On Second Order Linear Differential Equations with Inverse Square Singularities

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We study the differential equation

$$y'' + (a/x^2 + q(x))y = Ey$$

where $a, E \in \mathbb{C}$ and where q is a complex-valued function which is locally integrable on $\mathbb{R} - \{0\}$ and analytic in a neighborhood of $x = 0$ (or nearly so). In particular, we are interested in the behavior of solutions of initial value problems as the initial point varies and in the asymptotic behavior of solutions as the parameter E tends to infinity.

1. Introduction

The spectrum of Hill's differential operator, i.e., of the self-adjoint operator associated with the equation $y'' + qy = Ey$ where q is a real-valued, periodic, continuous function consists of a countable number of intervals, called (spectral) bands. Those potentials for which the number of bands is finite are called finite-band potentials. They have lately received a great deal of attention due to their connection with nonlinear integrable systems and the Korteweg-de Vries hierarchy (see e.g., Belokolos et al. [2] and the literature cited there). The earliest example for a finite-band potential was given by Ince [8]: when $\wp$ denotes Weierstrass' elliptic function with real and purely imaginary half-periods ω and ω' and when g is a nonnegative integer then the Lamé potential

$$q(x) = -g(g+1)\wp(x + \omega')$$

is a real-valued, continuous, and periodic function of the real variable x and the self-adjoint $L^2(\mathbb{R})$-operator associated with the differential expression $d^2/dx^2 + q$ gives rise to a spectrum consisting of g compact bands and one closed ray. These and other elliptic finite-band potentials have been the subject of investigation in the papers [4] – [7]. From the point of view taken in these papers the requirement that q should be real-valued and continuous is very unnatural. In fact, we prove in [7] that so called Picard potentials which are in general complex-valued and which, for instance, include the Lamé potentials even when ω' is not purely imaginary, have finite-band structure. According to Rofe-Beketov [9] the bands will then be analytic arcs which are not confined to the real axis anymore. The results of [7], however, should even extend to $q(x) = -g(g+1)\wp(x)$ where the potential is not even continuous but has singularities of the form $-g(g+1)/(x - 2n\omega)^2$ when x is close to a period $2n\omega$ of $\wp$. This fact is the motivation for the present investigation in which we treat perturbations of a potential with an inverse square singularity. Periodic, complex-valued potentials with such singularities and their relationships with integrable systems will be treated in a subsequent investigation [10].

The differential equation $y'' + qy = Ey$ where the potential q has singularities of the form a/x^2 has also been treated by Gesztesy and Kirsch [3] when a is real and not larger than $1/4$. However, they were interested in solutions which are square-integrable even near the singularity. This forces Dirichlet boundary conditions when $a \leq -3/4$ while arbitrary boundary conditions may be chosen when $-3/4 < a \leq 1/4$ (in which case the differential expression is in the limit circle case at $x = 0$). In the present treatment, however, we assume that q is analytic in a neighborhood of $x = 0$ with the nonpositive imaginary axis removed. This allows to continue solutions in a unique way from one side of the singular point to the other. In particular, no boundary conditions can be applied at $x = 0$.

In Section 2 we treat the case $q = a/x^2$ by reducing the equation $y'' + qy = Ey$ to Bessel's equation. In Section 3 we treat certain perturbations of a/x^2 by constructing and studying solutions of an appropriate integral equation.

2. The Unperturbed Case

The substitutions $z = kx$, $y(x) = x^{1/2} w(z)$, and $4\nu^2 = 1 - 4a$ transform the differential equation

$$Ly = y'' + \frac{a}{x^2} y = -k^2 y$$

into Bessel's equation

$$z^2 w'' + z w' + (z^2 - \nu^2) w = 0.$$

A fundamental system of solutions of Bessel's equation is given by the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$. Hence a fundamental system of solutions of $Ly = Ey$ is given by $c_0(E, x_0, \cdot)$ and $s_0(E, x_0, \cdot)$ where x_0 is any nonzero complex number and

$$\begin{aligned} c_0(-k^2, x_0, x) &= \frac{\pi}{4} (x/x_0)^{1/2} \left(J_\nu(kx) Y_\nu(kx_0) - J_\nu(kx_0) Y_\nu(kx) \right) \\ &\quad + k \frac{\pi}{2} (x x_0)^{1/2} \left(J_\nu(kx) Y'_\nu(kx_0) - J'_\nu(kx_0) Y_\nu(kx) \right), \\ s_0(-k^2, x_0, x) &= \frac{\pi}{2} (x x_0)^{1/2} \left(J_\nu(kx_0) Y_\nu(kx) - J_\nu(kx) Y_\nu(kx_0) \right). \end{aligned}$$

Note that $c_0(E, x_0, \cdot)$ and $s_0(E, x_0, \cdot)$ are those solutions of $Ly = Ey$ which satisfy initial conditions $y(x_0) = 1$, $y'(x_0) = 0$ and $y(x_0) = 0$, $y'(x_0) = 1$, respectively. In $\Omega_0 = \{x \in \mathbb{C} : \Re(x) = 0 \Rightarrow \Im(x) > 0\}$ these functions are single-valued and analytic. In particular, we thus have a unique solution of the initial value problem $Ly = Ey$, $y(x_0) = y_0$, $y'(x_0) = y'_0$ in $\mathbb{R} - \{0\}$.

For any choice of x and x_0 in Ω_0 the functions $c_0(\cdot, x_0, x)$ and $s_0(\cdot, x_0, x)$ as well as their derivatives with respect to x, i.e., $c_0'(\cdot, x_0, x)$ and $s_0'(\cdot, x_0, x)$ are entire functions of order $1/2$.

Upon interchanging the roles played by x and x_0 one obtains the following relationships:

$$\begin{aligned} c_0(E, x_0, x) &= s_0'(E, x, x_0), \\ s_0(E, x_0, x) &= -s_0(E, x, x_0), \\ c_0'(E, x_0, x) &= -c_0'(E, x, x_0). \end{aligned}$$

Note that for $a = 0$ we have $c_0(-k^2, x_0, x) = \cos(k(x - x_0))$ and $s_0(-k^2, x_0, x) = \sin(k(x - x_0))/k$.

3. Perturbations

Next we consider perturbations of the potential a/x^2 which are locally integrable and analytic near zero. More precisely, for some real number $b > 0$ let $U = \{x \in \mathbb{C} : |x| < b, \Re(x) = 0 \Rightarrow \Im(x) > 0\}$ and $\Omega = U \cup (\mathbb{R} - \{0\})$. Then suppose that $q : \Omega \to \mathbb{C}$ satisfies the following three conditions:
1. $q \in L^1_{\mathrm{loc}}(\mathbb{R} - \{0\})$,
2. q is analytic in U,
3. there exists an $\varepsilon \in (0, 1]$ and a $Q > 0$ such that $|q(x)| \leq Q|x|^{\varepsilon - 1}$ for all $x \in U$.

For $E, y_0, y_0' \in \mathbb{C}$ and $x_0, x \in \Omega$ define

$$\phi_0(E, y_0, y_0', x_0, x) = y_0 c_0(E, x_0, x) + y_0' s_0(E, x_0, x).$$

Theorem 3.1. *If q satisfies the above hypotheses then there exists a function $\phi : \mathbb{C}^3 \times \Omega^2 \to \mathbb{C}$ with the following properties:*
(a) $\phi(E, y_0, y_0', x_0, \cdot)$ is the unique solution of the integral equation

$$y(x) = \phi_0(E, y_0, y_0', x_0, x) - \int_{\gamma_s} s_0(E, x', x) q(x') y(x') dx' \tag{3.1}$$

where γ_s is a piecewise continuously differentiable simple path in Ω connecting x_0 and x. In particular, $\phi(E, y_0, y_0', x_0, x_0) = y_0$.
(b) $\phi(E, y_0, y_0', x_0, \cdot)$ is analytic in U and continuously differentiable on $\mathbb{R} - \{0\}$. In fact, ϕ' (where the prime denotes differentiation with respect to the last argument) is given by

$$\phi'(E, y_0, y_0', x_0, x) = \phi_0'(E, y_0, y_0', x_0, x) - \int_{\gamma_s} s_0'(E, x', x) q(x') \phi(E, y_0, y_0', x_0, x') dx'$$

and therefore $\phi'(E, y_0, y_0', x_0, x_0) = y_0'$. The function $\phi'(E, y_0, y_0', x_0, \cdot)$ is locally absolutely continuous in $\mathbb{R} - \{0\}$.

(c) The function ϕ'' is given by

$$\phi''(E,y_0,y_0',x_0,x)=(E-\frac{a}{x^2}-q(x))\phi(E,y_0,y_0',x_0,x)$$

(for $x\in\mathbb{R}-\{0\}$ this holds almost everywhere). Hence $\phi(E,y_0,y_0',x_0,\cdot)$ is the unique solution of the initial value problem

$$y''+(a/x^2+q(x))y=Ey,\ y(x_0)=y_0,\ y'(x_0)=y_0'.$$

(d) $\phi(E,y_0,y_0',\cdot,x)$ and $\phi'(E,y_0,y_0',\cdot,x)$ are locally absolutely continuous in $\mathbb{R}-\{0\}$. Moreover,

$$\frac{\partial\phi}{\partial x_0}(E,y_0,y_0',x_0,x)=\phi(E,-y_0',(a/x_0^2+q(x_0)-E)y_0,x_0,x), \tag{3.2}$$

$$\frac{\partial\phi'}{\partial x_0}(E,y_0,y_0',x_0,x)=\phi'(E,-y_0',(a/x_0^2+q(x_0)-E)y_0,x_0,x). \tag{3.3}$$

(e) There exist positive constants C and Λ depending on x_0 and x but not on E, y_0 and y_0' such that

$$\begin{aligned}|\phi(E,y_0,y_0',x_0,x)-\phi_0(E,y_0,y_0',x_0,x)| &\le C\mathrm{e}^{|\Re(\sqrt{E})(x-x_0)|}\frac{|y_0||\sqrt{E}|+|y_0'|}{|\sqrt{E}|^{1+\varepsilon}},\\ |\phi'(E,y_0,y_0',x_0,x)-\phi_0'(E,y_0,y_0',x_0,x)| &\le C\mathrm{e}^{|\Re(\sqrt{E})(x-x_0)|}\frac{|y_0||\sqrt{E}|+|y_0'|}{|\sqrt{E}|^{\varepsilon}}.\end{aligned}$$

when $x_0,x\in\mathbb{R}-\{0\}$ and when $|E|\ge\Lambda$. In particular, the functions $\phi(\cdot,y_0,y_0',x_0,x)$ and $\phi'(\cdot,y_0,y_0',x_0,x)$ are entire and have order $1/2$.

Proof. Let K be a compact subset of Ω and B a compact subset of $\mathbb{C}$. Since ϕ_0 is continuous in all its arguments there exists a positive constant M_1 such that $|\phi_0|$, $|s_0|$, and $|s_0'|$ are bounded by M_1 on $B^3\times K^2$ and $B\times K^2$, respectively.

Let $\gamma:[0,1]\to K$ be a piecewise continuously differentiable simple path with initial point $\gamma(0)=x_0$ and let Γ be a positive constant such that $|\gamma'(t)|\le\Gamma$. For $s\in[0,1]$, let $\gamma_s:[0,1]\to\Omega$ be defined by $\gamma_s(t)=\gamma(st)$.

Now, for $x=\gamma(s)$ and for $n\in\mathbb{N}$ define recursively

$$\phi_n(E,y_0,y_0',x_0,x)=-\int_{\gamma_s}s_0(E,x',x)q(x')\phi_{n-1}(E,y_0,y_0',x_0,x')dx'.$$

Since $s_0(E,\cdot,x)q(\cdot)\phi_0(E,y_0,y_0',x_0,\cdot)$ is analytic in U and since U is simply connected the induction principle shows that $\phi_n(E,y_0,y_0',x_0,x)$ is independent of the path chosen to connect x_0 and x and that it is analytic in U when it is regarded as a function of x.

Defining $R(t)=\Gamma M_1|q(\gamma(t))|$ one obtains the estimate

$$|\phi_n(E,y_0,y_0',x_0,x)|\le\int_0^s R(s_1)\int_0^{s_1}R(s_2)...\int_0^{s_{n-1}}R(s_n)M_1ds_n...ds_1.$$

Defining $\hat{R}(t) = \int_0^t R(t')dt'$ gives then

$$|\phi_n(E, y_0, y_0', x_0, x)| \leq M_1 \frac{\hat{R}(s)^n}{n!}.$$

Hence, by the Weierstrass M-test $\sum_{n=0}^{\infty} \phi_n$ converges absolutely and uniformly on $B^3 \times K^2$ and hence on any compact subset of $\mathbb{C}^3 \times \Omega^2$.

Because of uniform convergence it follows next that the function

$$\phi(E, y_0, y_0', x_0, \cdot) = \sum_{n=0}^{\infty} \phi_n(E, y_0, y_0', x_0, \cdot)$$

is a solution of the integral equation (3.1). In order to show uniqueness of solutions of (3.1) assume that there are two solutions y and $\tilde{y}$. Defining

$$L = \sup\{|y(x) - \tilde{y}(x)| : x \in \gamma([0,1])\}$$

we obtain firstly $|y(x) - \tilde{y}(x)| \leq L\hat{R}(s)$. A repeated approximation shows now that

$$|y(x) - \tilde{y}(x)| \leq L \frac{\hat{R}(s)^n}{n!}$$

for all $n \in \mathbb{N}$. Since the right hand side in this inequality becomes arbitrarily small this shows that $y = \tilde{y}$ and hence that solutions of (3.1) are unique. This completes the proof of part (a) of the theorem.

$\phi(E, y_0, y_0', x_0, \cdot)$ is analytic in U as the uniform limit of analytic functions. Equation (3.1) shows that $\phi(E, y_0, y_0', x_0, \cdot)$ is differentiable in $\mathbb{R} - \{0\}$ and that its derivative is given as stated in (b).

Since $\phi_0(E, y_0, y_0', x_0, \cdot)$ is infinitely often differentiable in $\mathbb{R} - \{0\}$ we have to show the locally absolute continuity of

$$g(x) = \int_{\gamma_s} s_0'(E, x', x) q(x') \phi(E, y_0, y_0', x_0, x') dx'$$

in order to finish the proof of part (b). Hence assume that $[\alpha, \beta]$ is a subset of $\mathbb{R} - \{0\}$ and that

$$\alpha \leq t_0 < t_1 \leq t_2 < \ldots < t_{2N-1} \leq t_{2N} < t_{2N+1} \leq \beta.$$

When γ_1 is a piecewise continuously differentiable simple path connecting x_0 and α define γ_2 by $\gamma_2(t) = \alpha + (\beta - \alpha)t$ and let γ be the product path $\gamma_1\gamma_2$. Then

$$\begin{aligned} \sum_{k=0}^{N} |g(t_{2k+1}) - g(t_{2k})| &\leq \sum_{k=0}^{N} \int_{s_{2k}}^{s_{2k+1}} M_1 \mathrm{e}^{\hat{R}(1)} R(s') ds' \\ &+ \int_0^1 \sum_{k=0}^{N} |s_0'(E, \gamma(s'), t_{2k+1}) - s_0'(E, \gamma(s'), t_{2k})| |q(\gamma(s'))| M_1 \mathrm{e}^{\hat{R}(1)} \Gamma ds' \end{aligned} \tag{3.4}$$

where $s_k = \gamma^{-1}(t_k)$. Because of the integrability of R the first term on the right hand side can be made arbitrarily small provided $\sum_{k=0}^{N}(t_{2k+1} - t_{2k})$ is suitably small. To treat the second term note that $\Re s_0'(E, x', \cdot)$ and $\Im s_0'(E, x', \cdot)$ are continuously differentiable on $[\alpha, \beta]$. Hence

$$|s_0'(E, x', t_{2k+1}) - s_0'(E, x', t_{2k})| \leq M_2(t_{2k+1} - t_{2k})$$

for all $x' \in \gamma([0, 1])$ when M_2 is a suitably chosen constant. Hence the second term on the right hand side of (3.4) can be bounded by a constant times $\sum_{k=0}^{N}(t_{2k+1} - t_{2k})$. This finishes the proof of part (b) of the theorem.

Differentiating the equation

$$\phi'(E, y_0, y_0', x_0, x) = \phi_0'(E, y_0, y_0', x_0, x) - \int_{\gamma_s} s_0'(E, x', x)q(x')\phi(E, y_0, y_0', x_0, x')dx'$$

with respect to x shows now part (c) of the theorem.

Again, let $[\alpha, \beta]$ be a subset of $\mathbb{R} - \{0\}$. Since

$$\frac{\partial \phi_0(E, y_0, y_0', x_0, x)}{\partial x_0} = \phi_0(E, -y_0', y_0(a/x_0^2 - E), x_0, x) \tag{3.5}$$

$\phi_0(E, y_0, y_0', \cdot, x)$ is continuously differentiable and hence absolutely continuous in $[\alpha, \beta]$. In fact, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=0}^{N} |\phi_0(E, y_0, y_0', t_{2k+1}, x) - \phi_0(E, y_0, y_0', t_{2k}, x)| < \varepsilon$$

whenever $x \in K$ and $t_0, ..., t_{2N+1}$ are points satisfying

$$\alpha \leq t_0 < t_1 \leq t_2 < ... < t_{2N-1} \leq t_{2N} < t_{2N+1} \leq \beta$$

and $\sum_{k=0}^{N}(t_{2k+1} - t_{2k}) < \delta$. Since

$$\begin{aligned}
& |\phi_n(E, y_0, y_0', t_{2k+1}, x) - \phi_n(E, y_0, y_0', t_{2k}, x)| \\
\leq \; & \int_{s_{2k}}^{s_{2k+1}} R(s')|\phi_{n-1}(E, y_0, y_0', t_{2k}, \gamma(s'))|ds' \\
& + \int_0^s R(s')|\phi_{n-1}(E, y_0, y_0', t_{2k+1}, \gamma(s')) - \phi_{n-1}(E, y_0, y_0', t_{2k}, \gamma(s'))|ds'
\end{aligned}$$

and because of the integrability of R one obtains for suitably small δ

$$\sum_{k=0}^{N} |\phi_1(E, y_0, y_0', t_{2k+1}, x) - \phi_1(E, y_0, y_0', t_{2k}, x)| \leq \varepsilon(M_1 + \hat{R}(s)).$$

An induction shows now that

$$\begin{aligned}
&\sum_{k=0}^{N} |\phi_n(E, y_0, y_0', t_{2k+1}, x) - \phi_n(E, y_0, y_0', t_{2k}, x)| \\
\leq \;& \varepsilon(M_1 \frac{(\hat{R}(1)+\hat{R}(s))^{n-1}}{(n-1)!} + \frac{\hat{R}(s)^n}{n!}) \leq \varepsilon M_3 \frac{M_3^{n-1}}{(n-1)!}
\end{aligned}$$

for a suitable positive constant M_3. In particular, for every $n \in \mathbb{N}_0$ the function $\phi_n(E, y_0, y_0', \cdot, x)$ is locally absolutely continuous in $\mathbb{R} - \{0\}$.

Now consider $\phi(E, y_0, y_0', \cdot, x)$. We have

$$\begin{aligned}
&\sum_{k=0}^{N} |\phi(E, y_0, y_0', t_{2k+1}, x) - \phi(E, y_0, y_0', t_{2k}, x)| \\
&= \sum_{k=0}^{N} | \lim_{M\to\infty} \sum_{n=0}^{M} (\phi_n(E, y_0, y_0', t_{2k+1}, x) - \phi_n(E, y_0, y_0', t_{2k}, x))| \\
&\leq \lim_{M\to\infty} \sum_{n=0}^{M} \sum_{k=0}^{N} |\phi_n(E, y_0, y_0', t_{2k+1}, x) - \phi_n(E, y_0, y_0', t_{2k}, x)| \\
&\leq \lim_{M\to\infty} (\varepsilon + \sum_{n=1}^{M} \varepsilon M_3 \frac{M_3^{n-1}}{(n-1)!}) = \varepsilon(1 + M_3 \exp(M_3))
\end{aligned}$$

which proves that $\phi(E, y_0, y_0', \cdot, x)$ is locally absolutely continuous in $\mathbb{R} - \{0\}$.

Differentiating the integral equation (3.1) with respect to x_0 and using (3.5) gives now:

$$\begin{aligned}
\frac{\partial \phi(E, y_0, y_0', x_0, x)}{\partial x_0} &= \phi_0(E, -y_0', (a/x_0^2 + q(x_0) - E)y_0, x_0, x) \\
&\quad - \int_{\gamma_s} s_0(E, x', x) q(x') \frac{\partial \phi(E, y_0, y_0', x_0, x')}{\partial x_0} dx'.
\end{aligned}$$

Because solutions of the integral equation are unique this proves the validity of (3.2). Equation (3.3) follows now straight from part (b). This concludes the proof of part (d) of the theorem.

The Bessel functions and their derivatives may be expressed by

$$\begin{aligned}
J_\nu(z) &= \sqrt{\frac{2}{\pi z}} \left(P_\nu(z)\cos\chi - Q_\nu(z)\sin\chi\right), \\
Y_\nu(z) &= \sqrt{\frac{2}{\pi z}} \left(P_\nu(z)\sin\chi + Q_\nu(z)\cos\chi\right), \\
J_\nu'(z) &= \sqrt{\frac{2}{\pi z}} \left(-R_\nu(z)\sin\chi - S_\nu(z)\cos\chi\right), \\
Y_\nu'(z) &= \sqrt{\frac{2}{\pi z}} \left(R_\nu(z)\cos\chi - S_\nu(z)\sin\chi\right)
\end{aligned}$$

where $\chi = z - \nu\pi/2 - \pi/4$. With these expressions one obtains

$$\begin{aligned} c_0(-k^2, x_0, x) &= \cos(k(x-x_0))(f_2(kx, kx_0) + g_1(kx, kx_0)/(2kx_0)) \\ &\quad + \sin(k(x-x_0))(g_2(kx, kx_0) - f_1(kx, kx_0)/(2kx_0)) \\ s_0(-k^2, x_0, x) &= \frac{\sin(k(x-x_0))}{k} f_1(kx, kx_0) - \frac{\cos(k(x-x_0))}{k} g_1(kx, kx_0) \end{aligned}$$

where

$$\begin{aligned} f_1(z, z_0) &= P_\nu(z)P_\nu(z_0) + Q_\nu(z)Q_\nu(z_0), \\ f_2(z, z_0) &= P_\nu(z)R_\nu(z_0) + Q_\nu(z)S_\nu(z_0), \\ g_1(z, z_0) &= P_\nu(z)Q_\nu(z_0) - Q_\nu(z)P_\nu(z_0), \\ g_2(z, z_0) &= P_\nu(z)S_\nu(z_0) - Q_\nu(z)R_\nu(z_0). \end{aligned}$$

The functions P_ν, R_ν, Q_ν, and S_ν have the following asymptotic behavior:

$$\begin{aligned} P_\nu(z), R_\nu(z) &= 1 + O(z^{-2}) \\ Q_\nu(z) &= \frac{4\nu^2 - 1}{8z} + O(z^{-3}) \\ S_\nu(z) &= \frac{4\nu^2 + 3}{8z} + O(z^{-3}) \end{aligned}$$

as z tends to infinity if $|\arg z| < \pi$ (see, e.g., Abramowitz and Stegun [1]). If $z = w\mathrm{e}^{i\pi}$ one finds

$$\begin{aligned} P_\nu(z) &= P_\nu(w) + i\cos(\nu\pi)\mathrm{e}^{-2iw}(P_\nu(w) - iQ_\nu(w)) \\ Q_\nu(z) &= -Q_\nu(w) - \cos(\nu\pi)\mathrm{e}^{-2iw}(P_\nu(w) - iQ_\nu(w)) \\ R_\nu(z) &= R_\nu(w) - i\cos(\nu\pi)\mathrm{e}^{-2iw}(R_\nu(w) - iS_\nu(w)) \\ S_\nu(z) &= -S_\nu(w) + \cos(\nu\pi)\mathrm{e}^{-2iw}(R_\nu(w) - iS_\nu(w)). \end{aligned}$$

This implies that $f_k(z, z_0)$ and $g_k(z, z_0)$, $k = 1, 2$, are bounded by a constant C_1 as long as z and z_0 are bounded away from zero and their arguments are in $[-\pi, \pi]$.

Let $k = \sqrt{-E}$ have its argument in $(-\pi, 0]$, i.e., $k = \kappa - i\eta$ where $\kappa \in \mathbb{R}$ and $\eta \geq 0$ and let $r = 1/|k|$. For $x_0 < 0$ and $x_1 > 0$ let $\gamma : [0, 1] \to \Omega_0$ be defined by

$$\gamma(t) = \begin{cases} x_0 - 3t(r + x_0) & \text{if } 0 \leq t \leq 1/3, \\ -r\exp(-i\pi(3t - 1)) & \text{if } 1/3 \leq t \leq 2/3, \\ r + (3t - 2)(x_1 - r) & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

Then $\arg(k\gamma(t)) \in (-\pi, \pi]$ and $|k\gamma(t)| \geq 1$ for all $t \in [0, 1]$. Therefore, if $x, x' \in \gamma([0, 1])$

$$\begin{aligned} |c_0(-k^2, x', x)| &\leq 4C_1 \exp(1 + \eta|\Re(x) - \Re(x')|), \\ |s_0(-k^2, x', x)| &\leq \frac{2C_1}{|k|} \exp(1 + \eta|\Re(x) - \Re(x')|). \end{aligned}$$

Now define

$$f(k,x) = \exp(-\eta(x-x_0))\phi(-k^2, y_0, y_0', x_0, x).$$

From (3.1) we get

$$\begin{aligned} f(k,x) &= \mathrm{e}^{-\eta(x-x_0)}\phi_0(-k^2, y_0, y_0', x_0, x) \\ &\quad - \int_{\gamma_s} s_0(-k^2, x', x) q(x') \mathrm{e}^{-\eta(x-x')} f(k,x') dx'. \end{aligned}$$

Therefore

$$|kf(k,x)| \le 4C_1 \mathrm{e}\, (|y_0||k| + |y_0'|) + 2C_1\mathrm{e} \int_0^1 |q(\gamma(s'))| F(k) |\gamma'(s')| ds'$$

where

$$F(k) = \max\{|f(k,x')| : x' \in \gamma([0,1])\}.$$

Now consider $\int_0^1 |q(\gamma(s'))||\gamma'(s')|ds'$. Since the integral associated with the part of γ which lies outside of U depends only on x_0 and x_1, since $|\gamma'(t)|$ is bounded by a constant depending only on x_0 and x_1, and since $|q(x')|$ is bounded by $Q|x'|^{\varepsilon-1} \le Q|k|^{1-\varepsilon}$ we obtain the existence of a constant $\tilde{Q}$ such that

$$\int_0^1 |q(\gamma(s'))||\gamma'(s')|ds' \le \tilde{Q}|k|^{1-\varepsilon}$$

for suitably large $|k|$. Hence

$$|f(k,x)| \le 4C_1\mathrm{e}\left(|y_0| + \frac{|y_0'|}{|k|}\right) + \frac{2C_1\mathrm{e}\tilde{Q}}{|k|^{\varepsilon}} F(k).$$

Since the right hand side of this inequality is independent of x it is, in fact, also a bound for $F(k)$. Hence

$$F(k) \le 4C_1\mathrm{e}\left(|y_0| + \frac{|y_0'|}{|k|}\right)\left(1 - 2C_1\mathrm{e}\tilde{Q}|k|^{-\varepsilon}\right)^{-1} \le 8C_1\mathrm{e}\left(|y_0| + \frac{|y_0'|}{|k|}\right)$$

provided $|k|$ is so large that $2C_1\mathrm{e}\tilde{Q}|k|^{-\varepsilon} \le 1/2$. Inserting this estimate into (3.1) and the derivative of this equation and taking into account that $s_0'(E,x',x) = c_0(E,x,x')$ gives the desired estimates. □

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