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On a theorem of Hochstadt

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The differential equation

$$Ly = y'' + qy = zy$$

where q is a periodic locally integrable function of period p and λ is a complex parameter is known as Hill's equation. When $y_1(z,\cdot)$ and $y_2(z,\cdot)$ are solutions of the equation satisfying $y_1(z,0) = y_2'(z,0) = 1$ and $y_1'(z,0) = y_2(z,0) = 0$ then the function

$$T(z) = y_1(z,p) + y_2'(z,p)$$

which is (not quite rightly) called the discriminant of the differential expression L determines the spectrum of the operator H on $H^{2,2}(\mathbb{R})$ defined by $Hy = y'' + qy$. In fact, according to a result of Rofe-Beketov [11],

$$\sigma(H) = \{z : -2 \leq T(z) \leq 2\}.$$

In 1963 Hochstadt [10] proved the following

Theorem 1. *When q is real-valued and periodic with period p and when $\sigma(H)$ consists of a finite number of closed intervals then the discriminant T may be uniquely determined from the endpoints of these spectral intervals.*

In fact, Hochstadt gives an explicit formula for the discriminant which will be reproduced below.

Since Hochstadt's work there has been a considerable interest in finite-band potentials, i.e., potentials q such that the spectrum of the operator associated with $d^2/dx^2 + q$ consists of finitely many regular analytic arcs, due to their close relationship with the Korteweg-de Vries hierarchy and hence with integrable systems. However, the interest is not restricted to real-valued potentials. For instance,

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Guillemin and Uribe [9] considered potentials of the form $\sum_{k=1}^{\infty} b_k e^{2ikx}$. Birnir in his work [3] on the complex version of Hill's equation has a generalization of Hochstadt's theorem for which he assumes that the algebraic multiplicities of the (semi-)periodic eigenvalues of the equation which are the zeros of the discriminant are equal to the geometric multiplicities of these eigenvalues and hence are never larger than two. (He also remarks that, in the complex case, any algebraic multiplicity is possible.) Gesztesy and myself have treated potentials which are elliptic functions. Many classical real-valued finite-band potentials, in particular the Lamé potentials, are elliptic functions restricted to some line in the complex plane but the methods used to treat them work just as well when the function is complex-valued (see [5]–[8]) or even when certain singularities are present (see [12]).

In this paper I will present a theorem (Theorem 3) which specifies information from which an entire function from a certain class may be recovered uniquely. Hochstadt's theorem follows then immediately from this theorem since the required pieces of information are well-known properties of discriminants of real-valued periodic differential expressions. However, the theorem applies to any entire function with a certain asymptotic behavior. In particular, it applies to discriminants of complex-valued periodic differential equations regardless what the multiplicities of the (semi-)periodic eigenvalues may be. As an illustration the new theorem is applied below to some simple Lamé potentials. In particular, Example 3 constructs explicitly a potential with a periodic eigenvalue whose algebraic multiplicity is three.

We start with a theorem on the counting of zeros of certain entire functions.

Theorem 2. *Let T be an entire function with the following asymptotic property: there exists a nonzero complex number p such that $|T(-k^2) - 2\cos kp|$ and $|kT'(-k^2) - p\sin kp|$ are of order $|k|^{-1}\exp(|Im(k)|p)$ as $|k|$ tends to infinity.*

Let $B_m = \{z : |z| < (2m+1)^2\pi^2/(4|p|^2)\}$. Then the number of zeros[1] of $T^2 - 4$ in B_m is $2m+1$ and the number of zeros of T' in B_m is m whenever m is a suitably large positive integer. Moreover, the number of zeros of $T - 2$ in B_m is odd and the number of zeros of $T + 2$ in B_m is even.

Proof. Let $T_0(-k^2) = 2\cos kp$ and note that $T_0'(-k^2) = pk^{-1}\sin kp$. Then $T_0 - 2$ has a simple zero at zero and double zeros at $-(2j)^2\pi^2/p^2$, $j \in \mathbb{N}$. Also, T_0+2 has double zeros at $-(2j-1)^2\pi^2/p^2$, $j \in \mathbb{N}$, and T_0' has simple zeros at $z = -j^2\pi^2/p^2$, $j \in \mathbb{N}$. Hence when $m \in \mathbb{N}_0$ the number of zeros of $T_0 - 2$, $T_0 + 2$, and T_0' in B_m equals $2\lfloor m/2\rfloor + 1$, $2\lfloor (m+1)/2\rfloor$, and m, respectively. (Here $\lfloor x\rfloor = \max\{n \in \mathbb{Z} : n \le x\}$.)

Thus the theorem holds when $T = T_0$. In general it follows from Rouché's theorem when m is so large that $|T - T_0| < |T_0 \pm 2|$ and $|T' - T_0'| < |T_0'|$ on the boundary of B_m.

Note that Rouché's theorem shows also that the zeros of $T^2 - 4$ outside a suitably large disk are such that always two of them are close to $-j^2\pi^2/p^2$. In

[1] Zeros are counted according to their multiplicities unless noted otherwise.

particular, these zeros are either simple or double. Also there is one zero of T' close to $-j^2\pi^2/p^2$ for all large $j \in \mathbb{N}$ and this accounts for all large zeros of T'. Hence, if we assume that only finitely many zeros of $T^2 - 4$ are simple then there exists $m \in \mathbb{N}$ such that all zeros of $T^2 - 4$ which have odd multiplicity are in B_m. In fact there is then necessarily an odd number of distinct zeros of $T^2 - 4$ which have odd multiplicity.

Given a contour γ and a continuous function f we denote subsequently the change in the argument of $f(z)$ az z moves along γ by $\Delta_\gamma \arg f$.

Theorem 3. *Let T be an entire function with asymptotic behavior as described in Theorem 2 and assume that only finitely many zeros of $T^2 - 4$ are simple. Then T is uniquely determined by the following data:*
1. the distinct zeros $z_0, \ldots, z_{2g}$ of $T^2 - 4$ which have odd multiplicity,
2. the number p,
3. the numbers $\Delta_{\gamma_l} \arg(T + \sqrt{T^2 - 4})$, $l = 1, \ldots, g$, where $\gamma_1, \ldots, \gamma_g$ are simple mutually nonintersecting contours which do not pass z_0 such that γ_j connects the points z_{2j-1} and z_{2j} (these numbers are always multiples of π).

Proof. If $\hat{z}$ is a zero of $T^2 - 4$ with multiplicity $n > 1$ then it is also a zero of T' with multiplicity $n - 1$. In particular, when $n = 2$ we find that a double zero of $T^2 - 4$ is a simple zero of T'. Hence there is an $m \in \mathbb{N}$ such that outside B_m the zeros of $T^2 - 4$ (which are double) coincide with the zeros of T' (which are simple) and thus, by Theorem 2, there exists an entire function D such that

$$T(z)^2 - 4 = c_1 \left(\prod_{j=0}^{2m} (z - \hat{z}_j) \right) D(z)^2$$

and

$$T'(z) = c_2 \left(\prod_{j=1}^{m} (z - \lambda_j) \right) D(z)$$

where $\hat{z}_j$, $j = 0, \ldots, 2m$ and λ_j, $j = 1, \ldots, m$ are respectively the zeros of $T^2 - 4$ and T' in B_m, repeated according to their multiplicities. Therefore

$$\frac{T'(z)}{\sqrt{T(z)^2 - 4}} = c_3 \frac{\prod_{j=1}^{m} (z - \lambda_j)}{2\sqrt{\prod_{j=0}^{2m} (z - \hat{z}_j)}} = c_3 \frac{\prod_{j=1}^{g} (z - \lambda_j)}{2\sqrt{\prod_{j=0}^{2g} (z - z_j)}} \tag{1}$$

where $c_3 = 2c_2c_1^{-1/2}$ and where, to obtain the last equality, we might have relabeled the λ_j. Let me emphasize here that the numbers $z_0, \ldots, z_{2g}$ in the right most member of (1) are pairwise distinct while some of the numbers $\lambda_1, \ldots, \lambda_g$ may still coincide. In fact, if z_j is a zero of $T^2 - 4$ of order $2n + 1$ then z_j appears n times among the λ_j.

Introduce the hyperelliptic curve

$$C = \{(z, w) \in \mathbb{C}^2 : w^2 = 4(z - z_0) \ldots (z - z_{2g})\}$$

and let M be the Riemann surface $C \cup \{P_\infty\}$ obtained upon compactification of C.

The function $\rho = \frac{1}{2}(T + \sqrt{T^2 - 4})$ which satisfies $T = \rho + 1/\rho$ is single-valued on C. We find

$$T' = \frac{\rho'}{\rho}(\rho - 1/\rho), \quad T^2 - 4 = (\rho - 1/\rho)^2$$

and hence

$$T'(z)/\sqrt{T(z)^2 - 4} = \rho'/\rho.$$

From the asymptotic behavior of T we may infer that

$$\rho(-k^2) = \exp(ikp + O(k^{-1})) \tag{2}$$

as k tends to infinity.

On M define the holomorphic differentials $\phi_j = z^{j-1}dz/w$ for $j = 1, \ldots, g$ and the meromorphic differential $\psi = z^g dz/w$ (with a second order pole at P_∞). Hence

$$d\beta = \frac{T'(z)dz}{\sqrt{T(z)^2 - 4}} = \frac{\rho' dz}{\rho}$$

is a meromorphic differential on M. Denoting by $(-1)^{j+1}\tau_{g+1-j}$ the elementary symmetric polynomials in the variables $\lambda_1, \ldots, \lambda_g$, i. e.,

$$\tau_g = \sum_{j=1}^{g} \lambda_j, \ \ldots, \ \tau_1 = (-1)^{g+1} \prod_{j=1}^{g} \lambda_j,$$

we may write

$$d\beta = p(\psi - \sum_{j=1}^{g} \tau_j \phi_j)$$

since $c_3 = p$ as will be shown next. Let $t = z^{-1/2} = -i/k$ be a local coordinate at P_∞ and $P_1 \neq P_\infty$ a point in the domain of t. Define $\beta(P) = c + \int_{P_1}^{P} d\beta$ where c is a constant which will be determined later. β is a single-valued function in the domain of t. If P tends to P_∞ then

$$\beta(P) = \frac{c_3}{t(P)} + O(1) = ikc_3 + O(1).$$

Since $d\beta = (\log\rho)'dz$ we obtain that $\rho = e^{\beta + O(1)}$ and hence, using (2), that

$$\exp(ikp + O(k^{-1})) = \exp(ikc_3 + O(1))$$

which implies that $c_3 = p$.

The numbers $\alpha_l = \Delta_{\gamma_l} \arg(T + \sqrt{T^2 - 4})/\pi$ are integers. They are even or odd depending on whether $T(z_{2l-1}) = T(z_{2l})$ or $T(z_{2l-1}) = -T(z_{2l})$. Hence, if $\alpha_1 + \ldots + \alpha_g$ is even then the number of points in $\{z_j : 1 \leq j \leq g, T(z_j) = 2\}$ is also even. Now Theorem 2 implies that $T(z_0) = 2$. Similarly, $T(z_0) = -2$ if $\alpha_1 + \ldots + \alpha_g$ is odd. In the first case we fix the constant c such that $\beta((z_0, 0)) = 0$ while we let $\beta((z_0, 0)) = i\pi$ in the second case. With this choice we have that

$T = \rho + 1/\rho = 2\cos(i\beta)$. The theorem is then proven once we have determined the vector $\tau = (\tau_1, \ldots, \tau_g)^t$ from the data given in the statement of the theorem. Define the closed curves

$$a_l(t) = \begin{cases} (\gamma_l(2t), w(\gamma_l(2t)) & \text{if } t \in [0, 1/2], \\ (\gamma_l(2-2t), -w(\gamma_l(2-2t)) & \text{if } t \in [1/2, 1] \end{cases}$$

and choose also closed curves $b_1, \ldots, b_g$ on M such that the set $\{a_1, \ldots, b_g\}$ forms a canonical homology basis for M.

It is well-known that the matrix 2Ω with entries $2\Omega_{l,j} = \int_{a_l} \phi_j$ is nonsingular. We introduce also the vectors H and α whose components are $H_l = \int_{a_l} \psi/2$ and $\alpha_l = \Delta_{\gamma_l} \arg(T + \sqrt{T^2 - 4})/\pi$, respectively. Then

$$2i\pi\alpha_l = \int_{a_l} (\log\rho)' dz = \int_{a_l} d\beta = 2p(H_l - \sum_{j=1}^{g} \tau_j \Omega_{l,j}). \tag{3}$$

Thus denoting the transpose of the vector $(\tau_1, \ldots, \tau_g)$ by τ we obtain

$$\Omega\tau = H - \frac{i\pi}{p}\alpha \tag{4}$$

and this determines the vector τ.

Remark 1. The present result is not directly concerned with differential equations. However, differential equations are its motivation and a major application. Therefore a few remarks comparing Theorems 2 and 3 with the work of Birnir, who, as mentioned above, gave a generalization of Hochstadt's theorem to complex potentials, seem to be in order. Firstly, there are no restrictions on the multiplicities of eigenvalues. Secondly, the theorem applies to any potential whose discriminant satisfies the hypotheses. This includes particularly potentials with certain inverse square singularities (see [12]), i.e., potentials which are not locally integrable. Thirdly, the straightforward application of Rouché's theorem in Theorem 2 replaces Birnir's more complicated deformation analysis when counting eigenvalues. Finally, some caution is necessary when reading Birnir's theorem which is Theorem 1.1 in [3]. It reads "The simple spectrum determines the double."[2] but its proof shows that the third piece of information in Theorem 3 is also required to determine the numbers n_1,..., n_g figuring there. (Birnir assumes that $p = 1$.)

Remark 2. Note that β is a multi-valued function which is determined only up to adding an integer linear combination of the a- and b-periods of $d\beta$. Since $T = 2\cos i\beta$ this fact does not affect T provided the periods of $d\beta$ are integer multiples of $2\pi i$. Since $\arg(T(z) + \sqrt{T(z)^2 - 4})$ is always a multiple of π when z equals one of the points $z_0, \ldots, z_{2g}$ the a-periods of $d\beta$ are indeed integer multiples

[2] The spectrum means here the collection of zeros of $T^2 - 4$. In view of Birnir's condition that no zero has multiplicity larger than two the simple spectrum gives therefore the collection of zeros of $T^2 - 4$ with odd multiplicity.

of $2\pi i$ according to equation (3). Defining the matrix Ω' and the vector H' by $2\Omega'_{l,j} = \int_{b_l} \phi_j$ and $2H'_l = \int_{b_l} \psi$ and integrating $d\beta$ along the b-curves rather than the a-curves gives

$$\int_{b_l} d\beta = 2p(H'_l - \sum_{j=1}^{g} \tau_j \Omega'_{l,j}). \tag{5}$$

Since the numbers $\tau_1, \ldots, \tau_g$ are already determined the fact that $\int_{b_l} d\beta$ is an integer multiple of $2\pi i$ gives a restriction on possible values of p. Hence when, in addition to the numbers $\alpha_1, \ldots, \alpha_g$ the number $\int_{b_l} d\beta = \Delta_{b_l} \arg(T + \sqrt{T^2 - 4})$ is known for one of the curves b_l it is generally possible to determine the value of p uniquely since equations (4) and (5) represent a system of $g + 1$ linear equations for the $g + 1$ variables τ and $1/p$.

Proof of Hochstadt's theorem: We next prove Hochstadt's theorem as a special case of Theorem 3 by obtaining the information required from well-known properties of discriminants of real, periodic differential expressions. Specifically, T has the asymptotic behavior required in Theorem 3, it is real on the real line, the zeros of $T^2 - 4$ are all real and have multiplicity not larger than two, $T(z) \geq 2$ when z is a maximum of T, and $T(z) \leq -2$ when z is a minimum of T (see, e.g., Eastham [4]).

Proof of Theorem 1. Let p be the period of q. Since $\sigma(H) = \{z : -2 \leq T(z) \leq 2\}$ the simple zeros of $T^2 - 4$ are precisely the endpoints of the spectral bands. Since there are only finitely many bands there are only finitely many simple zeros of $T^2 - 4$. When they are labeled such that $z_{2g} < \ldots < z_0$ then $T(z_0) = 2$ since $T(z)$ tends to infinity when z does. Also $T(z_{2j-1}) = T(z_{2j})$ for $j = 1, \ldots, g$. Assume that $T(z_{2j}) = 2$. As z moves from z_{2j} to z_{2j-1} we have that $T(z) \geq 2$ and hence that $\arg(T(z) + \sqrt{T(z)^2 - 4}) = 0$. Similarly, when $T(z_{2j}) = -2$ then $\arg(T(z) + \sqrt{T(z)^2 - 4}) = \pi$. Hence letting $\gamma_j = (1 - t)z_{2j} + tz_{2j-1}$ we have $\alpha_j = 0$ for $j = 1, \ldots, g$. Thus, by Theorem 3, the function T is uniquely determined. In fact, according to the proof of Theorem 3 we have

$$\beta(z) = p \int_{z_0}^{z} \frac{z^g - \tau_g z^{g-1} - \ldots - \tau_1}{w} dz$$

where $w = \sqrt{4(z - z_0)\ldots(z - z_{2g})}$ and where $\tau_1, \ldots, \tau_g$ are given by the system of equations

$$\sum_{j=1}^{g} \tau_j \Omega_{l,j} = H_l, \quad l = 1, \ldots, g$$

using

$$\Omega_{l,j} = \int_{z_{2l}}^{z_{2l-1}} \frac{z^{j-1} dz}{w} \quad \text{and} \quad H_l = \int_{z_{2l}}^{z_{2l-1}} \frac{z^g dz}{w}.$$

T is now given by $T(z) = 2\cos(i\beta(z))$.

As indicated in Remark 2 we may even find p from one more piece of information.

Theorem 4. *When q satisfies the hypothesis of Theorem 1 then the value of p may be determined from the number of zeros (counting multiplicities) of $T^2 - 4$ in any one of the finite spectral bands.*

Proof. Assume that $T^2 - 4$ has m' zeros (counting multiplicities) between z_{2l+1} and z_{2l} for some $l \in \{0, \ldots, g-1\}$. Let $\tilde{\gamma}_l(t) = (1-t)z_{2l+1} + tz_{2l}$. Then $\Delta_{\tilde{\gamma}_l} \arg(T + \sqrt{T^2 - 4}) = (m' + 2)\pi/2$. This gives

$$H_l' - \sum_{j=1}^{g} \tau_j \Omega_{l,j}' = \frac{(m'+2)\pi i}{2p} \tag{6}$$

where

$$\Omega_{l,j}' = \int_{z_{2l+1}}^{z_{2l}} \frac{z^{j-1} dz}{w} \quad \text{and} \quad H_l' = \int_{z_{2l+1}}^{z_{2l}} \frac{z^g dz}{w}.$$

Equation (6) determines p since its left hand side is independent of p.

Example 1. Assume that there is only one zero of $T^2 - 4$ with odd multiplicity, i.e., $g = 0$. Calling this zero z_0 we know that $T(z_0) = 2$ since $T - 2$ must have at least one zero. Note also that there is no information required from part 3. of Theorem 3 when $g = 0$. Hence we obtain

$$\beta(P) = \int_{(z_0,0)}^{P} \frac{p dz}{\sqrt{4(z - z_0)}} = p\sqrt{z - z_0}$$

choosing the branch of the root according to P. Now $T = 2\cos(ip\sqrt{z - z_0})$ (which is independent of the branch chosen for the root). This proves also that z_0 is necessarily a simple root of $T^2 - 4$ and that all other roots are double.

T is the discrimimant of any differential expression $d^2/dx^2 + q$ for which q has period p and whose conditional stability set consists of only one regular analytic arc which ends at z_0. In particular, $2\cos(i\pi\sqrt{z})$ is the discriminant for d^2/dx^2 (when $q = 0$ is regarded as a function of period π) as well as for $d^2/dx^2 + \mathrm{e}^{2ix}$. Both of these differential expressions have the nonpositive real axis as their conditional stabiltity set (see [12] for the second one).

Example 2. Next assume that there are three zeros of $T^2 - 4$ with odd multiplicity. We denote them by z_0, z_1, and z_2. Then we have $T = 2\cos(i\beta)$ where β is the elliptic integral

$$\beta = p\int_{(z_0,0)}^{P} \frac{z - \tau_1}{\sqrt{4(z - z_0)(z - z_1)(z - z_2)}} dz + ik\pi \tag{7}$$

and where $k = 0$ or $k = 1$ depending on whether $T(z_0) = 2$ or $T(z_0) = -2$. The number τ will have to be determined from the number $\alpha_1 = \Delta_{\gamma_1} \arg(T + \sqrt{T^2 - 4})/\pi$.

Introduce $a = (z_0 + z_1 + z_2)/3$,

$$e_1 = z_0 - a, \; e_2 = z_1 - a, \; e_3 = z_2 - a,$$

$$g_2 = 2(e_1^2 + e_2^2 + e_3^2) \text{ and } g_3 = 4e_1e_2e_3.$$

Let $\wp(u)$ denote Weierstrass's elliptic function with invariants g_2 and g_3 and choose fundamental periods $(2\omega, 2\omega')$ such that

$$\wp(\omega) = e_1, \quad \wp(\omega + \omega') = e_2, \quad \wp(\omega') = e_3.$$

Also recall that then

$$\zeta(\omega) = \eta, \quad \zeta(\omega + \omega') = \eta + \eta', \quad \zeta(\omega') = \eta'$$

(see, e.g., Abramowitz and Stegun [1] or Akhiezer [2]).

Using the substitution $z = \wp(u) + a$ yields

$$\beta = p \int_\omega^u \frac{\wp(u) + a - \tau_1}{\wp'(u)} \wp'(u) du$$

on account of the basic relation

$$\wp'(u)^2 = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3).$$

Now let $\gamma_1(t) = \wp(\omega' + t\omega) + a$ for $t \in [0, 1]$ which joins the points z_2 and z_1. Since the number of zeros of $T - 2$ is odd we obtain that $T(z_0) = 2$ or $T(z_0) = -2$ depending on whether $\alpha_1 = \Delta_{\gamma_1} \arg(T + \sqrt{T^2 - 4})/\pi$ is even or odd. We let $\alpha_1 = 2m_1 + k$ where k is the same as in (7).

It is now easy to compute that

$$\Omega_{1,1} = \omega \text{ and } H_1 = -\eta + a\omega.$$

Therefore equations (3) yields

$$\tau_1 = -\frac{(2m_1 + k)\pi i}{p\omega} - \frac{\eta}{\omega} + a$$

from which we get

$$\beta(z) = p \int_\omega^u (\wp(u') + a - \tau_1) du' = -p\zeta(u) - 2m_1\pi i + \frac{(2m_1 + k)\pi i}{\omega} u + \frac{p\eta}{\omega} u$$

and

$$T(z) = 2\cos(-(2m_1 + k)\pi u/\omega - ip(\zeta(u) - \eta u/\omega)) \tag{8}$$

where $z = \wp(u) + a$.

As expected comparison with (11) shows that this agrees with the discriminant of the Lamé equation $y'' - 2\wp(x)y = zy$ when $\wp(x)$ is considered as a function of period $p = 2m\omega + 2m'\omega'$ and when we choose $m' = -2m_1 - k$ and $a = 0$.

Now let $\tilde\gamma_1(t) = \wp((1 - t)\omega' + \omega) + a$ which connects the points z_1 and z_0. Given the additional information that $\Delta_{\tilde\gamma_1} \arg(T + \sqrt{T^2 - 4}) = \tilde m_1\pi$ we are able to determine p rather than having to provide it. Indeed, defining

$$b_1(t) = \begin{cases} (\tilde\gamma_1(2t), w(\gamma_1(2t)) & \text{if } t \in [0, 1/2], \\ (\tilde\gamma_1(2 - 2t), -w(\gamma_1(2 - 2t)) & \text{if } t \in [1/2, 1] \end{cases}$$

we obtain that $\Omega'_{1,1} = -\omega'$ and $H'_1 = \eta' - a\omega'$. Therefore, from (5), $2p(\eta' - a\omega' + \tau_1\omega') = 2i\tilde m_1\pi$ which implies, given that τ_1 has already been computed,

$$p = -2\tilde m_1\omega - (4m_1 + 2k)\omega'.$$

Example 3. If we let in the previous example $e_1 = -\eta/\omega$, $m_1 = 0$, and (for simplicity) $a = 0$ then $\tau_1 = z_0$. Since generally

$$e_1 = -\frac{\eta}{\omega} + \left(\frac{\pi}{2\omega}\right)^2 \left(1 + 8\sum_{k=1}^{\infty} \frac{h^{2k}}{(1+h^{2k})^2}\right)$$

where $h = \exp(i\pi\omega'/\omega)$ (see, e.g., [2], p. 204), we may enforce that $e_1 = -\eta/\omega$ by choosing the periods of $\wp$ such that

$$\sum_{k=1}^{\infty} \frac{h^{2k}}{(1+h^{2k})^2} = -1/8.$$

In fact, $2\omega_1 = 2$ and $2\omega_3 = 1 + 0.70945978i$ will approximately do. Note that the associated invariants g_2 and g_3 are real but that the Weierstrass discriminant $g_2^3 - 27g_3^2$ is negative.

For this choice of periods we have

$$\beta(z) = \frac{1}{3\sqrt{(z_0 - z_1)(z_0 - z_2)}}(z - z_0)^{3/2} + \dots$$

which implies that $T^2 - 4$ has a third order zero at $z_0 = e_1$. Hence, in this case, z_0 is a periodic eigenvalue of algebraic multiplicty three of $d^2/dx^2 - 2\wp(x)$.

Example 4. Consider now the discriminant of $Ly = y'' - 6\wp(x)y = zy$ when x is a real variable and $\wp$ has invariants $g_2 = 0$ and $g_3 > 0$ and hence fundamental periods $2\omega > 0$ and $2\omega' = \omega(1+i\sqrt{3})$. The conditional stability set of L is then the union of the interval $(-\infty, -3e_1]$ and an arc connecting $-3e_2$ and $-3e_3 = \overline{-3e_2}$ which passes through zero. According to (12) the discriminant of L is given by

$$T(z) = 2\cos(\sqrt{3}(2\omega\zeta(u_1/\varepsilon) + 2\eta u_1/\varepsilon)) \tag{9}$$

where $\varepsilon = (1 + i\sqrt{3})/2$ and $z = -3\wp(u_1/\varepsilon)$. This fact can be recovered from the results of Example 2: Choose $a = 0$ and let $\hat{\wp}$ and $\hat{\zeta}$, be Weierstrass's elliptic functions with invariants $\hat{g}_2 = 0$ and $\hat{g}_3 = t^6 g_3$. These functions have fundamental periods $\hat{\omega} = \omega/t$ and $\hat{\omega}' = \omega'/t$ and satisfy $\hat{\wp}(u) = t^2\wp(tu)$ and $\hat{\zeta}(u) = t\zeta(tu)$. In particular, $\hat{e}_j = t^2 e_j$, $\hat{\eta} = t\eta$ and $\hat{\eta}' = t\eta'$. Letting $t = i\sqrt{3}$ we have that the band edges of L are given by $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$. Also $\Delta_{\gamma_1} = -2\pi$ and $\Delta_{\tilde{\gamma}_1} = \pi$, i.e., $k = 0$, $m_1 = -1$, and $\tilde{m}_1 = 1$. We obtain therefore from (8) that

$$T(z) = 2\cos(2\pi u/\hat{\omega} - ip(\hat{\zeta}(u) - \hat{\eta}u/\hat{\omega}))$$

where $z = \hat{\wp}(u) = t^2\wp(tu)$. Since $p = 4\hat{\omega}' - 2\hat{\omega} = 2\omega$ and $\eta = \pi/(2\omega\sqrt{3})$ this becomes

$$T(z) = 2\cos(\sqrt{3}(2\eta tu + 2\omega\zeta(tu)))$$

which agrees with (9) when $tu = u_1/\varepsilon$.

Appendix A. Lamé's Equation

Lamé's differential equation

$$y'' - g(g+1)\wp(x)y = zy$$

has the solution

$$y(x) = \prod_{j=1}^{g} \frac{\sigma(x-u_j)}{\sigma(x)} \exp(x\zeta(u_j))$$

provided

$$z = (2g-1)\sum_{j=1}^{g} \wp(u_j)$$

and

$$\sum_{\substack{1\le j\le g\\ j\ne k}} (\zeta(u_j-u_k) - \zeta(u_j) + \zeta(u_k)) = 0, \quad \text{for } k = 1,\dots,g \tag{10}$$

(see [5]). Assuming that 2ω and $2\omega'$ are fundamental periods of $\wp$ one obtains that $y(u+2m\omega+2m'\omega') = \rho_{m,m'}y(u)$ where

$$\rho_{m,m'} = \prod_{j=1}^{g} \exp((2m\omega+2m'\omega')\zeta(u_j) - u_j(2m\eta+2m'\eta')),$$

i.e., y is a Floquet solution of Lamé's equation with respect to the period $2m\omega + 2m'\omega'$ with multiplier $\rho_{m,m'}$.

The discriminant of a periodic differential expression d^2/dx^2+q is given by $T = \rho + 1/\rho$ where ρ is a Floquet multiplier of the associated equation. Hence the discriminant of Lamé's equation with respect to the period $p = 2m\omega+2m'\omega'$ is given by

$$T(z) = 2\cos(i\sum_{j=1}^{g}((2m\omega+2m'\omega')\zeta(u_j) - u_j(2m\eta+2m'\eta')))$$

where the u_j satisfy the conditions in (10) and $m, m' \in \mathbb{Z}$.

We now consider the case $g = 1$. Using Legendre's relation $\eta\omega' - \eta'\omega = i\pi/2$ we find that the discriminant of $y'' - 2\wp(x)y = zy$ is given by

$$T(z) = 2\cos(m'\pi u/\omega - ip(\zeta(u) - \eta u/\omega)) \tag{11}$$

when $z = \wp(u)$ and $\wp(x)$ is considered as a function of period $p = 2m\omega+2m'\omega'$. Note that $T(e_1) = 2$ if m' is even and $T(e_1) = -2$ if m' is odd.

Finally we examine more closely the special case where $g = 2$, $g_2 = 0$, and $p = 2\omega$. Let $\varepsilon = (1+i\sqrt{3})/2$. Condition (10) is satisfied when $u_2 = \varepsilon u_1$ and we find that then

$$T(z) = 2\cos(\sqrt{3}(2\omega\zeta(u_1/\varepsilon) + 2\eta u_1/\varepsilon)) \tag{12}$$

and

$$z = 3\wp(u_1) + 3\wp(\varepsilon u_1) = -3\wp(u_1/\varepsilon).$$

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