ON THE INVERSE RESONANCE PROBLEM FOR JACOBI OPERATORS — UNIQUENESS AND STABILITY

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1. Introduction

In this paper we obtain conditional stability estimates for the inverse resonance problem for a wide class of Jacobi-type operators, assuming only finitely many eigenvalues and resonances are known. More precisely, given two operators in an appropriate class for which the eigenvalues and resonances in some disc are respectively close, we estimate in Theorem 4.1, our main result, the difference of their coefficients. As a corollary we also obtain a uniqueness result, by considering the case where the eigenvalues and resonances of the two operators coincide respectively in any disc.

With only finite data available, we may expect that an inverse spectral problem for a Jacobi matrix will have infinitely many solutions. It is important to be able to prove that these solutions are all close, in some suitable sense. In experiments, only a finite amount of data can ever be obtained; for this reason finite data inverse problems are the only type of inverse problem which are ever solved in the real world.

Our main result (vastly) generalizes recent work on discrete Schrödinger operators in [13] and can be seen as a discrete analogue of the results for Schrödinger operators on a half-line in [11].

There has been a lot of interest in inverse spectral problems, both discrete and continuous, in the last sixty years, going back to the 1946 paper of Borg [1] and continuing with the seminal works of Gel'fand, Levitan and Marchenko, which are reviewed, for instance, in the books by Levitan [9] and Marchenko [10]. Most of this work concerns uniqueness rather than stability, although stability results are given by Ryabushko [16], McLaughlin [14] and in [12] for Schrödinger equations on compact intervals with L^2 potentials, and for L^p potentials, $p \geq 2$, in Horvath and Kiss [3]. There are also forthcoming stability results of Savchuk and Shkalikov, and of Hryniv and Mykytyuk (private communications from Hryniv and Shkalikov).

For periodic problems, both discrete and continuous, Korotyaev has proved several results, the most recent being a Borg-type theorem for periodic Jacobi matrices, joint work with Kutsenko [7], for problems with a sufficiently small periodic potential.

Regarding inverse resonance problems, we mention the work of Korotyaev [5], which presents a rather delicate estimate requiring full knowledge of the asymptotic behavior of eigenvalues and resonances, for Schrödinger equations with compactly supported potentials in a special class, and that in [6], which gives a complete characterization of the allowable data required for unique determination of the

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potential. Some conditional stability estimates are implicit from the work in these articles but require full knowledge of the spectral data.

The content of this paper is arranged as follows.

In Section 2 we assemble the basic theory of polynomials orthogonal with respect to a weight function. The coefficients in the three-term recurrence relations satisfied by these polynomials define what we shall take to be our unperturbed Jacobi operators. We also define the Titchmarsh-Weyl m-function for such operators – the m-function is the Stieltjes transform of the weight – and we use m to define the Jost solution of the unperturbed Jacobi equations. For three different classes of weights we describe the analytic continuation of the m-function through the essential spectrum, which induces an analytic continuation for the Jost solution onto a suitable Riemann surface.

Section 3 introduces perturbations to finitely many coefficients in the base Jacobi operators in Section 2. These perturbations change the values of the Jost solutions at finitely many sites. We introduce the Gel'fand transformators which map solutions of the unperturbed Jacobi equations to the solutions of the perturbed equations, and vice-versa, and obtain bounds on their kernels. These bounds also allow us to obtain bounds on the Jost functions.

In Section 4 we tackle the inverse resonance problem explicitly. It addresses the following problem: suppose that a base problem is fixed and that we consider the class of all perturbations to the Jacobi coefficients for which the resonances inside a disc of large radius R are very close; how small are the perturbations to the Jacobi coefficients? The methods are adapted and generalized from [11] but involve substantial additional technicalities due to the fact that we treat here a wide class of different possible Jost functions. Note that nothing is assumed about the resonances outside the disc of radius R. We emphasize that we do not use the asymptotic behavior of resonances established in Section 5. As an immediate consequence we establish in Corollary 4.2 that the coefficients are uniquely determined by the eigenvalues and resonances. Our bounds are not as tight and explicit as those we obtained in [11] for the Schrödinger operator on the half-line, because the bounds on the transformator kernel in Section 3 are more crude. This is because there is no D'Alembert formula for a discrete wave equation with non-constant coefficients of the type which we consider here. It is likely that the bounds can be somewhat improved in special cases, but this would detract from the generality of the present work.

Finally, Section 5 reviews and generalizes the work in [2] on location of resonances. We obtain asymptotic formulae describing the location of the n-th resonance z_n for large n, in the cases where there are infinitely many resonances.

2. Base operators

2.1. **Weight functions.** Suppose w is a nonnegative function on $\mathbb{R}$ so that both $\int_{\mathbb{R}} w(t)dt = 1$ and $\int_{\mathbb{R}} |t|^n w(t)dt < \infty$ for all $n \in \mathbb{N}$. It is then well known that there is a sequence of polynomials $p(n,\cdot)$, $n \in \mathbb{N}_0$, with real coefficients, such that $\deg p(n,\cdot) = n$ and

$$\int_{\mathbb{R}} p(n,t)p(m,t)w(t)dt = \delta_{n,m}.$$

The $p(n,\cdot)$ are uniquely defined, if one also requires that their leading coefficients are positive. In particular, $p(0,t) = 1$. One may also define polynomials $q(n,\cdot)$,

$n \in \mathbb{N}_0$, by setting

$$q(n,z) = \int_{\mathbb{R}} \frac{p(n,t) - p(n,z)}{t - z} w(t) dt.$$

One sees easily that $q(n,\cdot)$ is a polynomial of degree $n-1$ whose leading coefficient coincides with that of $p(n,\cdot)$ provided $n \geq 1$ and that $q(0,\cdot) = 0$.

For suitable constants $\alpha_{n,k}$, we have $tp(n,t) = \sum_{k=0}^{n+1} \alpha_{n,k} p(k,t)$, where $\alpha_{n,n+1} \neq 0$. Define also $\alpha_{n,k} = 0$ for $k > n+1$. Then, for arbitrary $n, m \in \mathbb{N}_0$,

$$\begin{aligned} \alpha_{n,m} = \int_{\mathbb{R}} tp(n,t)p(m,t)w(t)dt &= \int_{\mathbb{R}} p(n,t)tp(m,t)w(t)dt \\ &= \sum_{k=0}^{m+1} \alpha_{m,k} \int_{\mathbb{R}} p(n,t)p(k,t)w(t)dt = \alpha_{m,n}. \end{aligned}$$

This implies immediately that $\alpha_{n,m} = 0$ unless $|n-m| \leq 1$. In particular, setting $a_{n-1} = \alpha_{n,n-1}$ and $b_n = \alpha_{n,n}$, the $p(n,\cdot)$ satisfy the three-term recurrence relation

$$a_{n-1}p(n-1,t) + b_n p(n,t) + a_n p(n+1,t) = tp(n,t), \quad n \in \mathbb{N}_0$$

ignoring the term $a_{n-1}p(n-1,t)$ when $n = 0$. Henceforth we will take the point of view that $a_{-1} = 1$ and $p(-1,\cdot) = 0$. Of course the polynomials $q(n,\cdot)$ must satisfy the same recurrence relation upon a proper definition of $q(-1,\cdot)$, i.e.,

$$q(-1,t) = -a_0 q(1,t) = -1.$$

The latter equality follows since $a_0 p(1,t) = t - b_0$ implying that the leading coefficient of $a_0 p(1,\cdot)$, which equals $a_0 q(1,t)$, is one.

We emphasize that the coefficients a_n and b_n are real. Also, for any $n \in \mathbb{N}_0$, the number a_n is different from zero since it is a factor of the leading coefficient of the polynomial $t \mapsto p(n,t)$.

2.2. **The Jacobi operator associated with a weight function.** In the following we denote by S_N, for $N = -1$ or $N = 0$, the set of all complex-valued sequences defined on $\{N, N+1, N+2, ...\}$, i.e., $S_0 = \mathbb{C}^{\mathbb{N}\cup\{0\}}$ and $S_{-1} = \mathbb{C}^{\mathbb{N}\cup\{-1,0\}}$. We embed S_0 into S_{-1} by appending a zero on the left, i.e., by the function $S_0 \to S_{-1} : y \mapsto \hat{y}$ where $\hat{y}(-1) = 0$ and $\hat{y}(n) = y(n)$ for $n \in \mathbb{N}_0$.

Given numbers a_n and b_n we define now the Jacobi difference expression $\mathcal{J} : S_{-1} \to S_0$ by setting

$$(\mathcal{J}y)(n) = a_{n-1}y(n-1) + b_n y(n) + a_n y(n+1).$$

Finally, given a Jacobi difference expression we define an operator J in $\ell^2(\mathbb{N}_0)$ with domain of definition

$$\mathcal{D} = \{y \in \ell^2(\mathbb{N}_0) : \mathcal{J}\hat{y} \in \ell^2(\mathbb{N}_0)\}$$

by setting $Jy = \mathcal{J}\hat{y}$. We emphasize that our embedding of S_0 into S_{-1} amounts to a Dirichlet boundary condition on the left.

2.3. **The Titchmarsh-Weyl function.** When $\mathrm{Im}(\lambda) \neq 0$ the function $t \mapsto 1/(t-\lambda)$ is in $L^2(wdt)$. Thus, introducing for $k \in \mathbb{N}_0$ the numbers

$$\psi(k,\lambda) = \int_{\mathbb{R}} \frac{p(k,t)}{t-\lambda} w(t) dt,$$

Bessel's inequality shows that

$$\sum_{k=0}^{\infty} |\psi(k,\lambda)|^2 \leq \int_{\mathbb{R}} \frac{w(t)}{|t-\lambda|^2} dt < \infty$$

and hence that $k \mapsto \psi(k,\lambda)$ is square summable. Moreover, defining

$$m(\lambda) = \int_{\mathbb{R}} \frac{w(t)}{t-\lambda} dt,$$

the Stieltjes transform of w, we obtain

$$\begin{aligned} \psi(k,\lambda) &= \int_{\mathbb{R}} \frac{p(k,t)}{t-\lambda} w(t) dt \\ &= \int_{\mathbb{R}} \frac{p(k,t)-p(k,\lambda)}{t-\lambda} w(t) dt + p(k,\lambda) \int_{\mathbb{R}} \frac{w(t)}{t-\lambda} dt \\ &= q(k,\lambda) + m(\lambda) p(k,\lambda). \end{aligned}$$

Defining also $\psi(-1,\lambda) = q(-1,\lambda) + m(\lambda)p(-1,\lambda) = -1$ we see that $\psi(\cdot,\lambda)$ is a square summable solution of the Jacobi difference equation

$$(\mathcal{J}\psi(\cdot,\lambda))(n) = \lambda\psi(n,\lambda).$$

The function m is called the Titchmarsh-Weyl m-function associated with the weight function w. The function m is analytic away from the support of w. It may be possible to analytically extend it across the support of w as a multi-valued function or, equivalently, as a single-valued function defined on some Riemann surface covering the complex plane. We will next describe three classes of weights for which this Riemann surface is particularly simple and which will be the topic of our investigation.

Below we denote the positively oriented unit circle by $\mathbb{K}$. The closed lower half of $\mathbb{K}$, with orientation inherited from $\mathbb{K}$, is denoted by $\mathbb{K}_-$, while the upper half is denoted by $\mathbb{K}_+$.

Definition 1. We shall say that the function w is of class (1), (2), or (3) if it is a nonnegative function on $\mathbb{R}$ so that $\int_{\mathbb{R}} w(t)dt = 1$, if $\int_{\mathbb{R}} |t|^n w(t)dt < \infty$ for all $n \in \mathbb{N}$, and if, respectively, one of the following conditions is satisfied:

(1) w is supported on all of $\mathbb{R}$ and extends to an entire function.
(2) w is supported on $[0,\infty)$ and the map $u : [0,\infty) \to [0,\infty) : s \mapsto w(s^2)$ extends to an odd entire function.
(3) w is supported on $[-2,2]$ and there is an entire function f vanishing at the origin such that $w(s+1/s) = f(s) - f(1/s)$ for all $s \in \mathbb{K}_-$.

Lemma 2.1. *If w is of class (1), then the associated Titchmarsh-Weyl m-function defined in the upper (lower) half of the λ-plane can be extended across the real line to an entire function. More precisely, there is a function $M : \mathbb{C} \times \{-1,+1\} \to \mathbb{C}$ such that $M(\cdot,-1)$ and $M(\cdot,+1)$ are entire and $m(\lambda) = M(\lambda, \operatorname{sgn}(\operatorname{Im}(\lambda)))$ when $\operatorname{Im}(\lambda) \neq 0$.*

Proof. It is clear that m is analytic on both the upper and lower half-plane. Fix $\nu > 0$. Let γ be the contour obtained from the real line by replacing the interval $[-\nu,\nu]$ by a semi-circle through the lower half-plane connecting $-\nu$ to ν. Since w is entire we have

$$m(\lambda) = \int_{\mathbb{R}} \frac{w(t)}{t-\lambda} dt = \int_{\gamma} \frac{w(t)}{t-\lambda} dt$$

as long as λ is in the upper half plane. But the integral on the right defines an analytic function for all λ above the contour. Since ν is arbitrary m can be extended analytically from the upper half-plane to the lower half-plane. This entire function is $M(\cdot,+1)$. $M(\cdot,-1)$ is defined similarly, starting in the lower half-plane. □

If λ is in the lower half plane, the residue theorem gives

$$M(\lambda,+1)=\int_{\mathbb{R}}\frac{w(t)}{t-\lambda}dt+\oint_K\frac{w(t)}{t-\lambda}dt=M(\lambda,-1)+2\pi i w(\lambda) \tag{1}$$

when K denotes a positively oriented circle of sufficiently large radius centered at zero. Of course, all constituents being entire, the formula $M(\lambda,+1)=M(\lambda,-1)+2\pi i w(\lambda)$ holds in fact for all λ.

Lemma 2.2. *Suppose that w is of class (2) and let m denote the associated Titchmarsh-Weyl m-function. Then the function M, defined by $M(z)=m(z^2)$ in the upper half-plane, extends across the real axis to an entire function.*

Proof. Define the function u by

$$u(s)=\begin{cases} w(s^2) & \text{if } s\ge 0\\ -w(s^2) & \text{if } s\le 0\end{cases}.$$

Then, by assumption, u has an odd entire extension to the complex plane. Let $\lambda=z^2$ with $\mathrm{Im}(z)>0$ and $t=s^2$. Then

$$M(z)=\int_0^\infty 2s\frac{w(s^2)}{s^2-z^2}ds=\int_0^\infty\left(\frac{w(s^2)}{s-z}+\frac{w(s^2)}{s+z}\right)ds=\int_{-\infty}^\infty\frac{u(s)}{s-z}ds.$$

One shows that M is entire by repeating the proof of Lemma 2.1. □

We note that $M(z)=m(z^2)+2\pi i u(z)$, if $\mathrm{Im}(z)<0$.

Lemma 2.3. *Suppose that w is of class (3) and let m denote the associated Titchmarsh-Weyl m-function. Then the function M, defined by $M(z)=m(z+1/z)$ in the punctured open unit disc, extends across the unit circle to an analytic function on the punctured complex plane. It has a removable singularity at zero so that, in fact, it represents an entire function.*

Proof. Let $\lambda=z+1/z$ with $0<|z|<1$ and $t=s+1/s$. Then

$$\frac{1-1/s^2}{t-\lambda}=\frac{1}{s-z}-\frac{1}{s-zs^2}$$

so that

$$\begin{aligned}M(z)&=\int_{-2}^2\frac{w(t)}{t-\lambda}dt\\ &=\int_{\mathbb{K}_-}\frac{f(s)}{s-z}ds-\int_{\mathbb{K}_-}\frac{f(1/s)}{s-z}ds-\int_{\mathbb{K}_-}\frac{f(s)}{s-zs^2}ds+\int_{\mathbb{K}_-}\frac{f(1/s)}{s-zs^2}ds.\end{aligned}$$

On the right hand side we change variables from s to $1/s$ in the second and forth integral to get

$$M(z)=\oint_{\mathbb{K}}\frac{f(s)}{s-z}ds-\oint_{\mathbb{K}}\frac{f(s)}{s-zs^2}ds.$$

Since f is entire, $f(0) = 0$, and $|z| < 1$ the second integral is zero so that

$$M(z) = \oint_{\mathbb{K}} \frac{f(s)}{s - z} ds = 2\pi i f(z)$$

which extends to an entire function. □

Note that, in each of the three cases, the function M is defined on a twofold cover $\mathcal{R}$ of the complex plane in which the spectral parameter varies. Denoting the canonical projection of $\mathcal{R}$ by π we have in the first case $\mathcal{R} = \mathbb{C} \times \{-1, +1\}$ and $\pi(z, \sigma) = z$; in the second case $\mathcal{R} = \mathbb{C}$ and $\pi(z) = z^2$; and in the last case $\mathcal{R} = \mathbb{C} - \{0\}$ and $\pi(z) = z + 1/z$.

Lemma 2.4. *If the entire function associated with the weight function (w, u, or f respectively in the cases (1), (2), and (3)) has finite growth order ρ then so does the function M (in case (1) both $M(\cdot, +1)$ and $M(\cdot, -1)$).*

Proof. There is nothing to prove in case (3) since M is a multiple of f. The proofs in cases (1) and (2) are almost identical and we present a proof for case (2) only.

First consider M in the upper half plane where we have trivially $|M(z)| \leq 1/\operatorname{Im}(z)$ since $\int_{-\infty}^{\infty} u ds = 1$. We obtain a better estimate for points on or near the real axis in the following way. If $0 \leq \operatorname{Im}(z) \leq 1$ let $A = \{s \in \mathbb{R} : |s - z| \leq 1\}$ and $g_z(s) = (u(s) - u(z))/(s - z)$. Then

$$|M(z)| \leq 1 + \Big| \int_A \frac{u(z) + (s - z) g_z(s)}{s - z} ds \Big| \leq 1 + \pi |u(z)| + \int_A |g_z| ds$$

since the integral $\int_A 1/(s-z) ds$ has to be considered a principal value integral when z is real.

Using Taylor's theorem $g_z(s)$ may be represented by an integral over a circle of radius 2 centered at z involving the values of u on that circle. In any case this proves that, in the closed upper half plane, M can not grow at a larger order than u. In the lower half plane we have, as mentioned above, $M(z) = M(-z) + 2\pi i u(z)$. □

2.4. **Examples.**

(1) An example of a weight of class (1) is $w(t) = \mathrm{e}^{-t^2/2} / \sqrt{2\pi}$, $t \in \mathbb{R}$, which gives rise to (scaled) Hermite polynomials $p(n, z) = H_n(z/\sqrt{2})/\sqrt{2^n n!}$. The associated Jacobi expression is given by $a_n = \sqrt{n + 1}$ and $b_n = 0$ for $n \in \mathbb{N}_0$.

(2) Associated Laguerre polynomials $L_n^{(\alpha)}$ where $\alpha = (2k+1)/2$ is half of an odd positive integer are related to the weight functions $w(t) = t^{k+1/2} \mathrm{e}^{-t} / \Gamma(k + 3/2)$, $t \in [0, \infty)$ which are examples for weights of class (2). In fact

$$p(n, t) = (-1)^n \sqrt{\frac{n! \, \Gamma(k + 3/2)}{\Gamma(k + 3/2 + n)}} L_n^{(k+1/2)}(t).$$

The associated Jacobi expression is given by

$$a_n = \sqrt{(n + 1)(n + k + 3/2)} \quad \text{and} \quad b_n = 2n + k + 3/2$$

for $n \in \mathbb{N}_0$.

(3) Ultraspherical or Gegenbauer polynomials $C_n^{(\alpha)}$ where $\alpha = k+1$ is a positive integer yield examples for weights of class (3). In this case we have

$$w(t) = \frac{(k + 1)!}{\pi 2^{k+1} (2k + 1)!!} (4 - t^2)^{k+1/2}, \quad t \in [-2, 2]$$

and

$$p(n,t) = \sqrt{\frac{2^k(2k+1)!!k!n!(n+k+1)}{(k+1)(n+2k+1)!}}\, C_n^{(k+1)}(t/2).$$

The coefficients for the Jacobi expression are

$$a_n = \sqrt{\frac{(n+1)(n+2k+2)}{(n+k+1)(n+k+2)}}, \quad \text{and} \quad b_n = 0.$$

In particular, $a_n = 1$ and $b_n = 0$, the discrete Schrödinger case, occurs when $k = 0$.

3. Perturbations of base operators

In this section we consider certain perturbations of a Jacobi operator J_0 (base operator). For the remainder of the paper we make the following assumption:

Hypothesis 1. The spectral density of J_0 falls in one of three classes introduced in the previous section. Moreover, the associated entire function has finite growth order ρ.

Subsequently any quantity associated with J_0 will have an index zero, in particular we have the weight function w_0, the polynomials $p_0(n,\cdot)$ and $q_0(n,\cdot)$, the coefficients of the recurrence relation $a_{0;n}$ and $b_{0;n}$, the Titchmarsh-Weyl function m_0 and its extension M_0 defined on the Riemann surface $\mathcal{R}_0$ with the projection π_0. We also introduce the function $\Psi_0 : \mathbb{N}_{0,-1} \times \mathcal{R}_0 \to \mathbb{C}$ by

$$\Psi_0(n,z) = q_0(n,\lambda) + M_0(z)p_0(n,\lambda),$$

where $\lambda = \pi_0(z)$, which may be viewed as an extension of the *Jost solution*

$$\psi_0(n,\lambda) = q_0(n,\lambda) + m_0(\lambda)p_0(n,\lambda)$$

of $\mathcal{J}_0 y = \lambda y$. Note that $\mathcal{J}_0\Psi_0(\cdot,z) = \lambda\Psi_0(\cdot,z)$ for both values of z lying above $\lambda = \pi_0(z)$. If λ is outside the spectrum of J_0, i.e., the support of the spectral density w_0, then for precisely one of the two points above λ we have $M_0(z) = m_0(\lambda)$. This part of $\mathcal{R}_0$ is called the physical sheet of $\mathcal{R}_0$ while the other one is called the unphysical sheet. The physical and the unphysical sheet are joined by the preimage of the spectrum of J_0 under the projection π_0.

Since M_0 is entire, it is clear, in cases (1) and (2) that the (components of the) functions $\Psi_0(m,\cdot)$ are entire for all $m \in \mathbb{N}_{0,-1}$. In case (3), however, the polynomials p_0 and q_0 are polynomials in $z+1/z$ so that $z = 0$ becomes a singularity. To investigate it we define $h(k,s) = p_0(k, s+1/s)f(s)$ so that $h(k,s) - h(k,1/s) = p_0(k,t)w_0(t)$ for $t = s+1/s$ and $s \in \mathbb{K}_-$. Repeating the idea in the proof of Lemma 2.3 shows

$$\Psi_0(k,z) = \oint_{\mathbb{K}} h(k,s)\Big(\frac{1}{s-z} - \frac{1}{s-zs^2}\Big)ds$$

so that the functions $\Psi_0(k,\cdot)$ turn out to have a removable singularity at $z = 0$ and thus may be thought of as entire in this case also.

The perturbations J of J_0 which are allowed are such that the equations $\mathcal{J}y = \lambda y$ have solutions which are asymptotic to $\psi_0(\cdot,\lambda)$ whenever λ is not real. This is certainly the case when only finitely many of the coefficients a_n and b_n of $\mathcal{J}$ are different from the respective $a_{0;n}$ and $b_{0;n}$. We will also require an a priori bound on the coefficients. More precisely, henceforth we assume that $\mathcal{J}$ is in the class $B_0(N,Q)$ defined next.

Definition 2. Suppose a Jacobi difference expression $\mathcal{J}_0$ with coefficients $a_{0;n}$ and $b_{0;n}$ is given. Then the Jacobi expression $\mathcal{J}$ is said to be in the class $B_0(N,Q)$ if the following two conditions on the coefficients a_n and b_n of $\mathcal{J}$ are satisfied. (1) $a_{n-1}=a_{0;n-1}$ and $b_n=b_{0;n}$ for all $n>N$. (2) $|b_n|\le Q$, $|a_n|\le Q$, and $1/|a_n|\le Q$ for all $n\in\{0,...,N\}$.

We emphasize that even for $N=0$ the coefficient b_0 may be different from $b_{0;0}$.

3.1. **Transformation operators.** Suppose two Jacobi difference expressions $\mathcal{J}$ and $\tilde{\mathcal{J}}$ with the respective coefficients a_n, b_n and $\tilde a_n$, $\tilde b_n$ are given. An operator $\mathcal{K}: S_{-1}\to S_{-1}$ such that $\tilde{\mathcal{J}}\mathcal{K}=\mathcal{K}\mathcal{J}$ is called a transformation operator for $(\mathcal{J},\tilde{\mathcal{J}})$, if it is of the form

$$(\mathcal{K}f)(n)=k(n)f(n)+\sum_{m=n+1}^{\infty}K(n,m)f(m)$$

with appropriate coefficients $k(n)$, $n\ge -1$, and $K(n,m)$, $m>n\ge -1$ (here we abuse notation slightly by not distinguishing $\mathcal{K}$ from its restriction to S_0). Henceforth we set $K(n,m)=0$ when $m\le n$ for convenience. As an abbreviation we also introduce the "d'Alembertian"

$$(\Box K)(n,m)=\tilde a_{n-1}K(n-1,m)+\tilde a_nK(n+1,m) \\ -a_{m-1}K(n,m-1)-a_mK(n,m+1).$$

A direct computation shows that, if $\mathcal{K}$ is a transformation operator for $(\mathcal{J},\tilde{\mathcal{J}})$, the conditions

$$\tilde a_{n-1}k(n-1)=a_{n-1}k(n)$$

and

$$(\Box K)(n,m)=(b_m-\tilde b_n)K(n,m) \\ +(k(n)a_n-k(n+1)\tilde a_n)\delta_{m,n+1}+(b_n-\tilde b_n)k(n)\delta_{m,n} \quad (2)$$

must necessarily hold. These conditions are also sufficient provided that the sums involved are convergent.

In the following we will deal with finite perturbations of $\mathcal{J}_0$ only. More precisely, we require that both $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are in the class $B_0(N,Q)$ introduced by Definition 2. Then a transformation operator $\mathcal{K}$ for $(\mathcal{J},\tilde{\mathcal{J}})$ exists and one shows by induction that

(1) $k(n)=1$ when $n\ge N$,
(2) $K(n,m)=0$ when $m+n\ge 2N$,
(3) $k(n)=\prod_{j=n}^{N-1}a_j/\tilde a_j$ when $n\le N-1$,
(4) and, setting $q_{n,j}=\sum_{\ell=n+1}^{j}(b_{\ell+1}-\tilde b_\ell)$,

$$K(n,n+1)=\frac{k(n)}{a_n}\sum_{j=n+1}^{N}(b_j-\tilde b_j),\quad n\le N-1 \quad (3)$$

and

$$K(n,n+2) = \frac{k(n)}{a_n a_{n+1}} \sum_{j=n+1}^{N-1} (q_{n,j}(b_{j+1} - \tilde{b}_{j+1}) + a_j^2 - \tilde{a}_j^2), \quad n \le N-2. \tag{4}$$

Next we are interested in the interrelationship between the transformation operator $\mathcal{K}_0$ for $(\mathcal{J}_0, \mathcal{J})$, the operator $\tilde{\mathcal{K}}_0$ for $(\mathcal{J}_0, \tilde{\mathcal{J}})$, and the operator $\mathcal{K}$ for $(\mathcal{J}, \tilde{\mathcal{J}})$. First one recognizes that, because of the triangular structure of the kernel, any transformation operator is one-to-one and onto. If $\mathcal{L}_0 = \mathcal{K}_0^{-1}$ denotes the transformation operator for $(\mathcal{J}, \mathcal{J}_0)$ we have $k(n) = \tilde{k}_0(n)\ell_0(n)$ and

$$K(n,m) = \tilde{k}_0(n) L_0(n,m) + \tilde{K}_0(n,m)\ell_0(m) + \sum_{k=n+1}^{m-1} \tilde{K}_0(n,k) L_0(k,m).$$

In particular, setting $\mathcal{J} = \tilde{\mathcal{J}}$, we see that $1 = k_0(n)\ell_0(n) = \tilde{k}_0(n)\tilde{\ell}_0(n)$ and

$$0 = k_0(n) L_0(n,m) + K_0(n,m)\ell_0(m) + \sum_{k=n+1}^{m-1} K_0(n,k) L_0(k,m).$$

Combining these expressions we arrive at

$$\begin{aligned} \tilde{\ell}_0(n) K(n,m) &= (\tilde{\ell}_0(n)\tilde{K}_0(n,m) - \ell_0(n) K_0(n,m))\ell_0(m) \\ &\quad + \sum_{k=n+1}^{m-1} (\tilde{\ell}_0(n)\tilde{K}_0(n,k) - \ell_0(n) K_0(n,k)) L_0(k,m). \end{aligned} \tag{5}$$

We have the following crude estimate for the numbers $K(n,m)$.

Lemma 3.1. *For $n < m < 2N - n$, $n \ge -1$, we have the bound*

$$|K(n,m)| \le \mathfrak{c}_1 {\mathfrak{M}_1}^{N-n-1}$$

in which

$$\mathfrak{M}_1 = \max\big\{1, \max_{n+2<m<2N-n} (|a_m| + |a_{m-1}| + |\tilde{a}_{n+1}| + |b_m - \tilde{b}_{n+1}|)/|\tilde{a}_{n-1}|\big\}$$

and

$$\mathfrak{c}_1 = \max_{n \ge -1} \max(|K(n,n+1)|, |K(n,n+2)|).$$

Proof. We rearrange the equation (2) in the form

$$\begin{aligned} K(n,m) = \frac{1}{\tilde{a}_n}\Big\{ &a_m K(n+1,m+1) + a_{m-1} K(n+1,m-1) - \tilde{a}_{n+1} K(n+2,m) \\ &+ (b_m - \tilde{b}_{n+1}) K(n+1,m)\Big\} \end{aligned}$$

in which we have shifted the index n by 1 and then eliminated the terms involving $\delta_{m,n+1}$ and $\delta_{m,n+2}$ since we shall not be using the corresponding equations. Thus $|K(n,m)| \le \mathfrak{M}_1 \max(|K(n{+}1,m{-}1)|, |K(n{+}1,m)|, |K(n{+}1,m{+}1)|, |K(n{+}2,m)|)$. Starting on the line $m - n = 3$ we find first that $|K(N-2,N+1)| \le \mathfrak{c}_1\mathfrak{M}_1$ and by induction

$$|K(N-1-k, N+2-k)| \le \mathfrak{c}_1 \mathfrak{M}_1^k$$

for $k = 1, ..., N$ noting that $K(n,m) = 0$ in the region $m + n \ge 2N$ (see Figure 1). One then sweeps down the line $m - n = 4$ from its top right-hand point for which we have $|K(N-3,N+1)| \le \mathfrak{c}_1\mathfrak{M}_1^2$, and so on; in general the values of $K(n,m)$ on

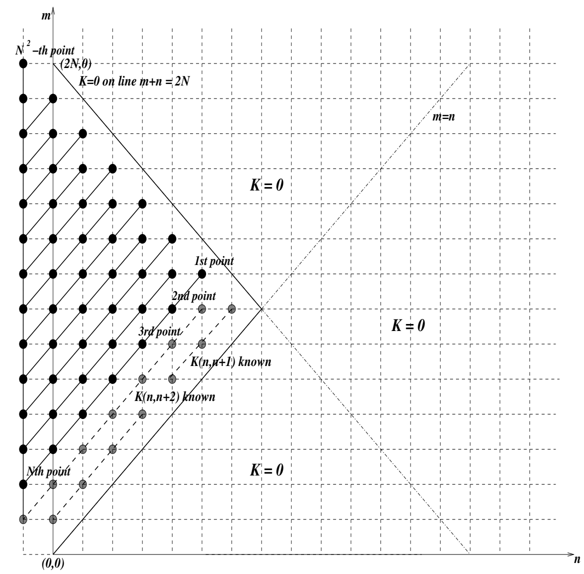

FIGURE 1. Grid of points for solving the difference equation for $K(n, m)$ for $N = 7$ from right to left.

the line $m - n = j$ can be filled in once all the values on the lines $m - n = j - 1$ and $m - n = j - 2$ are known. If $j = 2\ell + 1$ is odd one sees that

$$|K(N - \ell - k, N + 1 + \ell - k)| \leq \mathfrak{c}_1 \mathfrak{M}_1^{\ell - 1 + k}$$

for $k = 1, ..., N + 1 - \ell$. If $j = 2\ell$ is even we get instead

$$|K(N - \ell - k, N + \ell - k)| \leq \mathfrak{c}_1 \mathfrak{M}_1^{\ell - 1 + k}$$

for $k = 1, ..., N + 1 - \ell$. Thus the estimates for $|K(n, m)|$ depend on n only and are as we claimed. □

Corollary 3.2. *There is a constant C_0 depending only on N and Q such that $|K(n, m)| \leq C_0$ whenever $\mathcal{J} \in B_0(N, Q)$.*

In the previous Lemma we have obtained a crude bound on the coefficients $K(n, m)$ of the transformation operator $\mathcal{K}$ for $(\mathcal{J}, \tilde{\mathcal{J}})$. This bound was in terms of the values $K(n, n+1)$ and $K(n, n+2)$, i.e., in terms of the values of the coefficients a_n, b_n and $\tilde{a}_n$, $\tilde{b}_n$. Now we will establish a bound on the $K(n, m)$ in terms of the values on the line $n = -1$. This will be crucial for the solution of the inverse problem.

Lemma 3.3. *The transformation coefficients $K(n, m)$ admit the bound*

$$|K(n, m)| \leq \mathfrak{c}_2 \mathfrak{M}_2^{2N - m}$$

in which

$$\mathfrak{M}_2 = \max\left\{1, \max_{n+m \leq 2N-1} (|\tilde{a}_n| + |\tilde{a}_{n-1}| + |a_{m+1}| + |\tilde{b}_n - b_{m+1}|)/|a_m|\right\},$$

$$\mathfrak{c}_2 = \max_{2 \leq m \leq 2N} \frac{|\tilde{a}_{-1}||K(-1, m)|}{|a_{m-1}|}.$$

Proof. We rearrange the equation (2), i.e.,

$$(\Box K)(n,m) = (b_m - \tilde{b}_n)K(n,m) + (k(n)a_n - k(n+1)\tilde{a}_n)\delta_{n+1,m} + (b_n - \tilde{b}_n)k(n)\delta_{n,m}$$

as

$$K(n,m) = \frac{1}{a_m}\Big(\tilde{a}_{n-1}K(n-1,m+1) + \tilde{a}_n K(n+1,m+1) - a_{m+1}K(n,m+2) + (\tilde{b}_n - b_{m+1})K(n,m+1)\Big)$$

in which we shifted the index m by one and then removed the terms involving $\delta_{n,m+1}$ and $\delta_{n+1,m+1}$ since we shall not be using the corresponding equations. Thus

$$|K(n,m)| \le \mathfrak{M}_2 \max\{|K(n-1,m+1)|, |K(n,m+1)|, |K(n+1,m+1)|, |K(n,m+2)|\}.$$

Starting on the line $m+n = 2N-1$ we find $|K(0,2N-1)| \le \mathfrak{c}_2\mathfrak{M}_2$ and by induction

$$|K(n,2N-n-1)| \le \mathfrak{c}_2\mathfrak{M}_2^{n+1}$$

for $n = 0,...,N-1$ noting that $|K(-1,2N)| \le \mathfrak{c}_2$ and $K(n,m) = 0$ in the region $m+n \ge 2N$ (see Figure 2). One then sweeps down the line $m+n = 2N-2$ from its top left-hand corner for which we have $|K(0,2N-2)| \le \mathfrak{c}_2\mathfrak{M}_2^2$, and so on; in general the values of K on the line $m+n = 2N-j$ can be filled in once all the values on the lines $m+n = 2N-j+1$ and $m+n = 2N-j+2$ as well as those on the line $n = -1$ are known. A double induction establishes that

$$|K(n,2N-n-j)| \le \mathfrak{c}_2\mathfrak{M}_2^{n+j}$$

as long as $2n < 2N-j$. The largest bound thus encountered is the one for $|K(0,1)|$ which is bounded by $\mathfrak{c}_2\mathfrak{M}_2^{2N-1}$. $\Box$

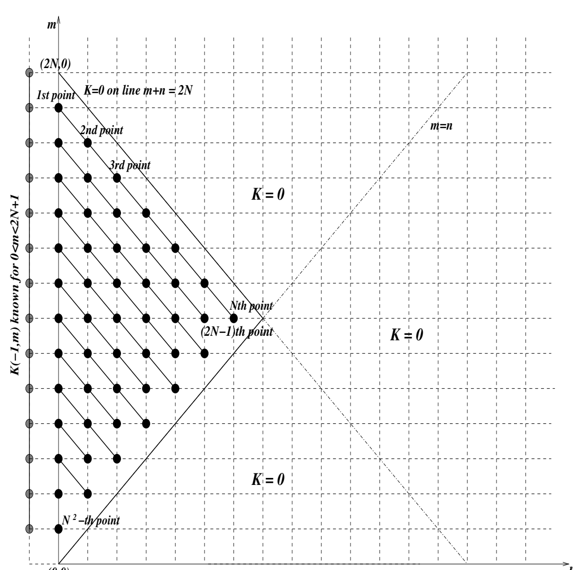

FIGURE 2. Grid of points for solving the difference equation for $K(n,m)$ from left to right.

3.2. **Jost solutions.** We now apply the transformation operator $\mathcal{K}_0$ for $(\mathcal{J}_0, \mathcal{J})$ to the sequence $\Psi_0(\cdot, z)$. This yields, for any $z \in \mathcal{R}_0$, a sequence $\Psi(\cdot, z)$. Specifically,

$$\Psi(n, z) = k_0(n)\Psi_0(n, z) + \sum_{m=n+1}^{2N-n-1} K_0(n, m)\Psi_0(m, z). \tag{6}$$

$\Psi(\cdot, z)$ solves $\mathcal{J}y = \lambda y$ with $\lambda = \pi_0(z)$ and it is square summable whenever $\Psi_0(\cdot, z)$ is, i.e., for z on the physical sheet of $\mathcal{R}_0$. It will be called the Jost solution for $\mathcal{J}$. It follows immediately from (6) that the functions $\Psi(n, \cdot)$ are all analytic on $\mathcal{R}_0$, i.e., they represent two entire functions if J_0 is in Class (1) and one entire function, if J_0 is in Class (2) or (3). The function $\Psi(-1, \cdot)$ is of particular importance and is called the Jost function for (J, J_0). While $\Psi_0(-1, \cdot)$, which is identically equal to -1, has no zeros, the zeros of $\Psi(-1, \cdot)$ play a significant role. If z is a zero of $\Psi(-1, \cdot)$ and lies on the physical sheet, then $\pi_0(z)$ is an eigenvalue of J and $\Psi(\cdot, z)$ is an eigenfunction. Conversely, if λ is an eigenvalue of J then there is point z on the physical sheet of $\mathcal{R}_0$ lying above λ so that $\Psi(-1, z) = 0$. On the other hand, the zeros of $\Psi(-1, \cdot)$ which do not lie on the physical sheet, or rather their images under π_0, are called *resonances*.

We now establish the following estimate for the Jost function.

Lemma 3.4. *For every nonnegative integer N and every positive number Q there is a positive constant $C_1 \geq 1$ such that the Jost function associated with any Jacobi difference expression $\mathcal{J} \in B_0(N, Q)$ has the following properties.*

(1) $|\Psi(-1, z)| \leq C_1 \,\mathrm{e}^{|z|^\kappa}$ *for all $z \in \mathcal{R}_0$ where κ is the smallest integer strictly larger than ρ.*
(2) $|\Psi(-1, z) \prod_{n=0}^{N-1} \frac{a_n}{a_{0;n}} + 1| \leq C_1/d$ *for all z in the physical sheet for which the distance of $\pi_0(z)$ to the support of w_0 is at least d.*

Remark 1. The restriction on κ in (1) to be the smallest integer strictly larger than ρ is made for later convenience. In general any κ larger than ρ would do here, sometimes it would be permissible to have $\kappa = \rho$.

Proof. The first statement is obvious in view of Corollary 3.2 and the fact that the functions $\Psi_0(m, \cdot)$ are constructed from polynomials and the function M_0 which has growth order ρ.

To prove the second statement we find from equation (6) for $n = -1$, Lemma 3.1, and

$$\Psi_0(m, z) = \int_{\mathbb{R}} \frac{p_0(m, t)}{t - \pi_0(z)} w_0(t) dt$$

that

$$|\Psi(-1, z) + \prod_{n=0}^{N-1} \frac{a_{0;n}}{a_n}| \leq \sum_{m=0}^{2N} C_0 \int_{\mathbb{R}} \frac{|p_0(m, t)| w_0(t)}{|t - \pi_0(z)|} dt.$$

The claim follows since, for $t \in \operatorname{supp}(w_0)$ we have $|t - \pi_0(z)| \geq d$. □

Corollary 3.5. *The second statement of the previous lemma implies the following statements in each of the three cases.*
For Case (1)

$$|\Psi(-1, (z, \sigma)) \prod_{n=0}^{N-1} \frac{a_n}{a_{0;n}} + 1| \leq C_1/|\operatorname{Im} z|, \quad \sigma \operatorname{Im} z > 0. \tag{7}$$

For Case (2)

$$|\Psi(-1,z)\prod_{n=0}^{N-1}\frac{a_n}{a_{0;n}}+1| \le C_1/(\operatorname{Im} z)^2, \quad \operatorname{Im} z > 0. \tag{8}$$

For Case (3)

$$|\Psi(-1,z)\prod_{n=0}^{N-1}\frac{a_n}{a_{0;n}}+1| \le 2C_1|z|, \quad |z| \le 1/4. \tag{9}$$

4. The inverse resonance problem

In this section we shall prove Theorem 4.1, our main result. In our estimates we will use two kinds of constants. The first kind, denoted by C or C_k, depends only on the base operator J_0 and on the numbers N and Q specifying the set $B_0(N,Q)$ of Definition 2. Note that this applies to the constant C_1 introduced in Lemma 3.4. The second kind, denoted by F, E, or E_1, may also depend on given positive numbers (of different meanings). We will make these dependencies explicit but suppress the dependence on J_0, N and Q in the notation as these are considered fixed throughout.

Theorem 4.1. *Suppose J_0 is a Jacobi operator with spectral density in one of the classes specified in Definition 1, $\Psi(-1,\cdot)$ and $\tilde\Psi(-1,\cdot)$ are Jost functions associated with two Jacobi difference expressions $\mathcal{J}$ and $\tilde{\mathcal{J}}$ in $B_0(N,Q)$, and $R_1 \ge 1$ is a given positive number. Then there are constants $F = F(R_1)$ and $E = E(R_1)$ so that the following statement is true. If $\epsilon \le E(R_1)$ and if all zeros of $\Psi(-1,\cdot)$ (on one of the sheets in Case (1)) in a disc of radius $R \ge F(R_1)$ are respectively ϵ-close to those of $\tilde\Psi(-1,\cdot)$ then the coefficients of $\mathcal{J}$ and $\tilde{\mathcal{J}}$ satisfy*

$$|b_n - \tilde b_n| \le \frac{1}{R_1}$$

and

$$|a_n^2 - \tilde a_n^2| \le \frac{1}{R_1}.$$

Remark 2. It follows from the subsequent proofs that, for some constant C, we have $F = CR_1^{\kappa+2}$ in Cases (1) and (2). In Case (3) an expression for F is not uniform for all possibilities but can be found explicitly for many interesting choices of the original unperturbed operator. For instance, in Example (3) in Section 2.4 F may be chosen as CR_1. Similarly E is a decreasing function of R_1.

Corollary 4.2. *If all zeros of $\Psi(-1,\cdot)$ coincide exactly with those of $\tilde\Psi(-1,\cdot)$, i.e., if $R=\infty$ and $\epsilon = 0$, then $b_n = \tilde b_n$ and $a_n^2 = \tilde a_n^2$.*

Proof. In this case we may choose R_1 arbitrarily large. □

Proof of Theorem 4.1. Henceforth we will use ψ as an abbreviation for $\Psi(-1,\cdot)$ or, in case (1), as an abbreviation of $\Psi(-1,(\cdot,+1))$ (we will have no need to consider the function $\Psi(-1,(\cdot,-1))$). We will prove in Lemma 4.4 that

$$\left|\psi(z)\prod_{n=0}^{N-1}\frac{a_n}{a_{0;n}} - \tilde\psi(z)\prod_{n=0}^{N-1}\frac{\tilde a_n}{a_{0;n}}\right| \le \frac{1}{R_1} \tag{10}$$

for all z in a certain disc $\mathcal{D}$ whose radius depends on the constant C_1 introduced in Lemma 3.4 provided that R is not smaller than a constant $F(R_1)$ and ϵ is not larger than a constant $E_1(R, R_1)$.

Setting

$$b(z) = \psi(z) \prod_{n=0}^{N-1} \frac{a_n}{a_{0;n}} - \tilde{\psi}(z) \prod_{n=0}^{N-1} \frac{\tilde{a}_n}{a_{0;n}}$$

and

$$x_{m+1} = \ell_0(-1)K_0(-1, m) - \tilde{\ell}_0(-1)\tilde{K}_0(-1, m)$$

one obtains from equation (6) that

$$\sum_{m=0}^{2N} \Psi_0(m, z)x_{m+1} = b(z).$$

We consider this equation for $2N + 1$ different values of z chosen in $\mathcal{D}$, which we denote by ζ_1, ..., ζ_{2N+1}. If we let A be the $(2N+1)\times(2N+1)$-matrix with entries $A_{j,m} = \Psi_0(m-1, \zeta_j)$ and if we let $x = (x_1, ...x_{2N+1})^\top$ and $b = (b(\zeta_1), ..., b(\zeta_{2N+1}))^\top$ we have

$$Ax = b$$

where all components of the vector b are of order $1/R_1$. Since the functions $\Psi_0(m, \cdot)$ are linearly independent, the matrix A is invertible and the norm of its inverse depends only on the base operator J_0 and the choice of the points ζ_j. Hence all entries of x are of order $1/R_1$ and we use this information in equation (5) to find that there is a constant C depending only on N and Q such that, for any $R_1 \geq 1$ and for $m = 0, ..., 2N$,

$$|\tilde{\ell}_0(-1)K(-1, m)| \leq C/R_1$$

provided R and ϵ satisfy the stated restrictions. The quantity $|\tilde{\ell}_0(-1)|$ is bounded by some constant depending only on N and Q so that the $|K(-1, m)|$ and hence the constant $\mathfrak{c}_2$ in Lemma 3.3 may be considered small. $\mathfrak{M}_2$ is also a priori bounded by a constant depending only on N and Q whence an application of Lemma 3.3 gives that all the coefficients $K(n, m)$, particularly the $K(n, n+1)$ and the $K(n, n+2)$, may be considered small. Using equations (3) we get now estimates on $|b_n - \tilde{b}_n|$ and, having those in place, equations (4) give estimates on $|a_n^2 - \tilde{a}_n^2|$. All these are of the form C'/R_1 for a suitable constant C'. If $C' > 1$ we call Lemma 4.4 with $C'R_1$ instead of R_1 to obtain a better estimate in (10).

The desired estimate has now been established for $R = F(R_1)$ and $\epsilon \leq E(R_1)$ where $E(R_1) = E_1(F(R_1), R_1)$. It will also hold for larger values of R since we have, in the course of the proof, no need to make an assumption on those resonances which are larger than $F(R_1)$. □

Lemma 4.3. *Suppose x_1, ..., x_n are complex numbers such that $|x_j - 1| \leq 1/(4n)$ for $j = 1, ...n$ and that $0 \leq k \leq n$. Then*

$$\left| \frac{\prod_{j=1}^k x_j}{\prod_{j=k+1}^n x_j} - 1 \right| \leq 4 \sum_{j=1}^n |x_j - 1|.$$

Proof. Letting $u = \log(\prod_{j=1}^k x_j / \prod_{j=k+1}^n x_j)$ one finds

$$|u| \leq \sum_{j=1}^n |\log(1 + x_j - 1)| \leq 2 \sum_{j=1}^n |x_j - 1|$$

using that $|\log(1+t)| \le 2|t|$ when $|t| \le 3/4$. The claim follows now since $|\,\mathrm{e}^u - 1| \le |u|\,\mathrm{e}^{|u|} \le 2|u|$, the latter inequality being valid as long as for $|u| \le 1/2$. □

Lemma 4.4. *Let*

$$\mathcal{D} = \begin{cases} \{z : |z - 3iC_1| \le C_1\} & \text{in Cases (1) or (2),} \\ \{z : |z| \le 1/(8C_1)\} & \text{in Case (3).} \end{cases}$$

Suppose $R_1 \ge 1$. Then there exist constants $F(R_1)$ and $E_1(R, R_1)$ such that

$$\left| \psi(z) \prod_{n=0}^{N-1} \frac{a_n}{a_{0;n}} - \tilde{\psi}(z) \prod_{n=0}^{N-1} \frac{\tilde{a}_n}{a_{0;n}} \right| \le \frac{1}{R_1}$$

for all $z \in \mathcal{D}$ provided that $R \ge F(R_1)$ and $\epsilon \le E_1(R, R_1)$.

Proof. The eigenvalues and resonances of J, i.e., the zeros of ψ are denoted by z_n. They are enumerated so that they are ordered by magnitude and repeated according to their multiplicities. Let $N_1 = N_1(R)$ be the number of zeros of ψ in the open disc of radius R centered at zero. Now we define the entire function

$$\varphi(z) = -\psi(z + z_0) \prod_{n=0}^{N-1} \tilde{a}_n / a_{0;n}$$

where z_0 is the center of $\mathcal{D}$, i.e. z_0 equals $3iC_1$ or zero. We also define r_0 to be the radius of $\mathcal{D}$, i.e., r_0 equals C_1 or $1/(8C_1)$. The zeros of φ are then $\mu_n = z_n - z_0$. These are never zero. In fact, using Corollary 3.5, we have, in Cases (1) and (2), that $\varphi(z) \ne 0$ as long as $\mathrm{Im}(z) > -2C_1$ and, in Case (3), that $\varphi(z) \ne 0$ while $|z| < 1/(4C_1)$. As a consequence $|\mu_n| \ge 2r_0$ for all n. Now we use Hadamard's factorization theorem to write

$$\varphi(z) = \exp(g(z)) \prod_{n=1}^{\infty} E_p(z/\mu_n) \tag{11}$$

where

$$E_p(z) = (1 - z)\exp(z + z^2/2 + ... + z^p/p),$$

$p \le \rho$ is the largest integer for which $\sum_{k=1}^{\infty} |\mu_k|^{-p}$ diverges, and g is a polynomial whose degree does not exceed ρ. Since $\prod_{k=1}^{\infty} \exp(z^j/\mu_k^j)$ converges when $j > p$ and since N_1 is finite we may rewrite equation (11) as

$$\varphi(z) = \exp(g(z)) \prod_{n=1}^{N_1} (z - \mu_n) \prod_{n=N_1+1}^{\infty} E_\kappa(z/\mu_n) \tag{12}$$

if $\kappa \ge p$ and if we properly redefine g. Note that the degree of g will not be more than κ. Just as in Lemma 3.4 we will henceforth reserve the letter κ for the smallest integer strictly larger than ρ. The analogous expression holds for $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(z) = -\tilde{\psi}(z + z_0) \prod_{n=0}^{N-1} \tilde{a}_n / a_{0;n}$$

except that the first N_1 of the zeros of $\tilde{\psi}$ are not necessarily ordered by magnitude but rather are the ones which are respectively ϵ-close to the zeros of ψ.

Now we may write

$$\frac{\varphi(z)}{\tilde{\varphi}(z)} = \mathrm{e}^{g(z) - \tilde{g}(z)}\, W(R, z) \frac{\Pi(R, z)}{\tilde{\Pi}(R, z)}$$

where

$$W(R,z) = \prod_{n=1}^{N_1} \frac{z-\mu_n}{z-\tilde{\mu}_n},$$

$$\Pi(R,z) = \prod_{k=N_1+1}^{\infty} E_\kappa(z/\mu_k)$$

and $\tilde{\Pi}$ is defined analogously to Π. We show in the Lemmas 4.5, 4.6 and 4.7 below that $|\Pi(R,z)-1|$, $|\tilde{\Pi}(R,z)-1|$, $|W(R,z)-1|$, and $|\,\mathrm{e}^{g(z)-\tilde{g}(z)}-1|$ are each less than $1/(24R_1)$ provided that $|z| \le r_0$, R is larger than some constant $F(R_1)$ defined as the maximum of such quantities defined in each of the Lemmas, and $E_1(R,R_1)$ is the minimum of the quantities $E_1(R,R_1)$ appearing in those lemmas. Lemma 4.3 gives now

$$\left|\frac{\varphi(z)}{\tilde{\varphi}(z)} - 1\right| \le \frac{16}{24R_1}.$$

Since, by part (2) of Lemma 3.4 or Corollary 3.5, $|\tilde{\varphi}(z)| \le 3/2$ for $|z| \le r_0$ we find next that

$$\left|\psi(z) \prod_{n=0}^{N-1} \frac{a_n}{a_{0;n}} - \tilde{\psi}(z) \prod_{n=0}^{N-1} \frac{\tilde{a}_n}{a_{0;n}}\right| \le \frac{1}{R_1}$$

for $z \in \mathcal{D}$ and assuming that R and ϵ satisfy the stated restrictions.. □

Lemma 4.5. *There exists a constant $C_2 \ge 1$ such that*

$$|\Pi(R,z)-1| \le \frac{C_2|z|^{\kappa+1}}{R} \exp(C_2|z|^{\kappa+1}/R)$$

provided that R is a positive number which exceeds $9C_1$ (where C_1 is given by Lemma 3.4) and $|z| \le R/2$. In particular,

$$|\Pi(R,z)-1| \le \frac{2C_2 r_0^{\kappa+1}}{R} \le \frac{1}{R_1}$$

for $|z| \le r_0$ if $R \ge F(R_1) = \max\{9C_1, 2C_2 r_0^{\kappa+1} R_1\}$.

Proof. We introduce the function $\nu(r)$ which counts the number of zeros of φ contained in the disc $|z| < r$ and note that $\nu(0) = 0$. Jensen's formula

$$\int_0^{\mathrm{e}\,r} \frac{\nu(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log|\varphi(\mathrm{e}\,r\,\mathrm{e}^{it})| dt - \log|\varphi(0)|,$$

the inequality $\nu(r) \le \int_0^{\mathrm{e}\,r} t^{-1}\nu(t)dt$, part (1) of Lemma 3.4, and the fact that $\nu(r) = 0$ for all sufficiently small r give

$$\nu(r) \le Cr^\kappa \tag{13}$$

for some positive constant C which depends only on N and Q.

Since $\kappa \ge 1$ the elementary factor $E_\kappa(w)$ satisfies $|\log E_\kappa(w)| \le |w|^{\kappa+1}$ as long as $|w| \le 1/2$. Therefore, thinking of w as z/μ_n, we are interested in an estimate on

$$S = \sum_{n=N_1+1}^{\infty} |\mu_n|^{-\kappa-1}.$$

It will be convenient to assume that $R \geq 3|z_0|$. Since $|z_n| \geq R$ we get $|\mu_n| \geq R/2$ so that

$$S \leq \int_{[R/2,\infty)} \frac{d\nu(t)}{t^{\kappa+1}} \leq (\kappa+1)\int_{R/2}^{\infty} \frac{\nu(t)dt}{t^{\kappa+2}}.$$

Now inequality (13) gives $S \leq C_2/R$ when $C_2 = 2C(\kappa+1)$. With the aid of the inequality $|\,\mathrm{e}^{u} - 1| \leq |u|\,\mathrm{e}^{|u|}$ we arrive at the statement of the lemma. □

Now we return to the case of two potentials q and $\tilde{q}$ in $B_0(N,Q)$.

Lemma 4.6. *Suppose that* $|z_n - \tilde{z}_n| = |\mu_n - \tilde{\mu}_n| \leq \epsilon \leq r_0/4$. *If* $|z| \leq r_0$, *then*

$$|W(R,z) - 1| \leq \frac{2\epsilon N_1(R)}{r_0} \exp\left(2\epsilon N_1(R)/r_0\right).$$

In particular,

$$|W(R,z) - 1| \leq \frac{4\epsilon N_1(R)}{r_0} \leq \frac{1}{R_1}$$

provided that $|z| \leq r_0$ *and* $\epsilon \leq E_1(R,R_1) = r_0/(4N_1(R)R_1)$.

In addition, in Cases (1) and (2) we have the following estimate for $2\,\mathrm{Im}(z) \geq |z| \geq r_0$:

$$|W(R,z) - 1| \leq \frac{4\epsilon N_1(R)}{|z|} \exp\left(4\epsilon N_1(R)/|z|\right).$$

Proof. Suppose that $|z| \leq r_0$. We use again that $|\,\mathrm{e}^{u} - 1| \leq |u|\,\mathrm{e}^{|u|}$ where $u = \log W(R,z)$. Our claim is therefore that $|u| \leq 2\epsilon N_1(R)/r_0$. The definition of W yields

$$|\log(W(R,z))| \leq \sum_{k=1}^{N_1} \left|\log\left(1 + \frac{\tilde{\mu}_k - \mu_k}{z - \tilde{\mu}_k}\right)\right|.$$

Since $|\tilde{\mu}_k| \geq 2r_0$ we get $|z - \tilde{\mu}_k| \geq r_0$ so that

$$\left|\frac{\tilde{\mu}_k - \mu_k}{z - \tilde{\mu}_k}\right| \leq \frac{\epsilon}{r_0} \leq \frac{1}{2}.$$

Now use the inequality $|\log(1+t)| \leq 2|t|$ valid for $|t| \leq 3/4$ and the fact that there are N_1 summands.

If we have Case (1) or Case (2) and $2\,\mathrm{Im}(z) \geq |z| \geq r_0$ we use that $\mathrm{Im}(\tilde{\mu}_k) < 0$ and obtain the estimate $|z - \tilde{\mu}_k| \geq \mathrm{Im}(z - \tilde{\mu}_k) \geq \mathrm{Im}(z) \geq |z|/2$ which yields

$$\left|\frac{\tilde{\mu}_k - \mu_k}{z - \tilde{\mu}_k}\right| \leq \frac{2\epsilon}{|z|} \leq \frac{1}{2}.$$

Then we proceed as before. □

Lemma 4.7. *Suppose* $R_1 \geq 1$ *and* $|z| \leq r_0$. *Then*

$$|\,\mathrm{e}^{g(z)-\tilde{g}(z)} - 1| \leq \frac{1}{R_1}$$

at least if $R \geq F(R_1)$ *and* $\epsilon \leq E_1(R,R_1)$ *where* F *and* E_1 *are suitable constants (specified in the proof).*

Proof. Note that

$$\mathrm{e}^{g(z)-\tilde g(z)} - 1 = \frac{\varphi(z)\tilde\Pi(R,z)}{\tilde\varphi(z)\Pi(R,z)W(R,z)} - 1.$$

If we can suitably estimate the quantities $\Pi(R,z)-1$, $\tilde\Pi - 1$, $\varphi(z)-1$, $\tilde\varphi(z)-1$, and $W(R,z)-1$ we will get our claim from Lemma 4.3. We will use Lemma 4.5 to estimate the first two quantities. estimates for the other three will be slightly different for each of the three cases.

Consider first Case (1) with z in the disc $|z-3iR_0|\le R_0$ where $R_0 = 20C_1R_1$. Note that we have $|z|\le 2\operatorname{Im}(z)$ in this disc guaranteeing the applicability of Lemma 4.6. Since we also need $|z|\le R/2$ for any point z in order to apply Lemma 4.5 we shall choose $R\ge 160C_1R_1$. To estimate $\varphi(z)-1$ and $\tilde\varphi(z)-1$ we use inequality (7). These considerations and Lemma 4.3 give

$$\left|\mathrm{e}^{g(z)-\tilde g(z)} - 1\right| \le 20\max\{\frac{2C_2(4R_0)^{\kappa+1}}{R}, \frac{8\epsilon N_1(R)}{2R_0}, \frac{C_1}{2R_0}\} \le \frac{1}{2R_1}$$

provided that each of the terms whose maximum we seek is less than $1/(40R_1)$. The first condition may be satisfied by choosing the constant $R\ge C_2(80C_1R_1)^{\kappa+2}$. The second condition is satisfied by choosing $\epsilon\le C_1/(8N_1(R))$. The last condition is satisfied by our choice of R_0.

The disc $|z-3iR_0|\le R_0$ does not include the disc $|z|\le r_0$ for which we claim the estimate to hold. To remedy this situation we first use $|\log(1+z)|\le 2|z|$ to see that $|g(z)-\tilde g(z)|\le 1/R_1$. Since $g(z)-\tilde g(z)$ is a polynomial we may use Cauchy's estimate to bound its Taylor coefficients about $3iR_0$, the binomial theorem to bound its Taylor coefficients about 0, and finally the inequality $|\mathrm{e}^u - 1|\le |u|\mathrm{e}^{|u|}$ to show that

$$\left|\mathrm{e}^{g(z)-\tilde g(z)} - 1\right| \le \frac{cr_0^\kappa}{R_1}\exp(cr_0^\kappa/R_1)$$

for $|z|\le r_0$ where c is a constant depending only on κ. After redefining R_1 (and thereby changing $F(R_1)$) we complete the proof of the lemma for Case (1).

In Case (2) we can repeat the same arguments with (8) instead of (7) since $C_1/\operatorname{Im}(z)^2\le C_1/\operatorname{Im}(z)$ for all z for which we use the estimate.

Now, we proceed to the Case (3). First, one shows by induction that, for $n\le N$,

$$\Psi(n,z) = \alpha_n(\lambda) + M_0(z)\beta_n(\lambda),$$

where $\lambda = \pi_0(z)$ and where α_n and β_n are polynomials of degree no more than $2N-n-1$ and $2N-n$, respectively.

We consider two subcases. Firstly, if the entire function f_0 determining the weight w_0 is a polynomial, then ψ and hence φ is a polynomial whose degree we denote by ℓ (in fact, $\ell\le 2N+1+\deg(f_0)$). In this case we have $\kappa=1$ and g may not be a constant because of our unifying notation introduced in equation (12). In fact, since

$$\varphi(z) = \mathrm{e}^{g(z)}\prod_{n=1}^{N_1}(z-z_n)\prod_{n=N_1+1}^{\ell}(1-z/z_n)\,\mathrm{e}^{z/z_n}$$

and $\varphi(0)=1$ we find

$$g(z) = -\sum_{n=1}^{N_1}\log(-z_n) - \sum_{n=N_1+1}^{\ell}\frac{z}{z_n}.$$

Hence, for $|z| \le r_0$,

$$|g(z) - \tilde g(z)| \le \sum_{n=1}^{N_1} |\log(z_n/\tilde z_n)| + \frac{2|z|\ell}{R} \le \frac{\epsilon N_1}{r_0} + \frac{2|z|\ell}{R}.$$

This gives

$$\left| \mathrm{e}^{g(z)-\tilde g(z)} - 1 \right| \le \frac{2\epsilon N_1}{r_0} + \frac{4|z|\ell}{R} \le \frac{1}{R_1}$$

provided $\epsilon \le E_1(R, R_1) = r_0/(4R_1 N_1(R))$ and $R \ge F(R_1) = 8r_0\ell R_1$.

In the second subcase f_0 has an essential singularity at infinity, a property we will make use of. Define

$$h(z) = -2\pi i(f_0(z) - f_0(1/z))\beta_{-1}(z + 1/z) \prod_{n=0}^{N-1} \frac{a_n}{a_{0;n}}$$

so that $\varphi(z) = \varphi(1/z) + h(z)$ and

$$|\varphi(z) - 1| \le |\varphi(1/z) - 1| + |h(z)|.$$

Since β_{-1} has degree $2N+1$ we have $|\pi\beta_{-1}(z+1/z) \prod_{n=0}^{N-1} a_n/a_{0;n}| \le C_3|z|^{2N+1}$ for some C_3 depending only on N and Q when $|z| \ge 1$. Suppose $R_0 = 40(C_1 + C_3)R_1$. Using properties of the essential singularity, we can find $\kappa+1$ points y_j (depending only on f_0, N, Q, and R_1) satisfying the following conditions:

(1) $2R_0 \le |y_1|$ and $2|y_j| \le |y_{j+1}|$ for $j = 1, ..., \kappa$.
(2) $|y_j^{2N+2}(f_0(y_j) - f_0(1/y_j))| \le 1/2$ for $j = 1, ..., \kappa + 1$.

Around each point y_j there is a disc of radius δ_j (again depending only on f_0, N, Q, and R_1) such that $|y^{2N+2}(f_0(y) - f_0(1/y))| \le 1$ for all y within that disc. Let $\delta = \min\{\delta_1, ..., \delta_{\kappa+1}, R_0\}$. Thus, if y is any point in the union of the δ-discs about the points y_j, then $|h(y)| \le 2C_3/|y|$. Estimate (9) gives $|\varphi(1/y) - 1| \le 2C_1/|y|$ so that

$$|\varphi(y) - 1| \le \frac{2C_1 + 2C_3}{|y|} \le \frac{1}{20R_1}.$$

This inequality shows, in particular, that none of the zeros of φ can be located in any of the δ-discs about the y_j. Thus $|y_j - z_n| \ge \delta$ and $|(\tilde z_n - z_n)/(y_j - z_n)| \le \epsilon/\delta$. Proceeding as in the proof of Lemma 4.6 we find $|W(R, y_j) - 1| \le 4\epsilon N_1(R)/\delta \le 1/(20R_1)$ provided $\epsilon \le \delta/(80N_1(R)R_1)$.

Now we have estimates for $\varphi(y_j)-1$, $\tilde\varphi(y_j)-1$, and $W(R, y_j)-1$. For $\Pi(R, y_j)-1$ and $\tilde\Pi(R, y_j) - 1$ we invoke again Lemma 4.5. Then, following the same strategy as for case (1), we find,

$$| \mathrm{e}^{g(y_j)-\tilde g(y_j)} - 1| \le \frac{1}{R_1}$$

assuming that $\epsilon \le \delta/(80N_1(R)R_1)$ and $R \ge 40C_2|y_j|^{\kappa+1}R_1$. Thus

$$|g(y_j) - \tilde g(y_j)| \le \frac{2}{R_1}.$$

Since $g - \tilde g$ is a polynomial of degree κ interpolation shows that

$$(g - \tilde g)(z) = \sum_{n=1}^{\kappa+1} (g - \tilde g)(y_n) \prod_{j\ne n} \frac{z - y_j}{y_n - y_j}$$

for all z. Thus, we obtain, using the first condition satisfied by the y_j,

$$|g(z) - \tilde{g}(z)| \le \frac{(\kappa+1)2^{2\kappa+1}}{R_1}$$

for $|z| \le 2R_0$. Thus

$$|\operatorname{e}^{g(z)-\tilde{g}(z)} - 1| \le \frac{(\kappa+1)2^{2\kappa+1}}{R_1} \exp((\kappa+1)2^{2\kappa+1}/R_1) \le \frac{(\kappa+1)4^{\kappa}}{R_1}$$

for $|z| \le r_0 \le 2R_0$ at least if $R_1 \ge (\kappa+1)4^{\kappa}$. This proves the lemma after a rescaling of R_1 upon a proper choice of $F(R_1)$ and $E_1(R, R_1)$. □

5. Location of resonances

In this section we consider the asymptotic behavior of resonances appearing in the case of finitely supported perturbations of Jacobi matrices generated by the weight functions from the first two classes (see Definition 1). In the presentation we plan to confine ourselves to the case of a diagonal perturbation only. This restriction is not strictly necessary, but convenient for us since it allows us to avoid detailed proofs of the statements and to refer to the paper [2] where a very special situation of $\mathcal{J}_0$ in case (1) (Hermite operator corresponding to the choice: $a_{0;n} = \sqrt{n+1}, b_{0;n} = 0, n \in \mathbb{N}_0$) and a diagonal perturbation have been considered.

5.1. **Freud-type weights.** Let us consider firstly the case of so-called Freud-type weights [17], [8], which in our situation fall into the case (1):

$$w_0(t) = P_l(t)\overline{P_l(t)}e^{-Q_M(t)}, t \in \mathbb{R}, \tag{14}$$

where
i) $P_l(t)$ is a complex polynomial of degree $l \ge 0$ having a leading coefficient $d_l \ne 0$,
ii) $Q_M(t)$ is a real polynomial of even degree $M \ge 2$ having a positive leading coefficient a,
iii) finally we suppose that $w_0(t)$ satisfies the standard normalization condition.

Since w_0 can be extended to the whole complex plane as an entire function such a weight w_0 belongs to the class (1). This weight generates an unbounded Jacobi matrix $\mathcal{J}_0$ for whose entries the asymptotic behavior can be analyzed [4]. The most famous case of such operators (Hermite operator) corresponds to the choice $P_0(t) = 1/\sqrt[4]{2\pi}$ and $Q_2(t) = t^2/2$. The asymptotic behavior of resonances for the perturbed Hermite operator has been considered in [2] and we plan to follow the same strategy below.

Consider the Jost function on the Riemann surface $\mathcal{R} = \mathbb{C} \times \{-1, 1\}$, $z = \pi(z, \sigma)$,

$$\Psi(-1, z, \sigma) \equiv \rho_{-1}(z) + M_0(z, \sigma)\tau_{-1}(z)$$

(here and later we use the notations of [2]). Straightforward calculation based on the recurrence equation for $\Psi(n, z, \sigma)$ shows that ρ_{-1} and τ_{-1} are polynomials of degree not greater than $(2N-1)$ and $2N$ respectively. They have precisely the above mentioned degree values provided the perturbation of $\mathcal{J}_0$ is purely diagonal ($a_{0;n} = a_n$, $n \in \mathbb{N}_0$, $b_N \ne b_{0;N}$). As is well known the resonances correspond to the roots of the analytic continuation $\Psi(-1, z, \sigma)$ from the "physical sheet" ($\sigma \operatorname{Im} z > 0$) to the "unphysical" one ($\sigma \operatorname{Im} z < 0$) through the spectrum of $\mathcal{J}_0$ covering the whole real line ($w(t) \ne 0$, a.e. $t \in \mathbb{R}$). In the following we discuss only the case $\sigma = +1$ (the other case being analogous) which allows us to drop σ from our notation.

Using the formula (1) for the analytic continuation of m_0 we arrive at the following resonance location equation

$$0 = \Psi(-1, z) = \rho_{-1}(z) + (m_0(z) + 2\pi i w_0(z))\tau_{-1}(z)$$

or its equivalent form:

$$2\pi i w_0(z) = -\frac{\rho_{-1}(z)}{\tau_{-1}(z)} - m_0(z), \operatorname{Im} z < 0. \tag{15}$$

To analyze the asymptotic behavior near infinity for the ratio $\rho_{-1}(z)/\tau_{-1}(z)$ we have to use the following three results.

Lemma 5.1. *For any $n \in \mathbb{N}_0$ there exists a constant C_n such that*

$$\left|\Psi_0(n, z) + \frac{A_n}{z^{n+1}}\right| \leq \frac{C_n}{|\operatorname{Im} z||z|^{n+1}}, \operatorname{Im} z \neq 0,$$

where

$$A_n := \int_{\mathbb{R}} w_0(t)p_0(n, t)t^n dt.$$

Lemma 5.2. *Assume that the perturbation is purely diagonal ($a_{0;n} = a_n$ for all $n \in \mathbb{N}_0$) and that $b_N \neq b_{0;N}$. Then there exist positive constants C_N and K_N such that*

$$|\Psi(n, z) - \Psi_0(n, z)| \leq \frac{2K_N C_N \|b - b_0\|_1}{|z|^{n+2}}$$

for all $n \in \mathbb{N}_0$ and all complex $z, \operatorname{Im} z \geq 1$,

$$|z| \geq 2K_N \|b - b_0\|_1.$$

Here $\|\cdot\|_1$ stands for the ℓ^1-norm of a sequence.

Lemma 5.3. *Assume that $a_n = a_{0;n}, n \in \mathbb{N}_0$ and $b_N \neq b_{0;N}$, then the asymptotic behavior of the rational function ρ_{-1}/τ_{-1} near infinity is given by*

$$\frac{\rho_{-1}(z)}{\tau_{-1}(z)} = \frac{1}{z} \sum_{k=0}^{2N-1} \frac{1}{z^k} \left(\int_{\mathbb{R}} w_0(t)t^k dt \right) - \frac{\prod_{j=0}^{N-1} a_j^2}{z^{2N}(b_{0;N} - b_N)} + O\left(\frac{1}{z^{2N+1}}\right). \tag{16}$$

Remark 3. 1. The proof of each of the three lemmas repeats almost precisely mutatis mutandis the proofs of Lemma 2.1, Lemma 4.2 and Lemma 4.3 from [2], respectively.
2. Note that the statement of Lemma 5.2 and therefore of Lemma 5.3 is not valid for general type of perturbation with nonzero off-diagonal perturbation terms. A counterexample can be obtained in the case $N = 1$ already.
3. The coefficient in front of z^{-2N} in (16) appears as a common leading coefficient for both polynomials $\rho_{-1}(z)$ and $\tau_{-1}(z)$ which is equal to $\left(\prod_{j=0}^{N-1} a_j^2\right)/(b_{0;N} - b_N)$.

Application of Lemma 5.2 and 5.3, due to the cancellation of $(2N - 1)$ terms in the right hand side of (15), gives another form of the resonance location equation. To that end it is enough to consider the standard asymptotic expansion into power series for $1/(t-z)$ and hence for $m_0(z)$ generated by its definition as Cauchy transform of the weight function and equation (16) in (15). This gives

$$2\pi i w_0(z) = \frac{\prod_{j=0}^{N-1} a_j^2}{z^{2N}(b_{0;N}-b_N)} + O\left(\frac{1}{z^{2N+1}}\right). \tag{17}$$

Note that the error term is a function of z analytic near infinity. Substitution of the formula (14) for the weight function w_0 into the last equation gives

$$Q_M(z) = 2\pi i n + (2N+2l)\ln(z) + \ln\left[2\pi i(b_{0;N}-b_N)|d_l|^2 / \prod_{j=0}^{N-1} a_j^2\right] + O\left(\frac{1}{z}\right) \tag{18}$$

for any $n \in \mathbb{Z}$. Since $Q_M(z) \sim az^M$ as $z \to \infty$ one gets the asymptotic distribution of the resonances from equation (18) after using the standard arguments of the perturbation theory [15] exactly like in the paper [2].

Theorem 5.4. *Assume that the unperturbed weight function has the form* (14)*, that the perturbation of the matrix entries is purely diagonal, and that* $b_N \neq b_{0;N}$*. Then the resonances* z_n *in* $\mathbb{C}_-$ *(the "unphysical sheet") part of the Riemann surface approach all* M *rays*

$$\left\{\arg z = -\frac{\pi}{2M} - \frac{\pi j}{M}; j = 0, 1, ..., (M-1)\right\}$$

and after a proper numeration will satisfy the asymptotic formula

$$Q_M(z_n) = 2\pi i n + \frac{2(N+l)}{M}\ln(\frac{2\pi i n}{a}) - \ln C + O\left(\frac{1}{n^{1/M}}\right), n \in \mathbb{Z}, n \to \infty, \tag{19}$$

where

$$C := \frac{\prod_{j=0}^{N-1} a_j^2}{2\pi i(b_{0;N}-b_N)|d_l|^2}.$$

The last formula certainly allows the asymptotic behavior of the resonances z_n to be calculated on each ray $j = 0, 1, ..., (M-1)$, but the form (19) is more transparent.

5.2. **Associated Laguerre polynomials case.** A similar reasoning can be applied to a weight of class (2). Let us consider a classical case of the background weights

$$w_0(t) = \begin{cases} t^{k+1/2}e^{-t}/\Gamma(k+3/2) & \text{if } t \geq 0, \\ 0 & \text{if } t \leq 0 \end{cases} \tag{20}$$

($k \in \mathbb{N}$), corresponding to the case of the associated Laguerre polynomial. This example may be considered as a key one for weights of class (2). Let us briefly sketch the proof of the asymptotic formula in that case. Now the variable z belongs to the 2-sheeted Riemann surface $\mathcal{R} = \mathbb{C}$ and the canonical projection $\pi_0(z) = \lambda := z^2$ we have the following equation for resonance location.

$$0 = \frac{\Psi(-1,z)}{\tau_{-1}(z^2)} \equiv \frac{\rho_{-1}(z^2)}{\tau_{-1}(z^2)} + M_0(z), \tag{21}$$

where for the "unphysical sheet" $\operatorname{Im} z < 0$ the analytic continuation of $m(z)$ from $\mathbb{C}_+$ has the form (as noted after the proof of Lemma 2.2):

$$M_0(z) = m_0(z^2) + 2\pi i u_0(z), \quad \operatorname{Im} z < 0.$$

Remember that in the last formula

$$m_0(z^2) = \int_{\mathbb{R}} \frac{w_0(t)}{t - z^2} dt$$

and

$$u_0(z) = z^{2k+1} e^{-z^2} / \Gamma(k + 3/2).$$

Precisely as in the Section 5.1, by an application of results analogous to Lemmas 5.2 and 5.3, condition (21) can be reduced to the following:

$$2\pi i u_0(z) = \frac{\prod_{j=0}^{N-1} a_j^2}{z^{4N}(b_{0;N} - b_N)} + O\left(\frac{1}{z^{4N+2}}\right).$$

Again the arguments of Olver [15] (see [2] for details) lead to the following statement.

Theorem 5.5. *Let the background weight function have the form* (20). *Assume that $a_k = a_{0;k}$ for all $k \in \mathbb{N}_0$ and $b_N \neq b_{0;N}$. Then the resonances $\lambda_n = z_n^2$ on the second sheet (*$\operatorname{Im} z < 0$*) will approach both of the rays* $\arg z = -\frac{\pi}{4}, \arg z = -\frac{3\pi}{4}$, *and the location of the resonances satisfies after a proper numeration the following asymptotic formula:*

$$\lambda_n = z_n^2 = 2\pi i n + \frac{(4N + 2k + 1)}{2} \ln(2\pi i n) - \ln C' + O\left(\frac{\ln n}{n}\right), n \in \mathbb{Z}, n \to \infty, \quad (22)$$

where the constant C' has the same value as in Theorem 5.4,we just have to replace $|d_l|^2$ in it by its explicit value $1/\Gamma(k + 3/2)$.

Note that the two rays mentioned above correspond to the choice $n \to +\infty$ ($n \to -\infty$) respectively.

Remark 4. 1. All considerations are valid for general weight functions from classes (1) and (2). It is enough to replace the explicit asymptotic formulas (19) and (22) by the more complicated equations (17) and (21) respectively.
2. As far as the weight class (3) is concerned we do not expect to obtain interesting asymptotic information in general. Indeed, in the key example in class (3) of the discrete Schrödinger operator ($f(s) = s$ in Definition 1) we arrive at polynomials for the Jost functions. Therefore we have only a few resonances in that special case.
3. In the general case of a finitely supported perturbation (nonzero perturbation of the off-diagonal entries) the asymptotic formulas for the resonances look similar to (15) and (18). The only difference is in the values of the coefficients in front of the logarithmic terms and the values of the constants. These now certainly depend on the off-diagonal perturbation.
4. To demonstrate the general situation of a three-diagonal finitely supported

perturbation let us consider an example.
Elementary example ($N = 1$). Straightforward calculation shows

$$\frac{\rho_{-1}(\lambda)}{\tau_{-1}(\lambda)} = \frac{1}{\lambda} + \frac{1}{\lambda^2}\left\{ b_{0;0} - \frac{a_{0;0}^2}{b_{0;1} - b_1} \right\} + O\left(\frac{1}{\lambda^3}\right), \quad \lambda \to \infty,$$

provided $b_{0;1} \neq b_1$, and

$$\frac{\rho_{-1}(\lambda)}{\tau_{-1}(\lambda)} = -\frac{a_0^2}{a_{0;0}^2 - a_0^2}\frac{1}{\lambda} - \frac{1}{\lambda^2}\frac{a_0^2}{a_{0;0}^2 - a_0^2} \times \left(b_0 - \frac{(b_{0;0} - b_0)a_0^2}{a_{0;0}^2 - a_0^2} \right) + O\left(\frac{1}{\lambda^3}\right), \lambda \to \infty,$$

if $b_{0;1} = b_1$ and $a_{0;0} \neq a_0$.

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