LAMÉ POTENTIALS AND THE STATIONARY (M)KDV HIERARCHY

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Abstract. A new method of constructing elliptic finite-gap solutions of the stationary Korteweg-de Vries (KdV) hierarchy, based on a theorem due to Picard, is illustrated in the concrete case of the Lamé-Ince potentials $-s(s+1)\mathcal{P}(z)$, $s \in \mathbb{N}$ ($\mathcal{P}(.)$ the elliptic Weierstrass function). Analogous results are derived in the context of the stationary modified Korteweg-de Vries (mKdV) hierarchy for the first time.

1. Introduction

This is the first in a series of papers on the characterization of all elliptic finite-gap solutions of the stationary Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) hierarchy. This problem (described, e.g., in [44], p. 152) has its origin in the 1940 paper of Ince [36] who studied the Lamé potential

$$q(x) = -s(s+1)\mathcal{P}(x+\omega_3+\alpha), \quad s \in \mathbb{N},\ \alpha \in \mathbb{R},\ x \in \mathbb{R} \tag{1.1}$$

in connection with the second-order ordinary differential equation

$$\psi''(E,x) + [q(x) - E]\psi(E,x) = 0, \quad E \in \mathbb{C}. \tag{1.2}$$

(Here $\mathcal{P}(x) \equiv \mathcal{P}(x;\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods (f.p.) $2\omega_1$, $2\omega_3$, $\mathrm{Im}(\omega_3/\omega_1) \neq 0$, see [1], Ch. 18). In the real-valued case, where $\omega_1 \in \mathbb{R}\backslash\{0\}$, $i\omega_3 \in \mathbb{R}\backslash\{0\}$, a modern spectral theoretic interpretation of Ince's result in [36] implies that the self-adjoint operator $L = \dfrac{d^2}{dx^2} + q$ in $L^2(\mathbb{R})$ has a finite-gap spectrum of the type

$$\sigma(L) = (-\infty, E_{2s}] \cup \bigcup_{m=1}^{s} [E_{2m-1}, E_{2(m-1)}], \quad E_{2s} < E_{2s-1} < \cdots < E_0. \tag{1.3}$$

In obvious notation, any potential q yielding a finite-gap spectrum of the type (1.3) is called a finite-gap potential (the proper extension of this notion to complex-valued and possibly singular q will be given in

Definition 2.2). Subsequent work by Dubrovin [18], Its and Matveev [39], McKean and van Moerbeke [41], and Novikov [43] then proved that every finite-gap potential q satisfies appropriate higher-order stationary KdV equations. The KdV flow $q_t = \frac{1}{4}q_{xxx} + \frac{3}{2}qq_x$ with initial condition $q(x,0) = -6\mathcal{P}(x)$ was explicitly integrated by Dubrovin and Novikov [20] (see also [22], [23], [24], [38]) and found to be of the type

$$q(x,t) = -2\sum_{j=1}^{3}\mathcal{P}(x - x_j(t)) \tag{1.4}$$

for an appropriate expression of $\{x_j(t)\}_{j=1}^3$. The first systematic study of the isospectral torus $I_{\mathbb{R}}(q_0)$ of real-valued smooth potentials q_0 of the form

$$q_0(x) = -2\sum_{j=1}^{M}\mathcal{P}(x - x_j) \tag{1.5}$$

with finite-gap spectrum as in (1.3) was undertaken by Airault, McKean, and Moser in 1977 in their celebrated paper [3]. Among a variety of results they proved that any element of $I_{\mathbb{R}}(q_0)$ is an elliptic function of the type (1.5) (for different sets $\{x_j\}_{j=1}^M$) with M constant throughout $I_{\mathbb{R}}(q_0)$ and $s = \dim I_{\mathbb{R}}(q_0) \leq M$. The next breakthrough occurred when Verdier's [55] new explicit examples of elliptic finite-gap potentials were published in 1988. This immediately led to further examples by Belokolos and Enol'skii [6] and Smirnov [50] employing the reduction process of Abelian integrals to elliptic integrals (see, e.g., [7]) and finally culminated in the recent results of Treibich and Verdier [51], [52], [53] that a general complex-valued potential of the form

$$q(z) = -\sum_{j=1}^{4} d_j\mathcal{P}(z - \omega_j), \quad d_j \in \mathbb{C},\ 1 \leq j \leq 4,\ z \in \mathbb{C} \tag{1.6}$$

($\omega_2 = \omega_1 + \omega_3$, $\omega_4 = 0$) is a finite-gap potential if and only if $d_j/2$ are triangular numbers, i.e., if and only if

$$d_j = s_j(s_j+1) \text{ for some } s_j \in \mathbb{Z},\ 1 \leq j \leq 4. \tag{1.7}$$

The methods of Treibich and Verdier are based on the notion of hyperelliptic tangential covers of the torus $\mathbb{C}/\Lambda$ (Λ the period lattice generated by $2\omega_1$, $2\omega_3$).

In contrast to the approaches described above, our own methods to characterize elliptic finite-gap solutions of the (m)KdV hierarchy rely on entirely different ideas. Our main new strategy is based on a systematic use of a powerful theorem of Picard (see Theorem 2.3) concerning ordinary differential equations with elliptic coefficients combined with explicit realizations of the isospectral manifold given an (elliptic) finite-gap base potential q_0 (see, e.g., [11], [26], [31]). This approach immediately recovers and extends the results of [6], [50], [51], [52], [53] and, in particular, yields a complete characterization of all even (i.e., $q(z) = q(-z)$) elliptic finite-gap potentials [33]. Moreover, it leads to a natural conjecture on the structure of general elliptic finite-gap potentials (see Section 2).

In the present first paper of our series we have singled out the case of the Lamé-Ince potential (1.1) for a variety of reasons. First of all, the application of Picard's theorem is most transparent in the Lamé-Ince case as shown in Section 3. Secondly, Lamé's equation (1.2) with coefficient $q(x)$ as in (1.1) has led to an abundance of applications in physics and engineering (too numerous to be listed here) and, especially, in the context of completely integrable models such as (generalized) Calogero-Moser-type systems discussed, e.g., in [3], [12], [13], [14], [15], [38], [40], [45], [46] and the references therein. Moreover, as shown in Section 4, all results on Lamé-Ince potentials in connection with the stationary KdV hierarchy carry over most naturally to analogous elliptic finite-gap solutions (for simplicity still called Lamé potentials) of the stationary mKdV hierarchy. Consequently, Section 4 lays the foundation on which analogous Calogero-Moser-type systems for the mKdV hierarchy can be built.

The far more complex situation of Treibich-Verdier potentials (1.6) has been discussed in [32], the case of all even elliptic finite-gap potentials in [33].

2. The (m)KdV hierarchy and Picard's theorem

In this section we briefly review the essentials of the stationary (m)KdV hierarchy, its connection with finite-gap solutions, and Picard's theorem as needed in Sections 3 and 4.

Consider the recursion relation

$$\hat{f}_{j+1,x} = \frac{1}{4}\hat{f}_{j,xxx} + q\hat{f}_{j,x} + \frac{1}{2}q_x\hat{f}_j, \quad 0 \le j \le n, \quad \hat{f}_0 = 1, \tag{2.1}$$

and the differential expressions (Lax pair)

$$L(t) = \frac{d^2}{dx^2} + q(x,t),$$
$$\hat{P}_{2n+1}(t) = \sum_{j=0}^{n} \left[-\frac{1}{2}\hat{f}_{j,x}(x,t) + \hat{f}_j(x,t)\frac{d}{dx} \right] L(t)^{n-j}, \quad n \in \mathbb{N} \cup \{0\} \tag{2.2}$$

One can prove (see, e.g., [4], [17], Ch. 12, [27], [31] and the references therein) that

$$[\hat{P}_{2n+1}, L] = 2\hat{f}_{n+1,x} \tag{2.3}$$

($[\,.\,,\,.\,]$ the commutator). Explicitly one computes from (2.1)

$$\hat{f}_0 = 1, \quad \hat{f}_1 = \frac{1}{2}q + c_1, \quad \hat{f}_2 = \frac{1}{8}q_{xx} + \frac{3}{8}q^2 + \frac{c_1}{2}q + c_2, \quad \text{etc.}, \tag{2.4}$$

where the c_j are integration constants. We shall use the convention that all homogeneous quantities, defined by $c_\ell \equiv 0$, $\ell \in \mathbb{N}$, are denoted by omitting the hat, i.e., $f_j := \hat{f}_j(c_\ell \equiv 0)$, $P_{2n+1} := \hat{P}_{2n+1}(c_j \equiv 0)$. Thus the homogeneous version of (2.4) reads

$$f_0 = 1, \; f_1 = \frac{1}{2}q, \; f_2 = \frac{1}{8}q_{xx} + \frac{3}{8}q^2, \quad \text{etc.} \tag{2.5}$$

The KdV hierarchy is then defined as the sequence of evolution equations

$$\mathrm{KdV}_n(q) := q_t - [P_{2n+1}, L] = q_t - 2f_{n+1,x} = 0, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.6}$$

Explicitly,

$$\mathrm{KdV}_0(q) = q_t - q_x, \quad \mathrm{KdV}_1(q) = q_t - \frac{1}{4}q_{xxx} - \frac{3}{2}qq_x, \quad \text{etc.} \tag{2.7}$$

with $\mathrm{KdV}_1(.)$ the usual KdV functional. The inhomogeneous version of (2.6) is then given by

$$q_t - [\hat{P}_{2n+1}, L] = q_t - 2\hat{f}_{n+1,x} = q_t - 2\sum_{j=0}^{n} c_{n-j} f_{j+1,x} = 0, \quad c_0 = 1. \tag{2.8}$$

The special case of the n-th-order stationary KdV equation characterized by $q_t = 0$ then reads

$$f_{n+1,x} = 0 \text{ respectively } \hat{f}_{n+1,x} = \sum_{j=0}^{n} c_{n-j} f_{j+1,x} = 0, \quad c_0 = 1. \tag{2.9}$$

Next, introducing the polynomial in $E \in \mathbb{C}$

$$\hat{F}_n(E, x, t) = \sum_{j=0}^{n} E^j \hat{f}_{n-j}(x, t) \tag{2.10}$$

(2.8) becomes

$$q_t = \frac{1}{2}\hat{F}_{n,xxx} + 2(q - E)\hat{F}_{n,x} + q_x \hat{F}_n \tag{2.11}$$

and the stationary (inhomogeneous) KdV hierarchy reads

$$\frac{1}{2}\hat{F}_{n,xxx} + 2(q-E)\hat{F}_{n,x} + q_x\hat{F}_n = 0. \tag{2.12}$$

Integrating (2.12) times $\hat{F}_n$ once results in

$$\frac{1}{4}\hat{F}_{n,x}^2 - \frac{1}{2}\hat{F}_{n,xx}\hat{F}_n - (q-E)\hat{F}_n^2 = \hat{R}_{2n+1}(E), \tag{2.13}$$

where the integration constant $\hat{R}_{2n+1}(E)$ is a polynomial in E of degree $2n+1$ with leading coefficient 1 and hence can be written as

$$\hat{R}_{2n+1}(E) = \prod_{m=0}^{2n}(E-E_m), \quad \{E_m\}_{m=0}^{2n} \subset \mathbb{C}. \tag{2.14}$$

Since by (2.8) and (2.9), $q_t = 0$ is equivalent to the commutativity of $\hat{P}_{2n+1}$ and L,

$$[\hat{P}_{2n+1}, L] = 0, \tag{2.15}$$

a celebrated result of Burchnall and Chaundy [9] implies that $\hat{P}_{2n+1}$ and L satisfy an algebraic equation of the form

$$\hat{P}_{2n+1}^2 = \hat{R}_{2n+1}(L) = \prod_{m=0}^{2n}(L-E_m). \tag{2.16}$$

This naturally leads to an underlying hyperelliptic curve K_n of (arithmetic) genus n given by

$$K_n : y^2 = \hat{R}_{2n+1}(E) = \prod_{m=0}^{2n}(E-E_m). \tag{2.17}$$

In the self-adjoint case where $\{E_m\}_{m=0}^{2n} \subset \mathbb{R}$ and $E_{2n} < E_{2n-1} < \cdots < E_0$, the zeros E_m, $0 \le m \le 2n$ of $\hat{R}_{2n+1}(E)$ are precisely the spectral band edges in the sense of (1.3). For subsequent purposes we mention the following

Example 2.1.

(i) $$n = 1: \quad q(z) = -2\mathcal{P}(z), \ \mathrm{KdV}_1(q) = 0, \tag{2.18}$$

$$P_3^2 = L^3 - \frac{g_2}{4}L - \frac{g_3}{4} = \prod_{m=0}^{2}(L - E_m),$$

$$E_0 = e_1 = \mathcal{P}(\omega_1), \ E_1 = e_2 = \mathcal{P}(\omega_2), \ E_2 = e_3 = \mathcal{P}(\omega_3).$$

(ii) $$n = 2: \quad q(z) = -6\mathcal{P}(z), \ \mathrm{KdV}_2(q) - \frac{21}{8}g_2\,\mathrm{KdV}_0(q) = 0, \tag{2.19}$$

$$\Big(P_5 - \frac{21}{8}g_2 P_1\Big)^2 = (L^2 - 3g_2)\Big(L^3 - \frac{9g_2}{4}L + \frac{27g_3}{4}\Big) = \prod_{m=0}^{4}(L - E_m),$$

$$E_0 = (3g_2)^{1/2}, \ E_1 = -3e_3, \ E_2 = -3e_2, \ E_3 = -3e_1, \ E_4 = -(3g_2)^{1/2}.$$

(Here g_2, g_3 are the invariants associated with Λ, see [1], Ch. 18.)

The mKdV hierarchy can now be obtained as follows. One introduces the Lax pair

$$\mathcal{M}(t) = \begin{pmatrix} 0 & \frac{d}{dx} + \phi(x,t) \\ \frac{d}{dx} - \phi(x,t) & 0 \end{pmatrix}, \ \hat{Q}_{2n+1}(t) = \begin{pmatrix} \hat{P}_{2n+1}(t) & 0 \\ 0 & \hat{\tilde{P}}_{2n+1}(t) \end{pmatrix}, \tag{2.20}$$

where $\hat{P}_{2n+1}(t)$ respectively $\hat{\tilde{P}}_{2n+1}(t)$ are defined as in (2.2), with q respectively $\tilde{q}$ given by

$$q = -\phi_x - \phi^2, \quad \tilde{q} = \phi_x - \phi^2. \tag{2.21}$$

One then shows that

$$[\hat{Q}_{2n+1}, \mathcal{M}] = \hat{g}_{n+1,x}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.22}$$

where

$$\hat{g}_{j+1,x} = \frac{1}{4}\hat{g}_{j,xxx} - \phi^2\hat{g}_{j,x} - \phi_x\Big[\int^x dx'\phi\hat{g}_{j,x'} - c_j\Big], \quad 0 \le j \le n, \quad \hat{g}_0 = 1. \tag{2.23}$$

Here the c_j, $j \in \mathbb{N}$ are the integration constants from (2.4) and $c_0 = 1$. Also, since $\phi\hat{g}_{j,x}$ is the derivative of a certain differential polynomial in ϕ, the integral in (2.23) is understood to be homogeneous. The first few of the $\hat{g}_j$ are

$$\hat{g}_0 = 1, \quad \hat{g}_1 = \phi + c_1, \quad \hat{g}_2 = \frac{1}{4}\phi_{xx} - \frac{1}{2}\phi^3 + c_1\phi + c_2, \quad \text{etc.} \tag{2.24}$$

By g_j and Q_{2j+1} we denote the homogeneous versions of $\hat{g}_j$ and $\hat{Q}_{2j+1}$.

The mKdV hierarchy is then defined as the sequence of evolution equations

$$\text{mKdV}_n(\phi)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} := \frac{d}{dt}\mathcal{M} - [Q_{2n+1}, \mathcal{M}] = 0, \quad n \in \mathbb{N}\cup\{0\} \tag{2.25}$$

or equivalently, by

$$\text{mKdV}_n(\phi) := \phi_t - g_{n+1,x} = 0, \quad n \in \mathbb{N}\cup\{0\}. \tag{2.26}$$

Explicitly,

$$\text{mKdV}_0(\phi) = \phi_t - \phi_x, \quad \text{mKdV}_1(\phi) = \phi_t - \frac{1}{4}\phi_{xxx} + \frac{3}{2}\phi^2\phi_x, \quad \text{etc.} \tag{2.27}$$

We emphasize the symmetry of the mKdV hierarchy: with ϕ also $-\phi$ is a solution of (2.26). The special case of the n-th-order stationary mKdV equations characterized by $\phi_t = 0$ then reads

$$g_{n+1,x} = 0 \text{ respectively } \hat{g}_{n+1,x} = \sum_{j=0}^{n} c_{n-j} g_{j+1,x} = 0, \quad c_0 = 1. \tag{2.28}$$

Miura's identity [42] then connects the two hierarchies

$$\text{KdV}_n(\mp\phi_x - \phi^2) = [-2\phi \mp \partial_x]\,\text{mKdV}_n(\phi), \quad n \in \mathbb{N}\cup\{0\}. \tag{2.29}$$

The analogs of (2.16) and (2.17) then read

$$\hat{Q}_{2n+1}^2 = \prod_{m=0}^{2n}(\mathcal{M} - E_m^{1/2})(\mathcal{M} + E_m^{1/2}) = \prod_{m=0}^{2n}(\mathcal{M}^2 - E_m), \tag{2.30}$$

$$y^2 = \prod_{m=0}^{2n}(w - E_m^{1/2})(w + E_m^{1/2}) = \prod_{m=0}^{2n}(w^2 - E_m). \tag{2.31}$$

This leads to

Definition 2.2. *Any solution q (respectively ϕ) of one of the stationary equations* (2.9) *(respectively* (2.28)*) is called an* **(algebro-geometric) finite-gap potential** *associated with the KdV (respectively mKdV) hierarchy.*

The potentials q (respectively ϕ) then can be expressed in terms of the Riemann theta function associated with the (possibly singular) hyperelliptic curve K_n of (arithmetic) genus n as pioneered by Its and Matveev [39] (see also [29] and the references therein).

In the particular case where q (respectively ϕ) are elliptic functions (see, e.g., (1.1)), the following theorem of Picard plays a crucial role in their analysis. (For brevity, we only state it in the second-order case.)

Theorem 2.3. (Picard, see, e.g., [37], p. 375–376). *Consider the differential equation*

$$\psi''(z) + Q(z)\psi(z) = 0, \quad z \in \mathbb{C} \tag{2.32}$$

with Q an elliptic function with f.p. $2\omega_1$, $2\omega_3$. Suppose the general solution of (2.32) *is meromorphic. Then there exists at least one solution ψ_1 which is elliptic of the second kind, i.e., ψ_1 is meromorphic and*

$$\psi_1(z + 2\omega_j) = \rho_j \psi_1(z), \quad j = 1, 3 \tag{2.33}$$

for some constants ρ_1, $\rho_3 \in \mathbb{C}$. If in addition the characteristic equation corresponding to the substitution $z + 2\omega_1$ (or $z + 2\omega_3$) (see [37]*, p. 376 and 358) has distinct roots, then there exists a fundamental system of elliptic functions of the second kind of* (2.32)*.*

By the theory of elliptic functions, ψ_1 is elliptic of the second kind if and only if it is of the form

$$\psi_1(z) = Ce^{\lambda z} \prod_{j=1}^{m} [\sigma(z - a_j)/\sigma(z - b_j)] \tag{2.34}$$

for suitable $m \in \mathbb{N}$ and constants C, λ, a_j, b_j, $1 \leq j \leq m$. (Here $\sigma(z)$ is the Weierstrass sigma function associated with Λ, see [1], Ch. 18.)

Theorem 2.3 motivates

Definition 2.4. *Let q be elliptic. Then q is called a* **Picard potential** *if and only if*

$$\psi'' + q\psi = E\psi \tag{2.35}$$

has a meromorphic fundamental system of solutions for each $E \in \mathbb{C}$.

It can be shown [33] that q is a Picard potential whenever (2.35) has a meromorphic fundamental system of solutions for a sufficiently large number of distinct values of E.

The connection between Picard potentials and elliptic finite-gap potentials is now the following: By the Its-Matveev formula [39] for q and the corresponding Baker-Akhiezer function in terms of the associated Riemann theta function one can prove

Theorem 2.5. *Every elliptic finite-gap potential q is Picard (in the sense of Definition 2.4).*

(For more details see, e.g., [19], Ch. III, [49], Thm.6.10, [34].) Conversely, this naturally leads to the following

Conjecture . *Every Picard potential is finite-gap.*

By a systematic use of Picard's Theorem 2.3 we have recently proven this conjecture in [34]. This covers and extends in particular the cases of Lamé-Ince and Treibich-Verdier potentials (1.1) and (1.6) (and all other examples in [6], [50]). It is worthwhile to point out that this characterization of elliptic finite-gap potentials as Picard potentials yields the most effective criterion to date for deciding whether or not a given elliptic potential is actually finite-gap.

A key element in proving this conjecture turned out to be the following characterization of general periodic finite-gap potentials (not necessarily elliptic) and their associated diagonal Green's function.

Theorem 2.6 ([34]). *Assume that $q(x)$ is a periodic continuous function of period $\Omega > 0$ on $\mathbb{R}$ and that $L\psi = \psi'' + q(x)\psi = E\psi$ has two linearly independent Floquet solutions for all $E \in \mathbb{C}\backslash\{\hat{E}_j\}_{j=0}^{\hat{M}}$ for some $\hat{M} \in \mathbb{N} \cup \{0\}$ and precisely one Floquet solution for each $E = \hat{E}_j$ (assuming $\hat{E}_j \neq \hat{E}_{j'}$ for $j \neq j'$). Denote by $\hat{d}(E)$ the algebraic multiplicity of E as an (anti)periodic eigenvalue and by $\hat{p}(E)$ the minimal algebraic multiplicity of E as a Dirichlet eigenvalue on $[x_0, x_0 + \Omega]$ as x_0 varies in $\mathbb{R}$. Let $\hat{q}(E) = \hat{d}(E) - 2\hat{p}(E)$. Then*

(i). $\hat{q}(E)$ is positive on a finite set $\{\hat{E}_0, ..., \hat{E}_M\}$, $M \geq \hat{M}$ and zero elsewhere. Let $\hat{q}_j = \hat{q}(\hat{E}_j)$, $j = 0, ..., M$. Then $\sum_{j=0}^{M} \hat{q}_j = 2n + 1$ for some nonnegative integer n, i.e., $\sum_{j=0}^{M} \hat{q}_j$ is an odd positive integer. The Wronskian of two nontrivial Floquet solutions which are linearly independent on some punctured disk $0 < |E - \lambda| < \varepsilon$ tends to zero as E tends to λ if and only if $\lambda \in \{\hat{E}_0, ..., \hat{E}_M\}$.

(ii). The diagonal Green's function $G(E, x, x)$ associated with L is of the type

$$G(E, x, x) = \frac{-1}{2}\hat{F}_n(E, x)/[\hat{R}_{2n+1}(E)]^{1/2}, \tag{2.36}$$

where

$$\hat{F}_n(E, x) = \prod_{\ell=1}^{n}[E - \mu_\ell(x)], \tag{2.37}$$

$$\hat{R}_{2n+1}(E) = \prod_{j=0}^{M}(E - \hat{E}_j)^{\hat{q}_j}, \tag{2.38}$$

and where $\mu_\ell(x)$ denote (some of the) Dirichlet eigenvalues of L on the interval $[x, x + \Omega]$.

(iii). $q(x)$ is an algebro-geometric finite-gap potential associated with the compact (possibly singular) hyperelliptic curve of (arithmetic) genus

n obtained upon one-point compactification of the curve

$$y^2 = \hat{R}_{2n+1}(E) = \prod_{j=0}^{M} (E - \hat{E}_j)^{\hat{q}_j}, \tag{2.39}$$

where $n = [(\sum_{j=0}^{M} \hat{q}_j) - 1]/2$. Equivalently, there exists an ordinary differential expression $\hat{P}_{2n+1}$ of order $2n+1$, i.e.,

$$\hat{P}_{2n+1} = \sum_{\ell=0}^{2n+1} p_\ell(x) \frac{d^\ell}{dx^\ell}, \quad p_{2n+1}(x) = 1 \tag{2.40}$$

which commutes with L, and satisfies the Burchnall-Chaundy polynomial relation $\hat{P}_{2n+1}^2 = \hat{R}_{2n+1}(L)$.

The proof of Theorem 2.6 in [34] is based on well-known identities for the diagonal Green's function $G(E,x,x)$ in terms of the Floquet discriminant $\Delta(E)$ and a fundamental system of solutions of $L\psi(E,y) = E\psi(E,y)$ with respect to a reference point $x \in \mathbb{R}$, Hadamard-type factorizations of such solutions with respect to E, the nonlinear second-order differential equation satisfied by $G(E,x,x)$ respectively $\hat{F}_n(E,x)$ in (2.13), and the recursion formalism displayed in (2.1) - (2.17).

3. Lamé-Ince potentials associated with the KdV Hierarchy

In this section we study in detail the Lamé-Ince potential (1.1)

$$q(z) = -s(s+1)\mathcal{P}(z), \quad s \in \mathbb{N}, \ z \in \mathbb{C} \tag{3.1}$$

and the associated linear problem

$$\psi''(E,z) + [q(z) - E]\psi(E,z) = 0, \quad E \in \mathbb{C}. \tag{3.2}$$

The Frobenius method (see, e.g., [16]) for regular singular points of ordinary differential equations yields the following result.

Theorem 3.1. *The potential $-s(s+1)\mathcal{P}(z)$, $s \in \mathbb{C}$ is a Picard potential if and only if $s \in \mathbb{Z}$.*

It is well known that the Lamé-Ince potential (3.1) is a finite-gap potential. In fact, in the real-valued case, where $g_2^3 - 27g_3^2 > 0$, this represents a celebrated result of Ince [36] (see also [2], [5], Sects. 9.1-9.3, [57], Sects. 23.41, 23.42) as mentioned in the beginning of the introduction. The general complex-valued case was proven more recently by Treibich and Verdier [51]. Our main goal in this section (see Theorem 3.2) is to reproduce this result on the basis of the Picard property of the potential (3.1) as described in Theorem 3.1 thereby illustrating our new method in the simplest possible case.

Without loss of generality we shall assume $s \in \mathbb{N}$ in the following. By Picard's Theorem 2.3 and by (2.34), equation (3.2) has at least one solution $\psi_a(E,z)$ which is elliptic of the second kind, i.e., which is of the form

$$\psi_a(E,z) = e^{\lambda_a(E)z} \prod_{j=1}^{s} \frac{\sigma(z-a_j(E))}{\sigma(z)\sigma(-a_j(E))}, \ a(E) = (a_1(E), \dots, a_s(E)). \tag{3.3}$$

In particular, we note that near $z = 0$ any solution $\psi(E,z)$ of (3.2) behaves like

$$cz^{-s} + O(z^{-s+1}) \text{ or } cz^{s+1} + O(z^{s+2}) \tag{3.4}$$

for some nonzero constant c. One then computes

$$\begin{aligned} &-s(s+1)\mathcal{P}(z) = q(z) = E - \psi_a''(E,z)\psi_a(E,z)^{-1} \\ &= E - (2s-1)\sum_{j=1}^{s}\mathcal{P}(a_j(E)) - s(s+1)\mathcal{P}(z) + 2s\zeta(z)[\lambda_a(E) - \sum_{j=1}^{s}\zeta(a_j(E))] \\ &+ 2\sum_{j=1}^{s}\zeta(z-a_j(E))\Big[\sum_{\substack{\ell=1\\ \ell\neq j}}^{s}\zeta(a_\ell(E)-a_j(E)) + s\zeta(a_j(E)) - \lambda_a(E)\Big] \end{aligned} \tag{3.5}$$

and hence ψ_a solves (3.2) if and only if

$$E = (2s-1)\sum_{j=1}^{s}\mathcal{P}(a_j(E)), \tag{3.6}$$

$$\lambda_a(E) = \sum_{j=1}^{s}\zeta(a_j(E)), \tag{3.7}$$

$$\begin{aligned} 0 &= \sum_{\substack{\ell=1\\ \ell\neq j}}^{s}[\zeta(a_\ell(E)-a_j(E)) - \zeta(a_\ell(E)) + \zeta(a_j(E))] \\ &= \frac{1}{2}\sum_{\substack{\ell=1\\ \ell\neq j}}^{s}\frac{\mathcal{P}'(a_\ell(E)) + \mathcal{P}'(a_j(E))}{\mathcal{P}(a_\ell(E)) - \mathcal{P}(a_j(E))}, \quad 1 \le j \le s, \end{aligned} \tag{3.8}$$

where $\zeta(z)$ denotes the Weierstrass ζ function associated with Λ (see [1], Ch. 18). In order to derive (3.5) we used

$$\psi_a'/\psi_a = \lambda_a + \sum_{j=1}^{s}\zeta(z-a_j(E)) - s\zeta(z), \tag{3.9}$$

$$(\psi_a'/\psi_a)' = s\mathcal{P}(z) - \sum_{j=1}^{s}\mathcal{P}(z-a_j(E)), \tag{3.10}$$

$$
\begin{aligned}
(\psi_a'/\psi_a)^2 = (2s-1)\sum_{j=1}^{s}\mathcal{P}(a_j(E)) + \sum_{j=1}^{s}\mathcal{P}(z-a_j(E)) + s^2\mathcal{P}(z) \\
+ 2s\zeta(z)\Big[\sum_{j=1}^{s}\zeta(a_j(E)) - \lambda_a(E)\Big] \qquad (3.11) \\
- 2\sum_{j=1}^{s}\zeta(z-a_j(E))\Big[\sum_{\substack{\ell=1\\ \ell\neq j}}^{s}\zeta(a_\ell(E)-a_j(E)) + s\zeta(a_j(E)) - \lambda_a(E)\Big].
\end{aligned}
$$

(3.6), (3.7), and (3.8) show that together with ψ_a also ψ_{-a} is a solution of (3.2) for the same value of E, where $-a(E) = (-a_1(E), ..., -a_s(E))$. This fact is due to the reflection symmetry of the potential, i.e., to $q(-z) = q(z)$. Hence

$$
\psi''_{\pm a}(E,z) - [s(s+1)\mathcal{P}(z) + E]\psi_{\pm a}(E,z) = 0. \qquad (3.12)
$$

Their Wronskian $W(E) := W(\psi_a(E), \psi_{-a}(E))$ can be computed as

$$
\begin{aligned}
W(E) &= -\prod_{\ell=1}^{s}[\mathcal{P}(z)-\mathcal{P}(a_\ell)]\sum_{j=1}^{s}\mathcal{P}'(a_j)[\mathcal{P}(z)-\mathcal{P}(a_j)]^{-1} \\
&= -\sum_{j=1}^{s}\mathcal{P}'(a_j)\prod_{\substack{\ell=1\\ \ell\neq j}}^{s}[\mathcal{P}(z)-\mathcal{P}(a_\ell)] \qquad (3.13)
\end{aligned}
$$

(where $W(f,g)(z) := f(z)g'(z) - f'(z)g(z)$). Since $W(E)$ is independent of z we may evaluate (3.13) at $z = a_j$ to obtain

$$
W(E) = -\mathcal{P}'(a_j)\prod_{\substack{\ell=1\\ \ell\neq j}}^{s}[\mathcal{P}(a_j)-\mathcal{P}(a_\ell)], \quad 1 \leq j \leq s. \qquad (3.14)
$$

Moreover, $\psi_{\pm a}(E,z)$ are Floquet solutions of (3.2) since

$$
\psi_{\pm a}(E, z+2\omega_\ell) = \exp\Big\{\pm\sum_{j=1}^{s}[2\omega_\ell\zeta(a_j(E)) - 2a_j(E)\zeta(\omega_\ell)]\Big\}\psi_{\pm a}(E,z), \quad \ell = 1,3. \qquad (3.15)
$$

As described in Theorem 2.6, in order to show that $q(z)$ in (3.1) is a finite-gap potential and find the (arithmetic) genus n of the (possibly singular) hyperelliptic curve K_n, we need to find the E-values where $W(E)$ vanishes. (Note that for no value of E the functions $\psi_{\pm a}$ are identically equal to zero.) These values (together with the point at infinity upon one-point compactification) constitute the location of the branch points resp. singular points of the two-sheeted Riemann surface K_n. By (3.4) and the fact that $a_j(E) \neq a_\ell(E)$ for $j \neq \ell$ (since zeros in z of ψ different from the singularity $z = 0 (\operatorname{mod} \Delta)$ in (3.2) are necessarily simple) the Wronskian $W(E)$ vanishes if either a_j is a half-period (i.e., $a_j = \omega_\ell (\operatorname{mod} \Delta)$, $\ell \in \{1,2,3\}$, Δ the fundamental period parallelogram with vertices 0, $2\omega_1$, $2\omega_2$, $2\omega_3$) or there are pairs of the type $(a_{j_0}, -a_{j_0} (\operatorname{mod} \Delta))$ among $\{a_j\}_{j=1}^s$. Thus one obtains for even s either

$$\{a_1, \dots, a_s\} = \{a_{j_1}, -a_{j_1}, a_{j_2}, -a_{j_2}, \dots, a_{j_{s/2}}, -a_{j_{s/2}}\} \tag{3.16}$$

or

$$\{a_1, \dots, a_s\} = \{\omega_{\ell_1}, \omega_{\ell_2}, a_{j_1}, -a_{j_1}, \dots, a_{j_{(s-2)/2}}, -a_{j_{(s-2)/2}}\} \tag{3.17}$$

with $\omega_{\ell_1}, \omega_{\ell_2} \in \{\omega_1, \omega_2, \omega_3\}$ and $\omega_{\ell_1} \neq \omega_{\ell_2}$.

Similarly one has for odd s either

$$\{a_1, \dots, a_s\} = \{\omega_{\ell_1}, a_{j_1}, -a_{j_1}, \dots, a_{j_{(s-1)/2}}, -a_{j_{(s-1)/2}}\} \tag{3.18}$$

with $\omega_{\ell_1} \in \{\omega_1, \omega_2, \omega_3\}$ or

$$\{a_1, \dots, a_s\} = \{\omega_1, \omega_2, \omega_3, a_{j_1}, -a_{j_1}, \dots, a_{j_{(s-3)/2}}, -a_{j_{(s-3)/2}}\}. \tag{3.19}$$

(Here we abbreviated $-a_\ell (\operatorname{mod} \Delta)$ as $-a_\ell$ for notational convenience.) We emphasize again that the parameters $a_j(E)$ need to satisfy the conditions (3.8). In the cases $s = 1$ and $s = 2$ this information suffices to determine the points E where $W(E)$ vanishes. For $s > 2$, however, the conditions (3.8) appear to be too difficult to be handled directly. How this problem is circumvented will now be illustrated in the case (3.16). Similar considerations work in the other cases. Equations (3.3) and (3.16) yield for any $\hat{E}$ such that $W(\hat{E}) = 0$,

$$\begin{aligned} \psi_{\pm a}(\hat{E}, z) &= (-1)^{s/2} \prod_{j=1}^{s/2} \frac{\sigma(z - a_j(\hat{E}))\sigma(z + a_j(\hat{E}))}{\sigma(z)^2 \sigma(a_j(\hat{E}))^2} \\ &= \prod_{j=1}^{s/2} [\mathcal{P}(z) - \mathcal{P}(a_j(\hat{E}))] = \sum_{j=0}^{s/2} \nu_j(\hat{E}) \mathcal{P}(z)^j \end{aligned} \tag{3.20}$$

for appropriate constants $\nu_j(\hat{E})$ since

$$\lambda_{\pm a}(\hat{E}) = 0 \tag{3.21}$$

in this case. We therefore make the (slightly refined) ansatz

$$\psi_{\pm a}(E, z) = \sum_{j=0}^{s/2} \mu_j(E) [\mathcal{P}(z) - e_2]^j \tag{3.22}$$

and insert it into (3.2). Defining

$$\mu_{-1}(E) = \mu_{(s+2)/2}(E) = 0 \tag{3.23}$$

then yields (with $\mathcal{P}'^2 = 4(\mathcal{P}-e_2)^3 + 12e_2(\mathcal{P}-e_2)^2 + 4(e_2-e_1)(e_2-e_3)(\mathcal{P}-e_2)$)

$$\sum_{j=0}^{d}\{(1-2j-s)(2-2j+s)\mu_{j-1} + [e_2(12j^2 - s(s+1)) - E]\mu_j + 2(e_2-e_1)(e_2-e_3)(1+j)(1+2j)\mu_{j+1}\}[\mathcal{P}(z)-e_2]^j = 0, \quad d = s/2 \tag{3.24}$$

and hence is equivalent to the eigenvalue problem

$$J\underline{\mu} = E\underline{\mu}, \quad \underline{\mu} = (\mu_d, \dots, \mu_0)^T \tag{3.25}$$

where J is the $(d+1)\times(d+1)$ Jacobi matrix

$$J = \begin{pmatrix} \beta_d & \alpha_d & 0 & \cdots & \cdots & 0 \\ \gamma_{d-1} & \beta_{d-1} & \alpha_{d-1} & \ddots & & \vdots \\ 0 & \gamma_{d-2} & \beta_{d-2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \gamma_1 & \beta_1 & \alpha_1 \\ 0 & \cdots & \cdots & 0 & \gamma_0 & \beta_0 \end{pmatrix} \tag{3.26}$$

with

$$\begin{aligned} &\alpha_j = (1-2j-s)(2-2j+s),\ \beta_j = e_2[12j^2 - s(s+1)], \\ &\gamma_j = 2(e_2-e_1)(e_2-e_3)(1+j)(1+2j), \quad 0 \le j \le d = s/2 \end{aligned} \tag{3.27}$$

in case (3.16). Repeating the analysis (3.20)–(3.26) in the three remaining cases (3.17)–(3.19) one again obtains an eigenvalue problem of the type (3.25) for determining the number of the corresponding branch points and singular points. More precisely, J is given by (3.26) with

$$\begin{aligned} &\alpha_j = (2j-s)(1+2j+s),\ \beta_j = -e_{\ell_3}(3+4j) + e_2[6+4j(4+3j) - s(s+1)], \\ &\gamma_j = 2(e_2-e_1)(e_2-e_3)(1+j)(3+2j-2\delta_{\ell_3,2}), \quad 0 \le j \le d = (s-2)/2 \end{aligned} \tag{3.28}$$

in case (3.17) with (ℓ_1,ℓ_2,ℓ_3) a cyclic permutation of $(1,2,3)$,

$$\begin{aligned} &\alpha_j = (2j-1-s)(2_j+s),\ \beta_j = e_{\ell_1}(1+4j) + e_2[2+4j(2+3j) - s(s+1)], \\ &\gamma_j = 2(e_2-e_1)(e_2-e_3)(1+j)(1+2j+2\delta_{\ell_1,2}), \quad 0 \le j \le d = (s-1)/2 \end{aligned} \tag{3.29}$$

in case (3.18),

$$\begin{aligned} &\alpha_j = (2j+1-s)(2j+2+s),\ \beta_j = e_2[12(1+j)^2 - s(s+1)], \\ &\gamma_j = 2(e_2-e_1)(e_2-e_3)(1+j)(3+2j), \quad 0 \le j \le d = (s-3)/2 \end{aligned} \tag{3.30}$$

in case (3.19). (Here $\delta_{\ell,m} = \left\{\begin{smallmatrix} 1, & \ell=m \\ 0, & \ell\neq m \end{smallmatrix}\right\}$.)

Hence if s is even, (3.27) and (3.28) yield the following: there is a $((s+2)/2) \times ((s+2)/2)$ eigenvalue problem from (3.27) and there are three possibilities to pick two half-periods ω_{ℓ_1}, ω_{ℓ_2} out of $\{\omega_1, \omega_2, \omega_3\}$ each yielding an $(s/2) \times (s/2)$ eigenvalue problem from (3.28). Altogether this yields generically $[(s+2)/2] + 3(s/2) = 2s+1$ eigenvalues. Similarly, if s is odd, (3.29) yields three $((s+1)/2) \times ((s+1)/2)$ eigenvalue problems (one for each ω_ℓ, $1 \leq \ell \leq 3$) and one $((s-1)/2) \times ((s-1)/2)$ eigenvalue problem from (3.30). Altogether we get again $3((s+1)/2) + ((s-1)/2) = 2s+1$ eigenvalues.

In the real-valued case where $g_2^3 - 27g_3^2 > 0$, one can apply the well known fact (see, e.g., [5], p. 21) that the (real) Jacobi matrix J in (3.26) has real and distinct eigenvalues if $\gamma_j \alpha_{j+1} > 0$, $0 \leq j \leq d-1$. Also an eigenvalue cannot occur simultaneously in any two of the four matrices associated to a given problem for the following reason. Assume on the contrary that an eigenvalue E appears in two of the four matrices, say in J_1 and J_2. Then the eigenvector associated with E must be the same for both J_1 and J_2 since otherwise there would be two different Floquet solutions at this particular value E while, by construction, all the eigenvalues of these matrices refer to points where only one Floquet solution exists. Hence zero is an eigenvalue of $J_1 - J_2$ and, in particular, J_1 and J_2 have the same size. This may happen in the cases (3.17) and (3.18). Now consider the latter case and assume that J_1 is the matrix associated with ℓ_1 and J_2 the matrix associated with ℓ_2. (3.29) implies then that $J_1 - J_2$ is upper triangular and that its diagonal elements are given by $(e_{\ell_1} - e_{\ell_2})(1+4j)$. The diagonal of $J_1 - J_2$ must contain zero (zero is an eigenvalue) and hence we conclude that $e_{\ell_1} = e_{\ell_2}$ which is impossible. Since this argument works in the same way in all other cases we have recovered Ince's result [36] (see also [2], [5], Sects. 9.1–9.3, [57], Sects. 23.41, 23.42) that, in the case $g_2^3 - 27g_3^2 > 0$, all $2s+1$ values of E for which $W(E)$ vanishes are real and different from each other, $E_{2s} < E_{2s-1} < \cdots < E_0$ (see (1.3)). For alternative methods to compute $\{E_m\}_{m=0}^{2s}$ in the Lamé-Ince case see, e.g., [54], [56].

The corresponding solutions $\psi_{\pm a}(E_m, z)$ in (3.22) with $\mu_j(E_m)$ determined from the eigenvector $\underline{\mu}$ in (3.25), subject to one of (3.27)–(3.30), are the so called Lamé polynomials, see [5], Ch. IX and [57], Ch. XXIII, vital in the study of ellipsoidal harmonics in connection with Laplace's equation in ellipsoidal coordinates.

We sum up the findings of this section in

Theorem 3.2. *The Lamé potential $q(z) = -s(s+1)\mathcal{P}(z)$, $s \in \mathbb{C}$ is a finite-gap potential associated with the stationary KdV hierarchy, or equivalently, a Picard potential if and only if $s \in \mathbb{Z}$. If $s \in \mathbb{N} \cup \{0\}$ the underlying hyperelliptic curve K_s is of the form $y^2 = \prod_{m=0}^{2s}(E - E_m)$ and hence q satisfies a stationary KdV equation of the type*

$$\sum_{j=0}^{s} c_{s-j}\, \mathrm{KdV}_j(q) = 0, \quad c_0 = 1 \tag{3.31}$$

with the c_ℓ depending on g_2, g_3. For generic values of g_2, g_3 the curve K_s is nonsingular (i.e., $E_m \neq E_\ell$ for $m \neq \ell$). In particular, in the real-valued case where $g_2^3 - 27g_3^2 > 0$, K_s is nonsingular and all (finite) branch points are in real position, i.e., $\{E_m\}_{m=0}^{2s} \subset \mathbb{R}$.

Proof. If $s \notin \mathbb{Z}$ then clearly q is neither a Picard nor a finite-gap potential by Theorems 3.1 and 2.5.

If $s \in \mathbb{Z}$ then q is a Picard potential by Theorem 3.1 and there exist two linearly independent Floquet solutions for all but finitely many values of the spectral parameter E. Hence q is finite-gap by Theorem 2.6.

Now let $J = J_1 \oplus ... \oplus J_4$ where $J_1, ..., J_4$ are the Jacobi matrices (3.26) in the appropriate cases. Theorem 2.6 now implies that the eigenvalues of J are precisely the numbers $\hat{E}_0, ..., \hat{E}_M$. Hence the remaining statements follow if we can show that the multiplicity $\hat{q}_j$ for $j = 0, ..., M$ is precisely the multiplicity with which $\hat{E}_j$ appears in J. For $g_2^3 - 27g_3^2 > 0$ ordinary Floquet theory shows that $\hat{M} = M$ and $\hat{q}_j = 1$ for $j = 0, ..., M$. Also, according to earlier considerations, the eigenvalues of J are all distinct. Hence, in this case, the multiplicities do indeed coincide. Since the eigenvalues of J vary continuously with g_2 and g_3 their multiplicities must always coincide with the multiplicities of the corresponding $\hat{E}_j$. □

Given (3.31), q then satisfies appropriate stationary KdV equations of all orders higher than s.

While for generic values of g_2 and g_3 the numbers $\hat{E}_0, ..., \hat{E}_M$ will all be different from each other (and hence the underlying curve K_s of genus s will be nonsingular), for particular values of g_2 and g_3 some solutions can in fact coincide as shown, e.g., in (2.19) of Example 2.1 (ii) for $g_2 = 0$ (in which case K_s is singular).

Remark 3.3. (i) Our approach to Lamé's equation (3.2) is based on Picard's theorem and on the ansatz (3.3) which can be found, e.g., in Burkhardt's monograph [10], p. 343–353 from 1899 and in Halphen's monograph [35], p. 494–498 from 1888. In particular, the crucial condition (3.8) appears in [10] and [35]. Curiously enough, the section on Lamé's equation has been eliminated from the 1920 edition of [10]. Moreover, condition (3.8) is also not mentioned in Whittaker and Watson's monograph [57] and apparently in none of the other standard texts containing a discussion of Lamé's equation published in this century. (We note however, that an analog of (3.8) is mentioned on p. 574 of [57] in the context of Jacobi elliptic functions.) In particular, none of the references cited in the Introduction mentions condition (3.8) for general $s \in \mathbb{N}$.

(ii) Our main contribution to the circle of ideas in this section (besides reviving Picard's theorem in the manner of [10] and [35]) consists of establishing the eigenvalue problems (3.25)–(3.30) suitable modifications of which extend to all Treibich-Verdier [32] and, more generally, to all even Picard potentials as proven in [33]. This establishes, in particular, the analogs of Lamé polynomials for Treibich-Verdier and all even Picard potentials.

Finally, it should be mentioned that the fact that all Floquet solutions and multipliers can be parametrized by meromorphic functions on a compact Riemann surface is a typically one-dimensional phenomenon which does not extend to two dimensions as shown in [25].

4. Lamé potentials associated with the mKdV hierarchy

In this section we briefly indicate how to transfer the results of Section 3 to the stationary mKdV hierarchy.

Assuming in accordance with (2.21) and (3.1) that

$$q(z) = -s(s+1)\mathcal{P}(z) = -\phi'(z) - \phi(z)^2, \quad s \in \mathbb{N} \cup \{0\},\ z \in \mathbb{C} \tag{4.1}$$

we need to compute $\phi(z)$. By Miura's identity (2.29) and the commutation results of [21], [28], [30] ϕ will then solve appropriate stationary mKdV equations of all orders greater than or equal to s.

Let $\psi_\pm(E, z, z_0)$ be the normalized Floquet solutions

$$\psi_\pm(E, z, z_0) = \psi_{\pm a}(E, z)/\psi_{\pm a}(E, z_0) \tag{4.2}$$

for some appropriate $z_0 \in \mathbb{C}$ and $\psi_{\pm a}$ given by (3.3). General Floquet theory (see, e.g., [28], Appendix F) then yields the identity

$$W(\psi_-(E, \,.\,, z_0), \psi_+(E, \,.\,, z_0)) = 2i\phi_I(E, z_0), \tag{4.3}$$

where

$$\phi_\pm(E,z) = -\frac{1}{2}\frac{d}{dz}\ln[\phi_I(E,z)] \pm i\phi_I(E,z), \tag{4.4}$$

$$\phi_\pm(E,z) = \frac{d}{dz}\ln[\psi_\pm(E,z,z_0)], \tag{4.5}$$

and

$$q(z) - E = -\phi'_\pm(E,z) - \phi_\pm(E,z)^2. \tag{4.6}$$

While $\phi_\pm(E,z)$ can be obtained directly from (3.7) and (3.9) we indicate next how to compute ϕ_I which then also immediately gives rise to $\phi_\pm(E,z)$ by (4.4). The reason for this procedure is that ϕ_I is a fundamental quantity in Floquet theory (it determines the Floquet solutions and hence the Green's function of L) as well as in the context of complete integrability of the KdV-hierarchy in the periodic case (it yields the infinite sequence of conserved densities as coefficients in a high-energy asymptotic expansion of ϕ_I).

Evaluating (3.13) at $z = z_0$ and observing that $W(E)$ is independent of z then yields

$$W(E) = -\sum_{j=1}^{s} \frac{\mathcal{P}'(a_j(E))}{\mathcal{P}(z_0) - \mathcal{P}(a_j(E))} \prod_{\ell=1}^{s} (\mathcal{P}(z_0) - \mathcal{P}(a_\ell(E))) \tag{4.7}$$

$$= (-1)^s \sum_{j=1}^{s} \mathcal{P}'(a_j(E)) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{s} \mathcal{P}(a_\ell(E)). \tag{4.8}$$

Combining this with (4.2), (4.3), and

$$\psi_a(E,z)\psi_{-a}(E,z) = \prod_{j=1}^{s} [\mathcal{P}(z) - \mathcal{P}(a_j(E))], \tag{4.9}$$

one infers

$$\begin{aligned} \phi_I(E,z_0) &= \frac{1}{2i} \frac{-W(E)}{\psi_a(E,z_0)\psi_{-a}(E,z_0)} \\ &= \frac{1}{2i} \sum_{j=1}^{s} \frac{\mathcal{P}'(a_j(E))}{\mathcal{P}(z_0) - \mathcal{P}(a_j(E))} \qquad (4.10) \\ &= \frac{i}{2}(-1)^s \sum_{\ell=1}^{s} \mathcal{P}'(a_\ell(E)) \prod_{\substack{m=1 \\ m \neq \ell}}^{s} \mathcal{P}(a_m(E)) \prod_{j=1}^{s} [\mathcal{P}(z_0) - \mathcal{P}(a_j(E))]^{-1}. \qquad (4.11) \end{aligned}$$

Thus (4.4) yields

$$\begin{aligned}\phi_{\pm}(E,z) &= \frac{1}{2}\frac{d}{dz}\ln\{\prod_{j=1}^{s}[\mathcal{P}(z)-\mathcal{P}(a_j(E))]\} \pm \frac{1}{2}\sum_{j=1}^{s}\frac{\mathcal{P}'(a_j(E))}{\mathcal{P}(z)-\mathcal{P}(a_j(E))} \\ &= \frac{1}{2}\sum_{j=1}^{s}\frac{\mathcal{P}'(z)\pm\mathcal{P}'(a_j(E))}{\mathcal{P}(z)-\mathcal{P}(a_j(E))}. \end{aligned} \tag{4.12}$$

The commutation methods in [21], [28], [30] relating $L = \frac{d^2}{dx^2} + q$, $q = -\phi' - \phi^2$, $\tilde{L} = \frac{d^2}{dx^2} + \tilde{q}$, $\tilde{q} = \phi' - \phi^2$, and $\mathcal{M} = \left(\begin{smallmatrix} 0 & \frac{d}{dx}+\phi \\ \frac{d}{dx}-\phi & 0 \end{smallmatrix}\right)$ together with (2.29), Theorem 3.2, (4.6) and (4.12) then yield the following

Theorem 4.1. *The Lamé potential* $\phi_\epsilon(z) = \pm\frac{1}{2}\sum_{\ell=1}^{s}\frac{\mathcal{P}'(z)+\epsilon\mathcal{P}'(a_\ell(0))}{\mathcal{P}(z)-\mathcal{P}(a_\ell(0))}$, $\epsilon \in \{+,-\}$ *is a finite-gap potential associated with the stationary mKdV hierarchy if and only if* $a_j(0)$, $1 \leq j \leq s$ *satisfy* (3.6) *and* (3.8) *for* $E = 0$. *The underlying hyperelliptic curve* K_{2s} *is of the form* $y^2 = \prod_{m=0}^{2s}(w - E_m^{1/2})(w + E_m^{1/2}) = \prod_{m=0}^{2s}(w^2 - E_m)$ *and* ϕ_ϵ *satisfies a stationary mKdV equation of the type*

$$\sum_{j=0}^{s} c_{s-j}\,\mathrm{mKdV}_j(\phi_\epsilon) = 0, \quad c_0 = 1 \tag{4.13}$$

with c_ℓ *as in* (3.31).

As discussed in Section 3, the curve K_{2s} is nonsingular for generic values of g_2, g_3. Moreover, ϕ_ϵ automatically satisfies appropriate stationary mKdV equations of all orders higher than s.

It should be mentioned that ϕ_ϵ for $s = 1$ appears in a relativistic version of Calogero-Moser-type systems discussed in [8], [47], [48].

Finally, given ϕ_ϵ by (4.12), one computes for the finite-gap potential $\tilde{q}_\epsilon = q_\epsilon + 2\phi'_\epsilon$ in $\tilde{L}_\epsilon$

$$\tilde{q}_\epsilon(z) = \phi'_\epsilon(z) - \phi_\epsilon(z)^2 = -s(s-1)\mathcal{P}(z) - 2\sum_{j=1}^{s}\mathcal{P}(z - \epsilon a_j(0)), \quad \epsilon \in \{+,-\}. \tag{4.14}$$

$\tilde{q}_\epsilon(z)$ is isospectral to $q(z) = -s(s+1)\mathcal{P}(z)$, i.e., corresponds to the same hyperelliptic curve $K_s : y^2 = \prod_{m=0}^{2s}(E - E_m)$.

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