_{-1}(z) = -1 - b_N p_0(n,z)q_0(N,z)$$

and

$$\tau_{-1} = -b_N p_0(n,z)^2.$$

Recall (see, e.g., van der Waerden [27]) that the resultant of two polynomials

$$f(z) = c(z-\alpha_1)...(z-\alpha_n)$$

and

$$g(z) = d(z-\beta_1)...(z-\beta_m)$$

is given by

$$R(f,g) = c^m d^n \prod_{j=1}^n \prod_{k=1}^m (\alpha_j - \beta_k) = c^m \prod_{j=1}^n g(\alpha_j) = (-1)^{nm} d^n \prod_{k=1}^m f(\beta_k).$$

Since ρ_{-1} assumes the value -1 at each of the zeros of τ_{-1} this completes the proof. □

Corollary A.2. *If b is real valued and finitely supported then $\Psi(-1,\cdot,\pm1)$ does not have any real zeros. In other words, there are no real eigenvalues or resonances.*

Proof. Since b is real the polynomials ρ_{-1} and τ_{-1} have real coefficients and hence are real on the real line. Since the imaginary part of M_0 never vanishes on the real axis, a real zero of $\Psi(-1,\cdot,\pm1)$ would have to be a common zero of ρ_{-1} and τ_{-1}. But, according to the previous theorem, such common zeros do not exist. □

We remark here that the precise form of the resultant is not needed to arrive at the conclusion of the corollary. However, we find it striking that the determinant of a rather complicated matrix assumes such a simple form and feel that this result could be of independent interest.

Appendix B. The perturbation determinant

The basic object studied in this paper is the Jost function. It is well-known that in other models, particularly the discrete and continuous Schrödinger equation, the Jost function can be expressed as a perturbation determinant, see, for instance, Jost and Pais [17], Gohberg and Krein [15], Teschl [26], and Gesztesy and Makarov [14]. Here we prove that this fact holds true in the Hermite model, too. We emphasize that this connection between the Jost function and the perturbation determinant is a powerful source of deep analysis of operators and their spectra; exhibited, for instance, by the well-known trace formulas.

Assume that $b \in \mathbb{C}^{\mathbb{N}_0}$ has its support in $\{0,...,N\}$ with $b_N \neq 0$ and denote the Hermite operator with potential $(b_0,...,b_k,0,...)$ by H_k. Accordingly, H_{-1} denotes the operator associated with the potential $b = 0$ and $H_N = H$, the operator associated with the full potential b. We want to compute the determinant of the operator

$$(H_{-1} - z)^{-1}(H_N - z) = I + (H_{-1} - z)^{-1}b$$

where $z \in \mathbb{C} - \mathbb{R}$, the resolvent set of H_{-1}. Since the multiplication operator b is finite rank the determinant is well defined as the product of its eigenvalues, taking into account their algebraic multiplicities. We first note that

$$\det((H_{-1} - z)^{-1}(H_N - z)) = \prod_{k=0}^{N} \det((H_{k-1} - z)^{-1}(H_k - z)).$$

We define the sequence $p(\cdot, z)$ as the solution of $hy = zy$ satisfying initial conditions $p(-1, z) = 0$ and $p(0, z) = 1$. Also recall that $\psi(n, z) = \Psi(n, z, \sigma)$ for $\sigma\Im(z) > 0$.

Suppose now that λ is an eigenvalue of $(H_{k-1} - z)^{-1}(H_k - z)$. Then there is an vector $y \in \ell^2(\mathbb{N}_0)$ such that

$$b_k\delta_{k,n}y(n) = (\lambda - 1)((H_{k-1} - z)y)(n). \tag{6}$$

If $\lambda \neq 1$ this equation implies firstly that $y(n) = \alpha p(n, z)$ for all $n \leq k$ and secondly that $y(n) = \beta\psi_0(n, z)$ for $n \geq k$. Thus $\alpha p(k, z) = \beta\psi_0(k, z)$. Furthermore, evaluating (6) at k, we obtain also

$$b_k y(k) = (\lambda - 1)(a_{k-1}y(k - 1) - zy(k) + a_k y(k + 1))$$

which is equivalent to

$$\alpha b_k p(k, z) = a_{k-1}(\lambda - 1)(\alpha p(k - 1, z) - \beta\psi_0(k - 1, z)).$$

These two equations form a linear homogeneous system for α and β which has a nontrivial solution if and only if

$$\lambda = 1 + \frac{b_k p(k, z)\psi_0(k, z)}{[p(\cdot, z), \psi_0(\cdot, z)](k - 1)} = \frac{[p(\cdot, z), \psi_0(\cdot, z)](k)}{[p(\cdot, z), \psi_0(\cdot, z)](k - 1)}.$$

This is the only eigenvalue possibly different from one in which case it has algebraic multiplicity one. Hence

$$\det((H_{k-1} - z)^{-1}(H_k - z)) = \frac{[p(\cdot, z), \psi_0(\cdot, z)](k)}{[p(\cdot, z), \psi_0(\cdot, z)](k - 1)}$$

and, using $[p(\cdot, z), \psi_0(\cdot, z)](-1) = 1$,

$$\det((H_{-1} - z)^{-1}(H_N - z)) = [p(\cdot, z), \psi_0(\cdot, z)](N).$$

Since $\psi(n, z) = \psi_0(n, z)$ for $n \geq N$ we have that

$$[p(\cdot, z), \psi_0(\cdot, z)](N) = [p(\cdot, z), \psi(\cdot, z)](N) = [p(\cdot, z), \psi(\cdot, z)](-1) = -\psi(-1, z).$$

Thus we have proved the following theorem.

Theorem B.1. *If H is the Hermite operator associated with the finitely supported potential b and z is a non-real complex number, then*

$$\det((H_{-1} - z)^{-1}(H - z)) = -\psi(-1, z).$$

Appendix C. Elementary examples

We give here the Jost functions for the cases $N = 0$, $N = 1$, and $N = 2$ in order to provide some insight into the structure of Jost functions $\Psi(-1, z, \sigma) = \rho_{-1}(z) + \tau_{-1}(z)\sigma M_0(\sigma z)$.

$N = 0$:

$$\rho_{-1}(z) = -1$$
$$\tau_{-1}(z) = -b_0$$

$N = 1$:

$$\rho_{-1}(z) = -1 + b_0 b_1 - b_1 z$$
$$\tau_{-1}(z) = -b_0 + b_0 b_1 z - b_1 z^2$$

$N = 2$:

$$\rho_{-1}(z) = \frac{1}{2}\{-2 + 2b_0 b_1 + (b_2 - 2b_1 - b_0 b_1 b_2)z + (b_0 b_2 + b_1 b_2)z^2 - b_2 z^3\}$$

$$\begin{aligned}\tau_{-1}(z) = \frac{1}{2}\{&-2b_0 - b_2 + b_0 b_1 b_2 + (2b_0 b_1 - b_0 b_2 - b_1 b_2)z \\ &+ (-2b_1 + 2b_2 - b_0 b_1 b_2)z^2 + (b_0 b_2 + b_1 b_2)z^3 - b_2 z^4\}\end{aligned}$$

References

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Bernhard Beckermann. Complex Jacobi matrices. *J. Comput. Appl. Math.*, 127(1-2):17–65, 2001. Numerical analysis 2000, Vol. V, Quadrature and orthogonal polynomials.

[3] Bernhard Beckermann and Mirta Castro Smirnova. On the determinacy of complex Jacobi matrices. *Math. Scand.*, 95(2):285–298, 2004.

[4] B. M. Brown, I. Knowles, and R. Weikard. On the inverse resonance problem. *J. London Math. Soc. (2)*, 68(2):383–401, 2003.

[5] B. M. Brown, S. Naboko, and R. Weikard. The inverse resonance problem for Jacobi operators. *Bull. London Math. Soc.*, 37(5):727–737, 2005.

[6] B. M. Brown and R. Weikard. The inverse resonance problem for perturbations of algebro-geometric potentials. *Inverse Problems*, 20(2):481–494, 2004.

[7] T. Carleman. *Les fonctions quasi analytiques. Leçons professées au Collège de France.* Gauthier-Villars, 1926.

[8] Earl A. Coddington and Norman Levinson. *Theory of Ordinary Differential Equations.* Robert E. Krieger, Malabar, Florida, 1984.

[9] J. Dombrowski. Absolutely continuous measures for systems of orthogonal polynomials with unbounded recurrence coefficients. *Constr. Approx.*, 8(2):161–167, 1992.

[10] J. Dombrowski, J. Janas, M. Moszyński, and S. Pedersen. Spectral gaps resulting from periodic perturbations of a class of Jacobi operators. *Constr. Approx.*, 20(4):585–601, 2004.

[11] J. Dombrowski and S. Pedersen. Spectral measures and Jacobi matrices related to Laguerre-type systems of orthogonal polynomials. *Constr. Approx.*, 13(3):421–433, 1997.

[12] Joanne Dombrowski and Steen Pedersen. Spectral transition parameters for a class of Jacobi matrices. *Studia Math.*, 152(3):217–229, 2002.

[13] Jeffrey S. Geronimo, Oscar Bruno, and Walter Van Assche. WKB and turning point theory for second-order difference equations. In *Spectral methods for operators of mathematical physics*, volume 154 of *Oper. Theory Adv. Appl.*, pages 101–138. Birkhäuser, Basel, 2004.

[14] Fritz Gesztesy and Konstantin A. Makarov. (Modified) Fredholm determinants for operators with matrix-valued semi-separable integral kernels revisited. *Integral Equations Operator Theory*, 47(4):457–497, 2003.

[15] I. C. Gohberg and M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators.* Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.

[16] Jan Janas and Serguei Naboko. Jacobi matrices with power-like weights—grouping in blocks approach. *J. Funct. Anal.*, 166(2):218–243, 1999.

[17] R. Jost and A. Pais. On the scattering of a particle by a static potential. *Physical Rev. (2)*, 82:840–851, 1951.

[18] Evgeni Korotyaev. Inverse resonance scattering on the half line. *Asymptot. Anal.*, 37(3-4):215–226, 2004.

[19] Evgeny Korotyaev. Inverse resonance scattering on the real line. *Inverse Problems*, 21(1):325–341, 2005.

[20] Eli Levin and Doron S. Lubinsky. *Orthogonal polynomials for exponential weights*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 4. Springer-Verlag, New York, 2001.

[21] B. M. Levitan. *Inverse Sturm-Liouville problems*. VSP, Zeist, 1987. Translated from the Russian by O. Efimov.

[22] Vladimir A. Marchenko. *Sturm-Liouville operators and applications*. Birkhäuser Verlag, Basel, 1986. Translated from the Russian by A. Iacob.

[23] Marco Marletta and Rudi Weikard. Stability for the inverse resonance problem for a Jacobi operator with complex potential. *Inverse problems*, 23:1677–1688, 2007.

[24] M. A. Naĭmark. *Linear differential operators. Part II: Linear differential operators in Hilbert space*. With additional material by the author, and a supplement by V. È. Ljance. Translated from the Russian by E. R. Dawson. English translation edited by W. N. Everitt. Frederick Ungar Publishing Co., New York, 1968.

[25] F. W. J. Olver. *Asymptotics and special functions*. Academic Press, New York-London, 1974. Computer Science and Applied Mathematics.

[26] Gerald Teschl. *Jacobi operators and completely integrable nonlinear lattices*, volume 72 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[27] B. L. van der Waerden. *Algebra: Volume I*. Springer, New York, 1991.

[28] Rudi Weikard. A local Borg-Marchenko theorem for difference equations with complex coefficients. In *Partial differential equations and inverse problems*, volume 362 of *Contemp. Math.*, pages 403–410. Amer. Math. Soc., Providence, RI, 2004.

B.M.B.: Department of Computer Science, Cardiff University, Cardiff CF2 3XF, U.K.

E-mail address: Malcolm.Brown@cs.cf.ac.uk

S.N.: Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170, USA

E-mail address: naboko@math.uab.edu

R.W.: Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170, USA

E-mail address: rudi@math.uab.edu