Operators
and
Matrices
Volume 7, Number 2 (2013), 241–283 doi:10.7153/oam-07-15

INITIAL VALUE PROBLEMS AND WEYL–TITCHMARSH THEORY FOR SCHRÖDINGER OPERATORS WITH OPERATOR–VALUED POTENTIALS

FRITZ GESZTESY, RUDI WEIKARD AND MAXIM ZINCHENKO

Dedicated with great affection to the memory of W. Norrie Everitt (1924–2011)

(*Communicated by A. Böttcher*)

Abstract. We develop Weyl–Titchmarsh theory for self-adjoint Schrödinger operators H_α in $L^2((a,b);dx;\mathscr{H})$ associated with the operator-valued differential expression $\tau = -(d^2/dx^2) + V(\cdot)$, with $V : (a,b) \to \mathscr{B}(\mathscr{H})$, and $\mathscr{H}$ a complex, separable Hilbert space. We assume regularity of the left endpoint a and the limit point case at the right endpoint b. In addition, the bounded self-adjoint operator $\alpha = \alpha^* \in \mathscr{B}(\mathscr{H})$ is used to parametrize the self-adjoint boundary condition at the left endpoint a of the type

$$\sin(\alpha)u'(a) + \cos(\alpha)u(a) = 0,$$

with u lying in the domain of the underlying maximal operator $H_{\max}$ in $L^2((a,b);dx;\mathscr{H})$ associated with τ. More precisely, we establish the existence of the Weyl–Titchmarsh solution of H_α, the corresponding Weyl–Titchmarsh m-function m_α and its Herglotz property, and determine the structure of the Green's function of H_α.

Developing Weyl–Titchmarsh theory requires control over certain (operator-valued) solutions of appropriate initial value problems. Thus, we consider existence and uniqueness of solutions of 2nd-order differential equations with the operator coefficient V,

$$\begin{cases} -y'' + (V - z)y = f \text{ on } (a,b), \\ y(x_0) = h_0,\ y'(x_0) = h_1, \end{cases}$$

under the following general assumptions: $(a,b) \subseteq \mathbb{R}$ is a finite or infinite interval, $x_0 \in (a,b)$, $z \in \mathbb{C}$, $V : (a,b) \to \mathscr{B}(\mathscr{H})$ is a weakly measurable operator-valued function with $\|V(\cdot)\|_{\mathscr{B}(\mathscr{H})} \in L^1_{\text{loc}}((a,b);dx)$, and $f \in L^1_{\text{loc}}((a,b);dx;\mathscr{H})$. We also study the analog of this initial value problem with y and f replaced by operator-valued functions $Y, F \in \mathscr{B}(\mathscr{H})$.

Our hypotheses on the local behavior of V appear to be the most general ones to date.

Mathematics subject classification (2010): Primary: 34A12, 34B20, 34B24; Secondary: 47E05.
Keywords and phrases: Weyl–Titchmarsh theory, ODEs with operator coefficients, Schrödinger operators.
Based upon work partially supported by the US National Science Foundation under Grant No. DMS 0965411.

1. Introduction

The principal purpose of this paper is to derive a streamlined version of Weyl–Titchmarsh theory for Schrödinger operators on a finite or infinite interval $(a,b) \subset \mathbb{R}$ with operator-valued potentials $V \in \mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ a complex, separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the Banach space of bounded linear operators defined on $\mathcal{H}$) under very general conditions on the local behavior of V. We will work under the (simplifying) hypothesis that the underlying operator-valued differential expression

$$\tau = -d^2/dx^2 + V(x), \quad x \in (a,b), \tag{1.1}$$

is regular at the left endpoint a and in the limit point case at the right endpoint b. (For simplicity, the reader may think of the standard half-line case $(a,b) = (0,\infty)$.)

In performing this task, it is necessary to first study existence and uniqueness questions of the following initial value problems associated with τ in great detail. More precisely, in Section 2 we investigate the following two types of initial value problems: First, we consider existence and uniqueness of $\mathcal{H}$-valued solutions $y(z,\cdot,x_0) \in W^{2,1}_{\text{loc}}((a,b);dx;\mathcal{H})$ of the initial value problem

$$\begin{cases} -y'' + (V-z)y = f \text{ on } (a,b)\backslash E, \\ y(x_0) = h_0,\ y'(x_0) = h_1, \end{cases} \tag{1.2}$$

where the exceptional set E is of Lebesgue measure zero and independent of z. Here we suppose that $(a,b) \subseteq \mathbb{R}$ is a finite or infinite interval, $x_0 \in (a,b)$, $z \in \mathbb{C}$, $V : (a,b) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable operator-valued function with $\|V(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}((a,b);dx)$, and that $h_0, h_1 \in \mathcal{H}$, and $f \in L^1_{\text{loc}}((a,b);dx;\mathcal{H})$.
In particular, we prove for fixed $x_0, x \in (a,b)$ and $z \in \mathbb{C}$, that

- $y(z,x,x_0)$ depends jointly continuously on $h_0, h_1 \in \mathcal{H}$, and $f \in L^1_{\text{loc}}((a,b);dx;\mathcal{H})$,
- $y(z,x,x_0)$ is strongly continuously differentiable with respect to x on (a,b),
- $y'(z,x,x_0)$ is strongly differentiable with respect to x on $(a,b)\backslash E$,

and that

- for fixed $x_0, x \in (a,b)$, $y(z,x,x_0)$ and $y'(z,x,x_0)$ are entire with respect to z.

Second, again assuming $(a,b) \subseteq \mathbb{R}$ to be a finite or infinite interval, $x_0 \in (a,b)$, $z \in \mathbb{C}$, $Y_0, Y_1 \in \mathcal{B}(\mathcal{H})$, and $F, V : (a,b) \to \mathcal{B}(\mathcal{H})$ two weakly measurable operator-valued functions with $\|V(\cdot)\|_{\mathcal{B}(\mathcal{H})}, \|F(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}((a,b);dx)$, we consider existence and uniqueness of $\mathcal{B}(\mathcal{H})$-valued solutions $Y(z,\cdot,x_0) : (a,b) \to \mathcal{B}(\mathcal{H})$ of the initial value problem

$$\begin{cases} -Y'' + (V-z)Y = F \text{ on } (a,b)\backslash E, \\ Y(x_0) = Y_0,\ Y'(x_0) = Y_1, \end{cases} \tag{1.3}$$

where again the exceptional set E is of Lebesgue measure zero and independent of z.
For fixed $x_0 \in (a,b)$ and $z \in \mathbb{C}$, we prove that

- $Y(z,x,x_0)$ is continuously differentiable with respect to x on (a,b) in the $\mathcal{B}(\mathcal{H})$-norm,

- $Y'(z,x,x_0)$ is strongly differentiable with respect to x on $(a,b)\backslash E$,

and that

- for fixed $x_0, x \in (a,b)$, $Y(z,x,x_0)$ and $Y'(z,x,x_0)$ are entire in z in the $\mathscr{B}(\mathscr{H})$-norm.

In addition, Section 2 introduces the notion of regular endpoints of intervals, several notions of Wronskians, the variation of constants formula, and several versions of Green's formula.

Our principal Section 3 then develops Weyl–Titchmarsh theory associated with the operator-valued differential expression τ in (1.1) under the simplifying (yet most important) assumption that the left endpoint a is regular for τ and that the right endpoint b is of the limit point type for τ. We introduce minimal and maximal operators associated with τ, show that they are adjoint to each other, introduce the self-adjoint operators H_α in the underlying Hilbert space $L^2((a,b);dx;\mathscr{H})$, parametrized by the bounded self-adjoint operator $\alpha = \alpha^* \in \mathscr{B}(\mathscr{H})$ in the self-adjoint boundary condition at the left endpoint a of the type

$$\sin(\alpha)u'(a) + \cos(\alpha)u(a) = 0,$$

with u lying in the domain of the maximal operator $H_{\max}$ in $L^2((a,b);dx;\mathscr{H})$, establish the existence of the Weyl–Titchmarsh solution of H_α, introduce the corresponding Weyl–Titchmarsh m-function m_α and its Herglotz property, and determine the structure of the Green's function of H_α.

Appendix A then establishes basic facts on bounded operator-valued Herglotz functions (i.e., $\mathscr{B}(\mathscr{H})$-valued functions M analytic in the open upper complex half-plane $\mathbb{C}_+$ with $\operatorname{Im}(M(\cdot) \geqslant 0$ on $\mathbb{C}_+$).

While we restrict our attention to the case (a,b) with a a regular point for τ and τ in the limit point case at b, it is clear how to apply the standard 2×2 block operator formalism (familiar in the case of scalar and matrix-valued potentials V) to obtain the Weyl–Titchmarsh formalism for Schrödinger operators with both endpoints a and b in the limit point case (and hence Schrödinger operators on the whole real line $\mathbb{R}$, cf. Remark 3.18).

Of course, Schrödinger operators with bounded and unbounded operator-valued potentials $V(\cdot)$ have been studied in the past and we will briefly review the fundamental contributions in this area next. We note, however, that our hypotheses on the local behavior of $V(\cdot) \in \mathscr{B}(\mathscr{H})$ appear to be the most general to date.

The case of Schrödinger operators with operator-valued potentials under various continuity or smoothness hypotheses on $V(\cdot)$ and under various self-adjoint boundary conditions on bounded and unbounded open intervals received considerable attention in the past: In the special case where $\dim(\mathscr{H}) < \infty$, that is, in the case of Schrödinger operators with matrix-valued potentials, the literature is so voluminous that we cannot possibly describe individual references and hence we primarily refer to [3], [91], and the references cited therein. We also mention that the finite-dimensional case, $\dim(\mathscr{H}) < \infty$, as discussed in [23], is of considerable interest as it represents an important ingredient in some proofs of Lieb–Thirring inequalities (cf. [63]).

In addition, the constant coefficient case, where τ is of the form $\tau = -(d^2/dx^2) + A$, has received overwhelming attention. But since this is not the focus of this paper we just refer to [49], [50, Chs. 3, 4], [69], and the literature cited therein.

In the particular case of Schrödinger-type operators corresponding to the differential expression $\tau = -(d^2/dx^2) + A + V(x)$ on a bounded interval $(a,b) \subset \mathbb{R}$ with either $A = 0$ or A a self-adjoint operator satisfying $A \geqslant cI_{\mathscr{H}}$ for some $c > 0$, unique solvability of boundary value problems, the asymptotic behavior of eigenvalues, and trace formulas in connection with various self-adjoint realizations of $\tau = -(d^2/dx^2) + A + V(x)$ on a bounded interval (a,b) are discussed, for instance, in [11]–[13], [19], [46], [47], [51], [52], [74], [76] (for the case of spectral parameter dependent separated boundary conditions, see also [5], [7], [20]).

For earlier results on various aspects of boundary value problems, spectral theory, and scattering theory in the half-line case $(a,b) = (0,\infty)$, the situation closely related to the principal topic of this paper, we refer, for instance, to [6], [8], [35], [46]–[48], [51], [60], [74], [76], [87], [94], [103] (the case of the real line is discussed in [105]). While our treatment of initial value problems was inspired by the one in [94], we permit a more general local behavior of $V(\cdot)$. In addition, we also put particular emphasis on Weyl–Titchmarsh theory and the structure of the Green's function of H_α.

We should also add that this paper represents a first step in our program. Step two will be devoted to spectral properties of H_α, and step three will aim at certain classes of unbounded operator-valued potentials V, applicable to multi-dimensional Schrödinger operators in $L^2(\mathbb{R}^n; d^n x)$, $n \in \mathbb{N}$, $n \geqslant 2$, generated by differential expressions of the type $\Delta + V(\cdot)$. In fact, it was precisely the connection between multi-dimensional Schrödinger operators and one-dimensional Schrödinger operators with unbounded operator-valued potentials which originally motivated our interest in this program. This connection was already employed by Kato [58] in 1959; for more recent applications of this connection between one-dimensional Schrödinger operators with unbounded operator-valued potentials and multi-dimensional Schrödinger operators we refer, for instance, to [2], [32], [56], [64], [69], [71]–[73], [92], [93], [95]–[101], and the references cited therein.

Finally, we comment on the notation used in this paper: Throughout, $\mathscr{H}$ denotes a separable, complex Hilbert space with inner product and norm denoted by $(\cdot,\cdot)_{\mathscr{H}}$ (linear in the second argument) and $\|\cdot\|_{\mathscr{H}}$, respectively. The identity operator in $\mathscr{H}$ is written as $I_{\mathscr{H}}$. We denote by $\mathscr{B}(\mathscr{H})$ the Banach space of linear bounded operators in $\mathscr{H}$. The domain, range, kernel (null space) of a linear operator will be denoted by $\operatorname{dom}(\cdot)$, $\operatorname{ran}(\cdot)$, $\ker(\cdot)$, respectively. The closure of a closable operator S in $\mathscr{H}$ is denoted by $\overline{S}$.

2. The initial value problem of second-order differential equations with operator coefficients

In this section we provide some basic results about initial value problems for second-order differential equations of the form $-y'' + Qy = f$ on an arbitrary open interval $(a,b) \subseteq \mathbb{R}$ with a bounded operator-valued coefficient Q, that is, when $Q(x)$ is a bounded operator on a separable, complex Hilbert space $\mathscr{H}$ for a.e. $x \in (a,b)$. In fact, we are interested in two types of situations: In the first one $f(x)$ is an element of the Hilbert space $\mathscr{H}$ for a.e. $x \in (a,b)$, and the solution sought is to take values in $\mathscr{H}$.

In the second situation, $f(x)$ is a bounded operator on $\mathscr{H}$ for a.e. $x \in (a,b)$, as is the proposed solution y.

We start with some preliminaries: Let $(a,b) \subseteq \mathbb{R}$ be a finite or infinite interval and $\mathscr{X}$ a Banach space. Unless explicitly stated otherwise (such as in the context of operator-valued measures in Nevanlinna–Herglotz representations, cf. Appendix A), integration of $\mathscr{X}$-valued functions on (a,b) will always be understood in the sense of Bochner (cf., e.g., [15, p. 6–21], [39, p. 44–50], [54, p. 71–86], [70, Ch. III], [109, Sect. V.5] for details). In particular, if $p \geqslant 1$, the symbol $L^p((a,b);dx;\mathscr{X})$ denotes the set of equivalence classes of strongly measurable $\mathscr{X}$-valued functions which differ at most on sets of Lebesgue measure zero, such that $\|f(\cdot)\|_{\mathscr{X}}^p \in L^1((a,b);dx)$. The corresponding norm in $L^p((a,b);dx;\mathscr{X})$ is given by

$$\|f\|_{L^p((a,b);dx;\mathscr{X})} = \left(\int_{(a,b)} dx\, \|f(x)\|_{\mathscr{X}}^p\right)^{1/p} \tag{2.1}$$

and $L^p((a,b);dx;\mathscr{X})$ is a Banach space.

If $\mathscr{H}$ is a separable Hilbert space, then so is $L^2((a,b);dx;\mathscr{H})$ (see, e.g., [21, Subsects. 4.3.1, 4.3.2], [27, Sect. 7.1]).

One recalls that by a result of Pettis [84], if $\mathscr{X}$ is separable, weak measurability of $\mathscr{X}$-valued functions implies their strong measurability.

If $g \in L^1((a,b);dx;\mathscr{X})$, $f(x) = \int_{x_0}^x dx'\, g(x')$, $x_0, x \in (a,b)$, then f is strongly differentiable a.e. on (a,b) and

$$f'(x) = g(x) \text{ for a.e. } x \in (a,b). \tag{2.2}$$

In addition,

$$\lim_{t\downarrow 0} \frac{1}{t}\int_x^{x+t} dx'\, \|g(x') - g(x)\|_{\mathscr{X}} = 0 \text{ for a.e. } x \in (a,b), \tag{2.3}$$

in particular,

$$\operatorname*{s-lim}_{t\downarrow 0} \frac{1}{t}\int_x^{x+t} dx'\, g(x') = g(x) \text{ for a.e. } x \in (a,b). \tag{2.4}$$

Sobolev spaces $W^{n,p}((a,b);dx;\mathscr{X})$ for $n \in \mathbb{N}$ and $p \geqslant 1$ are defined as follows: $W^{1,p}((a,b);dx;\mathscr{X})$ is the set of all $f \in L^p((a,b);dx;\mathscr{X})$ such that there exists a $g \in L^p((a,b);dx;\mathscr{X})$ and an $x_0 \in (a,b)$ such that

$$f(x) = f(x_0) + \int_{x_0}^x dx'\, g(x') \text{ for a.e. } x \in (a,b). \tag{2.5}$$

In this case g is the strong derivative of f, $g = f'$. Similarly, $W^{n,p}((a,b);dx;\mathscr{X})$ is the set of all $f \in L^p((a,b);dx;\mathscr{X})$ so that the first n strong derivatives of f are in $L^p((a,b);dx;\mathscr{X})$. For simplicity of notation one also introduces $W^{0,p}((a,b);dx;\mathscr{X}) = L^p((a,b);dx;\mathscr{X})$. Finally, $W^{n,p}_{\text{loc}}((a,b);dx;\mathscr{X})$ is the set of $\mathscr{X}$-valued functions defined on (a,b) for which the restrictions to any compact interval $[\alpha,\beta] \subset (a,b)$ are in $W^{n,p}((\alpha,\beta);dx;\mathscr{X})$. In particular, this applies to the case $n = 0$ and thus defines

$L^p_{\rm loc}((a,b);dx;\mathscr{X})$. If a is finite we may allow $[\alpha,\beta]$ to be a subset of $[a,b)$ and denote the resulting space by $W^{n,p}_{\rm loc}([a,b);dx;\mathscr{X})$ (and again this applies to the case $n=0$).

Following a frequent practice (cf., e.g., the discussion in [14, Sect. III.1.2]), we will call elements of $W^{1,1}([c,d];dx;\mathscr{X})$, $[c,d]\subset(a,b)$ (resp., $W^{1,1}_{\rm loc}((a,b);dx;\mathscr{X})$), strongly absolutely continuous $\mathscr{X}$-valued functions on $[c,d]$ (resp., strongly locally absolutely continuous $\mathscr{X}$-valued functions on (a,b)), but caution the reader that unless $\mathscr{X}$ posseses the Radon–Nikodym (RN) property, this notion differs from the classical definition of $\mathscr{X}$-valued absolutely continuous functions (we refer the interested reader to [39, Sect. VII.6] for an extensive list of conditions equivalent to X having the RN property). Here we just mention that reflexivity of X implies the RN property.

In the special case where $\mathscr{X}=\mathbb{C}$, we omit $\mathscr{X}$ and just write $L^p_{(\rm loc)}((a,b);dx)$, as usual.

A Remark on notational convention. To avoid possible confusion later on between two standard notions of strongly continuous operator-valued functions $F(x)$, $x\in(a,b)$, that is, strong continuity of $F(\cdot)h$ in $\mathscr{H}$ for all $h\in\mathscr{H}$ (i.e., pointwise continuity of $F(\cdot)$), versus strong continuity of $F(\cdot)$ in the norm of $\mathscr{B}(\mathscr{H})$ (i.e., uniform continuity of $F(\cdot)$), we will always mean pointwise continuity of $F(\cdot)$ in $\mathscr{H}$. The same pointwise conventions will apply to the notions of strongly differentiable and strongly measurable operator-valued functions throughout this manuscript. In particular, and unless explicitly stated otherwise, for operator-valued functions Y, the symbol Y' will be understood in the strong sense; similarly, y' will denote the strong derivative for vector-valued functions y.

The following elementary lemma is probably well-known, but since we repeatedly use it below, and we could not quickly locate it in the literature, we include a detailed proof:

LEMMA 2.1. *Let $(a,b)\subseteq\mathbb{R}$. Suppose $Q:(a,b)\to\mathscr{B}(\mathscr{H})$ is a weakly measurable operator-valued function with $\|Q(\cdot)\|_{\mathscr{B}(\mathscr{H})}\in L^1_{\rm loc}((a,b);dx)$ and $g:(a,b)\to\mathscr{H}$ is (weakly) measurable. Then Qg is (strongly) measurable. Moreover, if g is strongly continuous, then there exists a set $E\subset(a,b)$ with zero Lebesgue measure, depending only on Q, such that for every $x_0\in(a,b)\backslash E$,*

$$\lim_{t\downarrow 0}\frac{1}{t}\int_{x_0}^{x_0+t}dx\,\|Q(x)g(x)-Q(x_0)g(x_0)\|_{\mathscr{H}}=0, \tag{2.6}$$

in particular,

$$\operatorname*{s-lim}_{t\downarrow 0}\frac{1}{t}\int_{x_0}^{x_0+t}dx\,Q(x)g(x)=Q(x_0)g(x_0), \tag{2.7}$$

in addition, the set of Lebesgue points of $Q(\cdot)g(\cdot)$ can be chosen independently of g.

Proof. Since by hypothesis, $Q(\cdot)$ on (a,b) is weakly measurable in $\mathscr{H}$, that is,

$$(f,Q(\cdot)g)_{\mathscr{H}} \text{ is (Lebesgue) measurable for all } f,g\in\mathscr{H}, \tag{2.8}$$

one infers that this is equivalent to $Q(\cdot)^*$ on (a,b) being weakly measurable in $\mathscr{H}$. An application of Pettis' theorem [84] then yields that $Q(\cdot)f$ (equivalently, $Q(\cdot)^* f$) on (a,b) is strongly measurable for all $f \in \mathscr{H}$.

Next, let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal system in $\mathscr{H}$. Then writing

$$\|Q(\cdot)f\|_{\mathscr{H}}^2 = \sum_{n\in\mathbb{N}} (Q(\cdot)f, e_n)_{\mathscr{H}} (e_n, Q(\cdot)f)_{\mathscr{H}}, \tag{2.9}$$

one concludes that $\|Q(\cdot)f\|_{\mathscr{H}}$ on (a,b) is measurable for all $f \in \mathscr{H}$. In addition, let $h(\cdot)$ on (a,b) be a weakly (and hence, strongly) measurable function in $\mathscr{H}$. Then

$$(f, Q(\cdot)h(\cdot))_{\mathscr{H}} = (Q(\cdot)^* f, h(\cdot))_{\mathscr{H}} = \sum_{n\in\mathbb{N}} (Q(\cdot)^* f, e_n)_{\mathscr{H}} (e_n, h(\cdot))_{\mathscr{H}}, \tag{2.10}$$

implies that $Q(\cdot)h(\cdot)$ on (a,b) is weakly measurable in $\mathscr{H}$. Another application of Pettis' theorem then yields the strong measurability of $Q(\cdot)h(\cdot)$ on (a,b) in $\mathscr{H}$.

Let $E_0 \subset (a,b)$ be a set of Lebesgue measure zero such that every $x_0 \in (a,b)\backslash E_0$ is a Lebesgue point for the function $\|Q(\cdot)\|_{\mathscr{B}(\mathscr{H})}$, implying,

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{x_0}^{x_0+t} dx\, \|Q(x)\|_{\mathscr{B}(\mathscr{H})} = \|Q(x_0)\|_{\mathscr{B}(\mathscr{H})}, \quad x_0 \in (a,b)\backslash E_0. \tag{2.11}$$

Next, let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of (a,b) such that each E_n is of Lebesgue measure zero and every $x_0 \in (a,b)\backslash E_n$ is a Lebesgue point for the vector-valued function $Q(\cdot)e_n$, that is,

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{x_0}^{x_0+t} dx\, \|Q(x)e_n - Q(x_0)e_n\|_{\mathscr{H}} = 0, \quad x_0 \in (a,b)\backslash E_n. \tag{2.12}$$

In addition, let $E = \bigcup_{n=0}^{\infty} E_n$, then every $x_0 \in (a,b)\backslash E$ is a Lebesgue point for $Q(\cdot)g(\cdot)$. Indeed, decomposing $g(x_0)$ with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$,

$$g(x_0) = \sum_{n\in\mathbb{N}} g_n(x_0)e_n, \quad g_n(x_0) = \big(e_n, g(x_0)\big)_{\mathscr{H}}, \; n \in \mathbb{N}, \tag{2.13}$$

and recalling that by Pettis' theorem, Qg is strongly measurable, yields (for $t > 0$)

$$
\begin{aligned}
&\left\| \frac{1}{t} \int_{x_0}^{x_0+t} dx \, [Q(x)g(x) - Q(x_0)g(x_0)] \right\|_{\mathcal{H}} \\
&\quad \leqslant \frac{1}{t} \int_{x_0}^{x_0+t} dx \, \|Q(x)g(x) - Q(x_0)g(x_0)\|_{\mathcal{H}} \\
&\quad \leqslant \frac{1}{t} \int_{x_0}^{x_0+t} dx \, \|Q(x)[g(x) - g(x_0)]\|_{\mathcal{H}} + \frac{1}{t} \int_{x_0}^{x_0+t} dx \, \|[Q(x) - Q(x_0)]g(x_0)\|_{\mathcal{H}} \\
&\quad \leqslant \frac{1}{t} \int_{x_0}^{x_0+t} dx \, \|Q(x)\|_{\mathcal{B}(\mathcal{H})} \sup_{x \in [x_0, x_0+t]} \|g(x) - g(x_0)\|_{\mathcal{H}} \\
&\qquad + \sum_{n=1}^{N} |g_n(x_0)| \left(\frac{1}{t} \int_{x_0}^{x_0+t} dx \, \|[Q(x) - Q(x_0)]e_n\|_{\mathcal{H}} \right) \\
&\qquad + \left(\frac{1}{t} \int_{x_0}^{x_0+t} dx \, [\|Q(x)\|_{\mathcal{B}(\mathcal{H})} + \|Q(x_0)\|_{\mathcal{B}(\mathcal{H})}] \right) \left\| \sum_{n=N+1}^{\infty} g_n(x_0)e_n \right\|_{\mathcal{H}}. \qquad (2.14)
\end{aligned}
$$

Finally, taking the limit $t \downarrow 0$ renders the first term on the right-hand side of (2.14) zero as $g(\cdot)$ is strongly continuous in $\mathcal{H}$ and x_0 is a Lebesgue point of $\|Q(\cdot)\|_{\mathcal{B}(\mathcal{H})}$ by (2.11). Similarly, taking $t \downarrow 0$ renders the second term on the right-hand side of (2.14) zero by (2.12). Again by (2.11), the third term on the right-hand side of (2.14) approaches $2\|Q(x_0)\|_{\mathcal{B}(\mathcal{H})} \left\| \sum_{n=N+1}^{\infty} g_n(x_0)e_n \right\|_{\mathcal{H}}$ as $t \downarrow 0$ and hence vanishes in the limit $N \to \infty$ (cf. (2.13)). $\square$

In connection with (2.7) we also refer to [39, Theorem II.2.9], [54, Subsect. III.3.8], [109, Theorem V.5.2].

DEFINITION 2.2. Let $(a,b) \subseteq \mathbb{R}$ be a finite or infinite interval, $Q: (a,b) \to \mathcal{B}(\mathcal{H})$ a weakly measurable operator-valued function with $\|Q(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}((a,b);dx)$, and suppose that $f \in L^1_{\text{loc}}((a,b);dx;\mathcal{H})$. Then the $\mathcal{H}$-valued function $y: (a,b) \to \mathcal{H}$ is called a (strong) solution of

$$-y'' + Qy = f \qquad (2.15)$$

if $y \in W^{2,1}_{\text{loc}}((a,b);dx;\mathcal{H})$ and (2.15) holds a.e. on (a,b).

We recall our notational convention that vector-valued solutions of (2.15) will always be viewed as strong solutions.

One verifies that $Q: (a,b) \to \mathcal{B}(\mathcal{H})$ satisfies the conditions in Definition 2.2 if and only if Q^* does (a fact that will play a role later on, cf. the paragraph following (2.33)).

THEOREM 2.3. *Let $(a,b) \subseteq \mathbb{R}$ be a finite or infinite interval and $V: (a,b) \to \mathcal{B}(\mathcal{H})$ a weakly measurable operator-valued function with $\|V(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}((a,b);dx)$. Suppose that $x_0 \in (a,b)$, $z \in \mathbb{C}$, $h_0, h_1 \in \mathcal{H}$, and $f \in L^1_{\text{loc}}((a,b);dx;\mathcal{H})$. Then there is a* *unique $\mathscr{H}$-valued solution $y(z,\cdot,x_0) \in W^{2,1}_{\rm loc}((a,b);dx;\mathscr{H})$ of the initial value problem*

$$\begin{cases} -y'' + (V-z)y = f \ \text{on} \ (a,b)\backslash E, \\ y(x_0) = h_0,\ y'(x_0) = h_1, \end{cases} \tag{2.16}$$

where the exceptional set E is of Lebesgue measure zero and independent of z.

Moreover, the following properties hold:

(*i*) *For fixed $x_0, x \in (a,b)$ and $z \in \mathbb{C}$, $y(z,x,x_0)$ depends jointly continuously on $h_0, h_1 \in \mathscr{H}$, and $f \in L^1_{\rm loc}((a,b);dx;\mathscr{H})$ in the sense that*

$$\begin{aligned} &\big\|y\big(z,x,x_0;h_0,h_1,f\big) - y\big(z,x,x_0;\widetilde{h}_0,\widetilde{h}_1,\widetilde{f}\big)\big\|_{\mathscr{H}} \\ &\quad \leqslant C(z,V)\big[\big\|h_0-\widetilde{h}_0\big\|_{\mathscr{H}} + \big\|h_1-\widetilde{h}_1\big\|_{\mathscr{H}} + \big\|f-\widetilde{f}\big\|_{L^1([x_0,x];dx;\mathscr{H})}\big], \end{aligned} \tag{2.17}$$

where $C(z,V) > 0$ is a constant, and the dependence of y on the initial data h_0, h_1 and the inhomogeneity f is displayed in (2.17).

(*ii*) *For fixed $x_0 \in (a,b)$ and $z \in \mathbb{C}$, $y(z,x,x_0)$ is strongly continuously differentiable with respect to x on (a,b).*

(*iii*) *For fixed $x_0 \in (a,b)$ and $z \in \mathbb{C}$, $y'(z,x,x_0)$ is strongly differentiable with respect to x on $(a,b)\backslash E$.*

(*iv*) *For fixed $x_0, x \in (a,b)$, $y(z,x,x_0)$ and $y'(z,x,x_0)$ are entire with respect to z.*

Proof. As discussed in the proof of Lemma 2.1, if $f : (a,b) \to \mathscr{H}$ is strongly measurable, then $Q(\cdot)f(\cdot)$ is also a strongly measurable $\mathscr{H}$-valued function.

As in the classical scalar case (i.e., $\mathscr{H} = \mathbb{C}$), one can show that a function $y(z,\cdot,x_0) \in W^{2,1}_{\rm loc}((a,b);dx;\mathscr{H})$ satisfies the initial-value problem (2.16) if and only if $y(z,\cdot,x_0)$ is strongly measurable, strongly locally bounded, and satisfies the integral equation,

$$\begin{aligned} y(z,x,x_0) &= \cos\big(z^{1/2}(x-x_0)\big)h_0 + z^{-1/2}\sin\big(z^{1/2}(x-x_0)\big)h_1 \\ &\quad + \int_{x_0}^x dx'\, z^{-1/2}\sin\big(z^{1/2}(x-x')\big)\big[V(x')y(z,x',x_0) - f(x')\big], \\ &\qquad z \in \mathbb{C},\ \operatorname{Im}(z^{1/2}) \geqslant 0,\ x_0, x \in (a,b). \end{aligned} \tag{2.18}$$

Thus, it suffices to verify existence and uniqueness for a solution of (2.18). For uniqueness it is enough to check that $y(z,\cdot,x_0) = 0$ is the only solution of

$$y(z,x,x_0) = \int_{x_0}^x dx'\, z^{-1/2}\sin\big(z^{1/2}(x-x')\big)V(x')y(x'). \tag{2.19}$$

Let $K \subset (a,b)$ be a compact subset containing x_0, then iterations of (2.19) yield

$$\sup_{x\in K}\|y(z,x,x_0)\|_{\mathscr{H}} \leqslant \frac{1}{n!}\bigg(C(z)\int_{x_0}^x \|V(x')\|_{\mathscr{B}(\mathscr{H})}dx'\bigg)^n \sup_{x'\in K}\|y(z,x',x_0)\|_{\mathscr{H}},\ n \in \mathbb{N}, \tag{2.20}$$

for an appropriate constant $C(z) > 0$. Since K and n are arbitrary, the only solution of (2.19) is the zero solution.

To show existence one uses the method of successive approximations. Define a sequence of vector-valued functions $y_n(z,\cdot,x_0) : (a,b) \to \mathscr{H}$, $n \in \mathbb{N}_0$, by

$$\begin{aligned} y_0(z,x,x_0) &= \cos\big(z^{1/2}(x-x_0)\big)h_0 + z^{-1/2}\sin\big(z^{1/2}(x-x_0)\big)h_1 \\ &\quad - \int_{x_0}^{x} dx'\, z^{-1/2}\sin\big(z^{1/2}(x-x')\big)f(x'), \\ y_n(z,x,x_0) &= \int_{x_0}^{x} dx'\, z^{-1/2}\sin\big(z^{1/2}(x-x')\big)V(x')y_{n-1}(z,x',x_0), \quad n\in\mathbb{N}. \end{aligned} \tag{2.21}$$

Then for each $n \in \mathbb{N}_0$, it follows inductively that for fixed $x_0 \in (a,b)$ and $z \in \mathbb{C}$, $y_n(z,x,x_0)$ is strongly locally absolutely continuous with respect to x on (a,b), and for fixed $x_0, x \in (a,b)$, $y_n(z,x,x_0)$, $y_n'(z,x,x_0)$ are entire with respect to z. The estimate

$$\begin{aligned} &\|y_n(z,x,x_0)\|_{\mathscr{H}} + \|y_n'(z,x,x_0)\|_{\mathscr{H}} \\ &\quad \leqslant \frac{1}{n!}\bigg(C\int_{x_0}^{x} dx'\, \|V(x')\|_{\mathscr{B}(\mathscr{H})} dx'\bigg)^n \bigg(\|h_0\|_{\mathscr{H}} + \|h_1\|_{\mathscr{H}} + \int_{x_0}^{x} \|f(x')\|_{\mathscr{H}}\bigg), \end{aligned} \tag{2.22}$$

holds uniformly in (z,x) on compact subsets of $\mathbb{C}\times(a,b)$, where C depends only on the compact subset of $\mathbb{C}\times(a,b)$. This yields convergence of the series,

$$y(z,x,x_0) = \sum_{n=0}^{\infty} y_n(z,x,x_0), \quad y'(z,x,x_0) = \sum_{n=0}^{\infty} y_n'(z,x,x_0), \tag{2.23}$$

with

$$\begin{aligned} \|y(z,x,x_0)\|_{\mathscr{H}} &\leqslant \exp\bigg(C\int_{x_0}^{x} dx'\, \|V(x')\|_{\mathscr{B}(\mathscr{H})}\bigg) \\ &\qquad \times \bigg(\|h_0\|_{\mathscr{B}(\mathscr{H})} + \|h_1\|_{\mathscr{H}} + \int_{x_0}^{x} dx'\, \|f(x')\|_{\mathscr{H}}\bigg), \end{aligned} \tag{2.24}$$

uniformly in (z,x) on compact subsets of $\mathbb{C}\times(a,b)$. Then (2.21), (2.23) imply that $y(z,\cdot,x_0)$ is a solution of the integral equation (2.18), and (2.23), (2.24) yield the properties (i) (taking into account linearity of (2.16)) and (iv).

Finally, by (2.18), for each $z \in \mathbb{C}$ and a.e. $x \in (a,b)$,

$$y''(z,x,x_0) = zy(z,x,x_0) - V(x)y(z,x,x_0) + f(x), \tag{2.25}$$

and hence

$$\begin{aligned} y(z,x,x_0) &= \cos\big(z^{1/2}(x-x_0)\big)h_0 + z^{-1/2}\sin\big(z^{1/2}(x-x_0)\big)h_1 \\ &\quad + \int_{x_0}^{x} dx'\bigg(\int_{x_0}^{x'} dx''\, \big[zy(z,x'',x_0) - V(x'')y(z,x'',x_0) + f(x'')\big]\bigg). \end{aligned} \tag{2.26}$$

This representation of $y(z,x,x_0)$ combined with Lemma 2.1 yields the properties (ii) and (iii). In particular, $y(z,\cdot,x_0)\in W^{2,1}_{\rm loc}((a,b);dx;\mathscr{H})$ and $y(z,\cdot,x_0)$ is a strong solution of the initial value problem (2.16). $\square$

For classical references on initial value problems we refer, for instance, to [33, Chs. III, VII] and [40, Ch. 10], but we emphasize again that our approach minimizes the smoothness hypotheses on V and f.

DEFINITION 2.4. Let $(a,b)\subseteq\mathbb{R}$ be a finite or infinite interval and assume that $F,Q:(a,b)\to\mathscr{B}(\mathscr{H})$ are two weakly measurable operator-valued functions such that $\|F(\cdot)\|_{\mathscr{B}(\mathscr{H})},\|Q(\cdot)\|_{\mathscr{B}(\mathscr{H})}\in L^1_{\rm loc}((a,b);dx)$. Then the $\mathscr{B}(\mathscr{H})$-valued function $Y:(a,b)\to\mathscr{B}(\mathscr{H})$ is called a solution of

$$-Y''+QY=F \tag{2.27}$$

if $Y(\cdot)h\in W^{2,1}_{\rm loc}((a,b);dx;\mathscr{H})$ for every $h\in\mathscr{H}$ and $-Y''h+QYh=Fh$ holds a.e. on (a,b).

COROLLARY 2.5. *Let $(a,b)\subseteq\mathbb{R}$ be a finite or infinite interval, $x_0\in(a,b)$, $z\in\mathbb{C}$, $Y_0,Y_1\in\mathscr{B}(\mathscr{H})$, and suppose $F,V:(a,b)\to\mathscr{B}(\mathscr{H})$ are two weakly measurable operator-valued functions with $\|V(\cdot)\|_{\mathscr{B}(\mathscr{H})},\|F(\cdot)\|_{\mathscr{B}(\mathscr{H})}\in L^1_{\rm loc}((a,b);dx)$. Then there is a unique $\mathscr{B}(\mathscr{H})$-valued solution $Y(z,\cdot,x_0):(a,b)\to\mathscr{B}(\mathscr{H})$ of the initial value problem*

$$\begin{cases}-Y''+(V-z)Y=F \text{ on } (a,b)\backslash E,\\ Y(x_0)=Y_0,\ Y'(x_0)=Y_1.\end{cases} \tag{2.28}$$

where the exceptional set E is of Lebesgue measure zero and independent of z. Moreover, the following properties hold:

(i) *For fixed $x_0\in(a,b)$ and $z\in\mathbb{C}$, $Y(z,x,x_0)$ is continuously differentiable with respect to x on (a,b) in the $\mathscr{B}(\mathscr{H})$-norm.*

(ii) *For fixed $x_0\in(a,b)$ and $z\in\mathbb{C}$, $Y'(z,x,x_0)$ is strongly differentiable with respect to x on $(a,b)\backslash E$.*

(iii) *For fixed $x_0,x\in(a,b)$, $Y(z,x,x_0)$ and $Y'(z,x,x_0)$ are entire in z in the $\mathscr{B}(\mathscr{H})$-norm.*

Proof. Applying Theorem 2.3 to $h_0=Y_0h$, $h_1=Y_1h$, and $f(x)=F(x)h$ with $h\in\mathscr{H}$ yields a unique vector-valued solution $y_h(z,x,x_0)$. Since $y_h(z,x,x_0)$ depends continuously on h by Theorem 2.3 (i), this yields a unique operator-valued solution $Y(z,\cdot,x_0):(a,b)\to\mathscr{B}(\mathscr{H})$ of the initial value problem (2.28), where $Y(z,x,x_0)h=y_h(z,x,x_0)$ for all $h\in\mathscr{H}$.

It follows from Theorem 2.3 (ii) that for fixed $x_0\in(a,b)$, $z\in\mathbb{C}$, and every $h\in\mathscr{H}$, $\|Y(z,\cdot,x_0)h\|_{\mathscr{H}}$ is continuous on (a,b) and hence bounded on every compact subset of (a,b). Thus, it follows from the uniform boundedness principle (cf. [59, Thm. III.1.3.29]) that $\|Y(z,\cdot,x_0)\|_{\mathscr{B}(\mathscr{H})}$ is bounded on every compact subset of (a,b).

Moreover, Theorem 2.3 (ii) and (iii) also imply that $Y(z,x,x_0)$ and $Y'(z,x,x_0)$ are differentiable with respect to x in the strong operator topology. Hence, using

$$\begin{aligned} Y(z,x,x_0)h &= \cos\big(z^{1/2}(x-x_0)\big)Y_0h + z^{-1/2}\sin\big(z^{1/2}(x-x_0)\big)Y_1h \\ &\quad + \int_{x_0}^{x} dx' \bigg(\int_{x_0}^{x'} dx'' \,[zY(z,x'',x_0)h - V(x'')Y(z,x'',x_0)h + F(x'')h]\bigg), \end{aligned} \tag{2.29}$$

one computes

$$\begin{aligned} &\Big\| \frac{1}{t}[Y(z,x+t,x_0) - Y(z,x,x_0)]h - Y'(z,x,x_0)h \Big\|_{\mathscr{H}} \\ &\quad \leqslant O(t)\|Y_0\|_{\mathscr{B}(\mathscr{H})}\|h\|_{\mathscr{H}} + O(t)\|Y_1\|_{\mathscr{B}(\mathscr{H})}\|h\|_{\mathscr{H}} \\ &\qquad + \frac{1}{|t|}\bigg(\int_{x}^{x+t} dx' \bigg(\int_{x}^{x'} dx'' \,[|z| + \|V(x'')\|_{\mathscr{B}(\mathscr{H})}]\|Y(z,x'',x_0)\|_{\mathscr{B}(\mathscr{H})}\bigg)\bigg)\|h\|_{\mathscr{H}} \\ &\qquad + \frac{1}{|t|}\bigg(\int_{x}^{x+t} dx' \bigg(\int_{x}^{x'} dx'' \,\|F(x'')\|_{\mathscr{B}(\mathscr{H})}\bigg)\bigg)\|h\|_{\mathscr{H}}. \end{aligned} \tag{2.30}$$

Since the right hand-side vanishes as $t \downarrow 0$ uniformly in $h \in \mathscr{H}$ with $\|h\|_{\mathscr{H}} \leqslant 1$, the solution $Y(z,x,x_0)$ is differentiable with respect to x in the $\mathscr{B}(\mathscr{H})$-norm topology. Similarly one uses (2.29) to verify continuity of $Y'(z,x,x_0)$ with respect to x in the $\mathscr{B}(\mathscr{H})$-norm topology, implying item (i).

Item (ii) follows directly from Theorem 2.3 (iii) with the set E possibly dependent on $h \in \mathscr{H}$. To remove the h-dependence one chooses an orthonormal basis $\{e_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ and let E_n be the corresponding exceptional sets. Then $E = \bigcup_{n=1}^{\infty} E_n$ can be used as the exceptional set in item (ii).

Finally, by Theorem 2.3 (iv), $Y(z,x,x_0)$ and $Y'(z,x,x_0)$ are entire with respect to z in the strong operator topology and hence by [59, Theorem III.1.37] also in the $\mathscr{B}(\mathscr{H})$-topology, implying item (iii). $\square$

Various versions of Theorem 2.3 and Corollary 2.5 exist in the literature under varying assumptions on V and f,F. For instance, the case where $V(\cdot)$ is continuous in the $\mathscr{B}(\mathscr{H})$-norm and $F=0$ is discussed in [53, Theorem 6.1.1]. The case, where $\|V(\cdot)\|_{\mathscr{B}(\mathscr{H}} \in L^1_{\rm loc}([a,c];dx)$ for all $c>a$ and $F=0$ is discussed in detail in [94] (it appears that a measurability assumption of $V(\cdot)$ in the $\mathscr{B}(\mathscr{H})$-norm is missing in the basic set of hypotheses of [94]). Our extension to $V(\cdot)$ weakly measurable and $\|V(\cdot)\|_{\mathscr{B}(\mathscr{H}} \in L^1_{\rm loc}([a,b);dx)$ may well be the most general one published to date, but we obviously claim no originality in this context.

DEFINITION 2.6. Pick $c \in (a,b)$. The endpoint a (resp., b) of the interval (a,b) is called *regular* for the operator-valued differential expression $-(d^2/dx^2)+Q(\cdot)$ if it is finite and if Q is weakly measurable and $\|Q(\cdot)\|_{\mathscr{B}(\mathscr{H})} \in L^1_{\rm loc}([a,c];dx)$ (resp., $\|Q(\cdot)\|_{\mathscr{B}(\mathscr{H})} \in L^1_{\rm loc}([c,b];dx)$) for some $c \in (a,b)$. Similarly, $-(d^2/dx^2)+Q(\cdot)$ is called *regular at* a (resp., *regular at* b) if a (resp., b) is a regular endpoint for $-(d^2/dx^2) + Q(\cdot)$.

We note that if a (resp., b) is regular for $-(d^2/dx^2)+Q(x)$, one may allow for x_0 to be equal to a (resp., b) in the existence and uniqueness Theorem 2.3.

If f_1, f_2 are strongly continuously differentiable $\mathscr{H}$-valued functions, we define the Wronskian of f_1 and f_2 by

$$W_*(f_1,f_2)(x) = (f_1(x), f_2'(x))_{\mathscr{H}} - (f_1'(x), f_2(x))_{\mathscr{H}}, \quad x \in (a,b). \tag{2.31}$$

If f_2 is an $\mathscr{H}$-valued solution of $-y''+Qy=0$ and f_1 is an $\mathscr{H}$-valued solution of $-y''+Q^*y=0$, their Wronskian $W_*(f_1,f_2)(x)$ is x-independent, that is,

$$\frac{d}{dx}W_*(f_1,f_2)(x) = 0, \text{ for a.e. } x \in (a,b). \tag{2.32}$$

Equation (2.55) will show that the right-hand side of (2.32) actually vanishes for all $x \in (a,b)$.

We decided to use the symbol $W_*(\cdot,\cdot)$ in (2.31) to indicate its conjugate linear behavior with respect to its first entry.

Similarly, if F_1, F_2 are strongly continuously differentiable $\mathscr{B}(\mathscr{H})$-valued functions, their Wronskian is defined by

$$W(F_1,F_2)(x) = F_1(x)F_2'(x) - F_1'(x)F_2(x), \quad x \in (a,b). \tag{2.33}$$

Again, if F_2 is a $\mathscr{B}(\mathscr{H})$-valued solution of $-Y''+QY=0$ and F_1 is a $\mathscr{B}(\mathscr{H})$-valued solution of $-Y''+YQ=0$ (the latter is equivalent to $-(Y^*)''+Q^*Y^*=0$ and hence can be handled in complete analogy via Theorem 2.3 and Corollary 2.5, replacing Q by Q^*) their Wronskian will be x-independent,

$$\frac{d}{dx}W(F_1,F_2)(x) = 0 \text{ for a.e. } x \in (a,b). \tag{2.34}$$

Our main interest is in the case where $V(\cdot) = V(\cdot)^* \in \mathscr{B}(\mathscr{H})$ is self-adjoint, that is, in the differential equation $\tau\eta = z\eta$, where η represents an $\mathscr{H}$-valued, respectively, $\mathscr{B}(\mathscr{H})$-valued solution (in the sense of Definitions 2.2, resp., 2.4), and where τ abbreviates the operator-valued differential expression

$$\tau = -(d^2/dx^2) + V(\cdot). \tag{2.35}$$

To this end, we now introduce the following basic assumption:

HYPOTHESIS 2.7. *Let $(a,b) \subseteq \mathbb{R}$, suppose that $V : (a,b) \to \mathscr{B}(\mathscr{H})$ is a weakly measurable operator-valued function with $\|V(\cdot)\|_{\mathscr{B}(\mathscr{H})} \in L^1_{\text{loc}}((a,b);dx)$, and assume that $V(x) = V(x)^*$ for a.e. $x \in (a,b)$.*

Moreover, for the remainder of this section we assume that $\alpha \in \mathscr{B}(\mathscr{H})$ is a self-adjoint operator,

$$\alpha = \alpha^* \in \mathscr{B}(\mathscr{H}). \tag{2.36}$$

Assuming Hypothesis 2.7 and (2.36), we introduce the standard fundamental systems of operator-valued solutions of $\tau y = zy$ as follows: Since α is a bounded self-adjoint operator, one may define the self-adjoint operators $A = \sin(\alpha)$ and $B = \cos(\alpha)$ via the spectral theorem. One then concludes that $\sin^2(\alpha) + \cos^2(\alpha) = I_{\mathscr{H}}$ and $[\sin\alpha, \cos\alpha] = 0$ (here $[\cdot,\cdot]$ represents the commutator symbol). The spectral theorem implies also that the spectra of $\sin(\alpha)$ and $\cos(\alpha)$ are contained in $[-1,1]$ and that the spectra of $\sin^2(\alpha)$ and $\cos^2(\alpha)$ are contained in $[0,1]$. Given such an operator α and a point $x_0 \in (a,b)$ or a regular endpoint for τ, we now define $\theta_\alpha(z,\cdot,x_0,),\phi_\alpha(z,\cdot,x_0)$ as those $\mathscr{B}(\mathscr{H})$-valued solutions of $\tau Y = zY$ (in the sense of Definition 2.4) which satisfy the initial conditions

$$\theta_\alpha(z,x_0,x_0) = \phi'_\alpha(z,x_0,x_0) = \cos(\alpha), \quad -\phi_\alpha(z,x_0,x_0) = \theta'_\alpha(z,x_0,x_0) = \sin(\alpha). \tag{2.37}$$

By Corollary 2.5 (iii), for any fixed $x,x_0 \in (a,b)$, the functions $\theta_\alpha(z,x,x_0)$ and $\phi_\alpha(z,x,x_0)$ as well as their strong x-derivatives are entire with respect to z in the $\mathscr{B}(\mathscr{H})$-norm. The same is true for the functions $z \mapsto \theta_\alpha(\bar z,x,x_0)^*$ and $z \mapsto \phi_\alpha(\bar z,x,x_0)^*$.

Since $\theta_\alpha(\bar z,\cdot,x_0)^*$ and $\phi_\alpha(\bar z,\cdot,x_0)^*$ satisfy the adjoint equation $-Y'' + YV = zY$ and the same initial conditions as θ_α and ϕ_α, respectively, one obtains the following identities from the constancy of Wronskians:

$$\theta'_\alpha(\bar z,x,x_0)^*\theta_\alpha(z,x,x_0) - \theta_\alpha(\bar z,x,x_0)^*\theta'_\alpha(z,x,x_0) = 0, \tag{2.38}$$
$$\phi'_\alpha(\bar z,x,x_0)^*\phi_\alpha(z,x,x_0) - \phi_\alpha(\bar z,x,x_0)^*\phi'_\alpha(z,x,x_0) = 0, \tag{2.39}$$
$$\phi'_\alpha(\bar z,x,x_0)^*\theta_\alpha(z,x,x_0) - \phi_\alpha(\bar z,x,x_0)^*\theta'_\alpha(z,x,x_0) = I_{\mathscr{H}}, \tag{2.40}$$
$$\theta_\alpha(\bar z,x,x_0)^*\phi'_\alpha(z,x,x_0) - \theta'_\alpha(\bar z,x,x_0)^*\phi_\alpha(z,x,x_0) = I_{\mathscr{H}}. \tag{2.41}$$

Equations (2.38)–(2.41) are equivalent to the statement that the block operator

$$\Theta_\alpha(z,x,x_0) = \begin{pmatrix} \theta_\alpha(z,x,x_0) & \phi_\alpha(z,x,x_0) \\ \theta'_\alpha(z,x,x_0) & \phi'_\alpha(z,x,x_0) \end{pmatrix} \tag{2.42}$$

has a left inverse given by

$$\begin{pmatrix} \phi'_\alpha(\bar z,x,x_0)^* & -\phi_\alpha(\bar z,x,x_0)^* \\ -\theta'_\alpha(\bar z,x,x_0)^* & \theta_\alpha(\bar z,x,x_0)^* \end{pmatrix}. \tag{2.43}$$

Thus the operator $\Theta_\alpha(z,x,x_0)$ is injective. It is also surjective as will be shown next: Let $(f_1,g_1)^\top$ be an arbitrary element of $\mathscr{H} \oplus \mathscr{H}$ and let y be an $\mathscr{H}$-valued solution of the initial value problem

$$\begin{cases} \tau y = zy, \\ y(x_1) = f_1,\ y'(x_1) = g_1, \end{cases} \tag{2.44}$$

for some given $x_1 \in (a,b)$. One notes that due to the initial conditions specified in (2.37), $\Theta_\alpha(z,x_0,x_0)$ is bijective. We now assume that $(f_0,g_0)^\top$ are given by

$$\Theta_\alpha(z,x_0,x_0)\begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix}. \tag{2.45}$$

The existence and uniqueness Theorem 2.3 then yields that

$$\Theta_\alpha(z,x_1,x_0)\begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}. \tag{2.46}$$

This establishes surjectivity of $\Theta_\alpha(z,x_1,x_0)$ which therefore has a right inverse too, also given by (2.43). This fact then implies the following identities:

$$\phi_\alpha(z,x,x_0)\theta_\alpha(\overline{z},x,x_0)^* - \theta_\alpha(z,x,x_0)\phi_\alpha(\overline{z},x,x_0)^* = 0, \tag{2.47}$$
$$\phi'_\alpha(z,x,x_0)\theta'_\alpha(\overline{z},x,x_0)^* - \theta'_\alpha(z,x,x_0)\phi'_\alpha(\overline{z},x,x_0)^* = 0, \tag{2.48}$$
$$\phi'_\alpha(z,x,x_0)\theta_\alpha(\overline{z},x,x_0)^* - \theta'_\alpha(z,x,x_0)\phi_\alpha(\overline{z},x,x_0)^* = I_{\mathcal{H}}, \tag{2.49}$$
$$\theta_\alpha(z,x,x_0)\phi'_\alpha(\overline{z},x,x_0)^* - \phi_\alpha(z,x,x_0)\theta'_\alpha(\overline{z},x,x_0)^* = I_{\mathcal{H}}. \tag{2.50}$$

Having established the invertibility of $\Theta_\alpha(z,x_1,x_0)$ we can now show that for any $x_1 \in (a,b)$, any $\mathcal{H}$-valued solution of $\tau y = zy$ may be expressed in terms of $\theta_\alpha(z,\cdot,x_1)$ and $\phi_\alpha(z,\cdot,x_1)$, that is,

$$y(x) = \theta_\alpha(z,x,x_1)f + \phi_\alpha(z,x,x_1)g \tag{2.51}$$

for appropriate vectors $f,g \in \mathcal{H}$ or $\mathcal{B}(\mathcal{H})$.

Next we establish a variation of constants formula.

LEMMA 2.8. *Suppose $F:(a,b)\to\mathcal{B}(\mathcal{H})$ is a weakly measurable operator-valued function such that $\|F(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}((a,b);dx)$, assume that $Y_0,Y_1 \in \mathcal{B}(\mathcal{H})$, and let $x_0 \in (a,b)$. Then the unique $\mathcal{B}(\mathcal{H})$-valued solution $Y(z,\cdot,x_0)$ of the initial value problem*

$$\begin{cases} (\tau - z)Y = F, \\ Y(x_0) = Y_0,\ Y'(x_0) = Y_1, \end{cases} \tag{2.52}$$

is given by $Y_h + Y_p$, where Y_p is the particular solution of $(\tau - z)Y = F$ $($in the sense of Definition 2.4$)$ of the form

$$\begin{aligned} Y_p(x) = \theta_\alpha(z,x,x_0)\int_{x_0}^{x} dx'\,\phi_\alpha(\overline{z},x',x_0)^*F(x') \\ - \phi_\alpha(z,x,x_0)\int_{x_0}^{x} dx'\,\theta_\alpha(\overline{z},x',x_0)^*F(x'), \end{aligned} \tag{2.53}$$

and Y_h is the unique solution of the homogeneous initial value problem $($again in the sense of Definition 2.4$)$

$$\begin{cases} \tau Y = zY, \\ Y(x_0) = Y_0,\ Y'(x_0) = Y_1. \end{cases} \tag{2.54}$$

The analogous statement holds when F is replaced by $f \in L^1_{\text{loc}}((a,b);dx;\mathcal{H})$ and Y_0,Y_1 are replaced by $y_0,y_1 \in \mathcal{H}$.

Proof. This follows from a direct computation taking into account the identities (2.47) and (2.49). □

Finally we establish several versions of Green's formula (also called Lagrange's identity) which will be used frequently in the following.

LEMMA 2.9. *Let $(a,b) \subseteq \mathbb{R}$ be a finite or infinite interval and $[x_1,x_2] \subset (a,b)$.*
(i) Assume that $f,g \in W^{2,1}_{\rm loc}((a,b);dx;\mathscr{H})$. Then

$$\int_{x_1}^{x_2} dx\,[((\tau f)(x),g(x))_{\mathscr{H}} - (f(x),(\tau g)(x))_{\mathscr{H}}] = W_*(f,g)(x_2) - W_*(f,g)(x_1). \quad (2.55)$$

(ii) Assume that $F:(a,b)\to\mathscr{B}(\mathscr{H})$ is absolutely continuous, that F' is again differentiable, and that F'' is weakly measurable. Also assume that $\|F''\|_{\mathscr{H}} \in L^1_{\rm loc}((a,b);dx)$ and $g \in W^{2,1}_{\rm loc}((a,b);dx;\mathscr{H})$. Then

$$\int_{x_1}^{x_2} dx\,[(\tau F^*)^*(x)g(x) - F(x)(\tau g)(x)] = (Fg'-F'g)(x_2) - (Fg'-F'g)(x_1). \quad (2.56)$$

(iii) Assume that $F,G:(a,b)\to\mathscr{B}(\mathscr{H})$ are absolutely continuous operator-valued functions such that F',G' are again differentiable and that F'', G'' are weakly measurable. In addition, suppose that $\|F''\|_{\mathscr{H}}, \|G''\|_{\mathscr{H}} \in L^1_{\rm loc}((a,b);dx)$. Then

$$\int_{x_1}^{x_2} dx\,[(\tau F^*)(x)^*G(x) - F(x)(\tau G)(x)] = (FG'-F'G)(x_2) - (FG'-F'G)(x_1). \quad (2.57)$$

Proof. The product rule for scalar products

$$\frac{d}{dx}(f(x),g(x))_{\mathscr{H}} = (f(x),g'(x))_{\mathscr{H}} + (f'(x),g(x))_{\mathscr{H}} \quad (2.58)$$

implies, as usual, the formula for integration by parts. Equation (2.55) is then an immediate consequence of the latter and the fact that V is self-adjoint so that $(Vf,g)_{\mathscr{H}} = (f,Vg)_{\mathscr{H}}$.

To prove (2.56), we first note that $g:(a,b)\to\mathscr{H}$ is strongly continuous so that, by Lemma 2.1 the function $F''g$ is (strongly) measurable and integrable. Lemma 2.1 then shows that also Fg'' and FVg are measurable. Consequently, the integral on the left-hand side of (2.56) is well-defined in the strong sense. The remainder of the proof relies again on a product rule. The product rule follows from the fact that each summand in

$$\begin{aligned} &\left\| F(x+\varepsilon)\left(\frac{g(x+\varepsilon)-g(x)}{\varepsilon} - g'(x)\right)\right\|_{\mathscr{H}} + \|(F(x+\varepsilon)-F(x))g'(x)\|_{\mathscr{H}} \\ &\quad + \left\|\left(\frac{F(x+\varepsilon)g(x)-F(x)g(x)}{\varepsilon} - F'(x)g(x)\right)\right\|_{\mathscr{H}} \end{aligned} \quad (2.59)$$

tends to zero as $\varepsilon \downarrow 0$, recalling that $x\in(a,b)$ is fixed.

Finally, to prove (2.57), we first note that $Gh : (a,b) \to \mathscr{H}$ is strongly continuous for any $h \in \mathscr{H}$. Again, Lemma 2.1 shows that $F''Gh$ is strongly measurable and integrable for any $h \in \mathscr{H}$. The same applies to the terms $FG''h$ and $FVGh$. Consequently, the integral on the left-hand side of (2.57) is well-defined in the strong sense. The stated equality (2.57) now follows from an integration by parts as before. $\square$

LEMMA 2.10. *Suppose that $y_0, y_1 \in \mathscr{H}$ and either $x_0 \in (a,b)$ or x_0 is a regular endpoint of τ. Let $y(z,\cdot,x_0)$ be the unique solution of*

$$\begin{cases} \tau y = zy, \\ y(x_0) = y_0,\ y'(x_0) = y_1. \end{cases} \tag{2.60}$$

Then there is a constant $c_0 > 0$ and a constant $C(z,V) \leqslant 1$ depending only on z and V such that

$$\int_{x_0}^{x} dx' \, \|y(x')\|_{\mathscr{H}}^2 \geqslant c_0^2 (x-x_0)^3 \big\|(y_0,y_1)^\top\big\|_{\mathscr{H}\oplus\mathscr{H}}^2 \tag{2.61}$$

provided $0 \leqslant x - x_0 \leqslant C(z,V)$. A similar estimate holds for $x < x_0$.

Proof. Define $r(t) = y(t) - y_0 - (t-x_0)y_1$. Then $-r'' = (z-V)y$ so that the vector version of the variation of constants formula (Lemma 2.8) treating $(z-V)y$ as the non-homogeneous term implies

$$r(x) = \int_{x_0}^{x} dx' \, (x'-x)[z - V(x')]y(x'). \tag{2.62}$$

Hence,

$$\begin{aligned} \|r(x)\|_{\mathscr{H}} \leqslant \sqrt{2}(x-x_0)\big\|(y_0,y_1)^\top\big\|_{\mathscr{H}\oplus\mathscr{H}} \int_{x_0}^{x} dx' \, \|z - V(x')\|_{\mathscr{B}(\mathscr{H})} \\ + (x-x_0)\int_{x_0}^{x} dx' \, \|z - V(x')\|_{\mathscr{B}(\mathscr{H})} \|r(x')\|_{\mathscr{H}}, \end{aligned} \tag{2.63}$$

provided $|x-x_0| \leqslant 1$. Gronwall's lemma then implies the estimate

$$\|r(x)\|_{\mathscr{H}} \leqslant C\big\|(y_0,y_1)^\top\big\|_{\mathscr{H}\oplus\mathscr{H}} (x-x_0) \int_{x_0}^{x} dx' \, \|z - V(x')\|_{\mathscr{B}(\mathscr{H})} \tag{2.64}$$

for an appropriate constant C depending on $V - z$. Thus, using an integration by parts,

$$\int_{x_0}^{x} dx' \, \|r(x')\|_{\mathscr{H}}^2 \leqslant \frac{1}{3}C^2 \big\|(y_0,y_1)^\top\big\|_{\mathscr{H}\oplus\mathscr{H}}^2 (x-x_0)^3 \bigg(\int_{x_0}^{x} dx' \, \|z - V(x')\|_{\mathscr{B}(\mathscr{H})}\bigg)^2. \tag{2.65}$$

On the other hand,

$$\begin{aligned} &\int_{x_0}^{x} dx' \, \|y_0 + (x'-x_0)y_1\|_{\mathscr{H}}^2 \\ &\quad \geqslant (x-x_0)\|y_0\|_{\mathscr{H}}^2 - (x-x_0)^2 \|y_0\|_{\mathscr{H}} \|y_1\|_{\mathscr{H}} + \frac{1}{3}(x-x_0)^3 \|y_1\|_{\mathscr{H}}^2 \\ &\quad \geqslant 4c_0^2 (x-x_0)^3 \big(\|y_0\|_{\mathscr{H}}^2 + \|y_1\|_{\mathscr{H}}^2\big) \end{aligned} \tag{2.66}$$

for some constant $c_0 > 0$, provided $x - x_0$ is sufficiently small (for instance, $c_0 = 1/10$ will do if $0 \leqslant x - x_0 \leqslant 1$). Combining this with (2.65) yields

$$\begin{aligned}\bigg(\int_{x_0}^{x} dx' \, \|y(x')\|_{\mathscr{H}}^2\bigg)^{1/2} &\geqslant (x-x_0)^{3/2} \big\|(y_0,y_1)^\top\big\|_{\mathscr{H}\oplus\mathscr{H}} \\ &\quad \times \bigg[2c_0 - C\int_{x_0}^{x} dx' \, \|z - V(x')\|_{\mathscr{B}(\mathscr{H})}\bigg].\end{aligned} \tag{2.67}$$

Finally, if x is sufficiently close to x_0 in (2.67), the term inside the square brackets will be larger than c_0. $\square$

3. Weyl–Titchmarsh theory

In this section we develop Weyl–Titchmarsh theory for self-adjoint Schrödinger operators H_α in $L^2((a,b);dx;\mathscr{H})$ associated with the operator-valued differential expression $\tau = -(d^2/dx^2) + V(\cdot)$, assuming regularity of the left endpoint a and the limit point case at the right endpoint b (see Definition 3.6). We prove the existence of Weyl–Titchmarsh solutions, introduce the corresponding Weyl–Titchmarsh m-function, and determine the structure of the Green's function of H_α.

The broad outline of our approach in this section follows to a certain degree the path taken in the scalar case by Bennewitz [24, Chs. 10, 11], Edmunds and Evans [42, Sect. III.10], and Weidmann [106, Sect. 8.4]. However, the operator-valued context also necessitates crucial deviations from the scalar approach as will become clear in the course of this section.

We note that the boundary triple approach (see, e.g., [36], [37], [68], [69], [50, Chs. 3, 4] and the extensive literature cited therein) constitutes an alternative way to introduce operator-valued Weyl–Titchmarsh functions. However, we are not aware that this approach has been established for potentials V satisfying our general Hypothesis 2.7. Moreover, we intend to derive the existence of Weyl–Titchmarsh solutions from first principles and with minimal technical efforts.

As before, $\mathscr{H}$ denotes a separable Hilbert space and (a,b) denotes a finite or infinite interval. One recalls that $L^2((a,b);dx;\mathscr{H})$ is separable (since $\mathscr{H}$ is) and that

$$(f,g)_{L^2((a,b);dx;\mathscr{H})} = \int_a^b dx \, (f(x),g(x))_{\mathscr{H}}, \quad f,g \in L^2((a,b);dx;\mathscr{H}). \tag{3.1}$$

Assuming Hypothesis 2.7 throughout this section, we are interested in studying certain self-adjoint operators in $L^2((a,b);dx;\mathscr{H})$ associated with the operator-valued differential expression $\tau = -(d^2/dx^2) + V(\cdot)$. These will be suitable restrictions of the *maximal* operator $H_{\max}$ in $L^2((a,b);dx;\mathscr{H})$ defined by

$$\begin{aligned}&H_{\max} f = \tau f, \\ &f \in \operatorname{dom}(H_{\max}) = \big\{g \in L^2((a,b);dx;\mathscr{H}) \,\big|\, g \in W^{2,1}_{\rm loc}((a,b);dx;\mathscr{H}); \\ &\hspace{6cm} \tau g \in L^2((a,b);dx;\mathscr{H})\big\}.\end{aligned} \tag{3.2}$$

We also introduce the operator $\dot{H}_{\min}$ in $L^2((a,b);dx;\mathscr{H})$ as the restriction of $H_{\max}$ to the domain

$$\operatorname{dom}(\dot{H}_{\min}) = \{g \in \operatorname{dom}(H_{\max}) \,|\, \operatorname{supp}(u) \text{ is compact in } (a,b)\}. \tag{3.3}$$

Finally, the *minimal* operator $H_{\min}$ in $L^2((a,b);dx;\mathscr{H})$ associated with τ is then defined as the closure of $\dot{H}_{\min}$,

$$H_{\min} = \overline{\dot{H}_{\min}}. \tag{3.4}$$

Next, we intend to show that $H_{\max}$ is the adjoint of $\dot{H}_{\min}$ (and hence that of $H_{\min}$), implying, in particular, that $H_{\max}$ is closed. To this end, we first establish the following two preparatory lemmas for the case where a and b are both regular endpoints for τ in the sense of Definition 2.6.

LEMMA 3.1. *In addition to Hypothesis 2.7 suppose that a and b are regular endpoints for τ. Then*

$$\begin{aligned} &\ker(H_{\max} - zI_{L^2((a,b);dx;\mathscr{H})}) \\ &\quad = \{[\theta_0(z,\cdot,a)f + \phi_0(z,\cdot,a)g] \in L^2((a,b);dx;\mathscr{H}) \,|\, f,g \in \mathscr{H}\} \end{aligned} \tag{3.5}$$

is a closed subspace of $L^2((a,b);dx;\mathscr{H})$.

Proof. It is clear that the set on the right-hand side of (3.5) is contained in $\ker(H_{\max} - zI_{L^2((a,b);dx;\mathscr{H})})$. The existence and uniqueness result, Theorem 2.3, also establishes the converse inclusion. Thus, we only need to show that $\ker(H_{\max} - zI_{L^2((a,b);dx;\mathscr{H})})$ is a closed subspace of $L^2((a,b);dx;\mathscr{H})$ (one recalls that we did not yet establish that $H_{\max}$ is a closed operator).

Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset \ker(H_{\max} - zI_{L^2((a,b);dx;\mathscr{H})})$ is a Cauchy sequence with respect to the topology in $L^2((a,b);dx;\mathscr{H})$. By Lemma 2.10 one has for some $\varepsilon > 0$,

$$\begin{aligned} \|u_n - u_m\|^2_{L^2((a,b);dx;\mathscr{H})} &\geqslant \int_a^{a+\varepsilon} dx\, \|u_n(x) - u_m(x)\|^2_{\mathscr{H}} \\ &\geqslant c_0^2 \varepsilon^3 \|(u_n(a) - u_m(a), u_n'(a) - u_m'(a))\|^2_{\mathscr{H}\oplus\mathscr{H}}. \end{aligned} \tag{3.6}$$

This implies that both $\{u_n(a)\}_{n\in\mathbb{N}}$ and $\{u_n'(a)\}_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathscr{H}$ and hence convergent. Denoting the limits by f and g, respectively, one concludes that $u = [\theta_0(z,\cdot,a)f + \phi_0(z,\cdot,a)g] \in \ker(H_{\max} - zI_{L^2((a,b);dx;\mathscr{H})})$. Since

$$\|u_n - u\|_{L^2((a,b);dx;\mathscr{H})} \leqslant [2(b-a)]^{1/2}\big[C_1(z)\|u_n(a) - f\|_{\mathscr{H}} + C_2(z)\|u_n'(a) - g\|_{\mathscr{H}}\big], \tag{3.7}$$

where

$$C_1(z) = \max_{x\in[a,b]} \|\theta_0(z,x,a)\|_{\mathscr{B}(\mathscr{H})}, \quad C_2(z) = \max_{x\in[a,b]} \|\phi_0(z,x,a)\|_{\mathscr{B}(\mathscr{H})}, \tag{3.8}$$

the element u is the strong limit if of the sequence u_n in $L^2((a,b);dx;\mathscr{H})$ and hence $\ker(H_{\max} - zI_{L^2((a,b);dx;\mathscr{H})})$ is closed. $\square$

REMARK 3.2. If $\mathscr{H}$ is finite-dimensional (e.g., in the scalar case, $\dim(\mathscr{H})=1$), then $\ker(H_{\max}-zI_{L^2((a,b);dx;\mathscr{H})})$ is finite-dimensional and hence automatically closed.

LEMMA 3.3. *In addition to Hypothesis 2.7 suppose that a and b are regular endpoints for τ. Denote by H_0 the linear operator in $L^2((a,b);dx;\mathscr{H})$ defined by the restriction of $H_{\max}$ to the space*

$$\operatorname{dom}(H_0)=\{g\in\operatorname{dom}(H_{\max})\,|\,g(a)=g(b)=g'(a)=g'(b)=0\}. \tag{3.9}$$

Then

$$\ker(H_{\max})=[\operatorname{ran}(H_0)]^{\perp}, \tag{3.10}$$

that is, the space of solutions u of $\tau u=0$ coincides with the orthogonal complement of the collection of elements τu_0 satisfying $u_0\in\operatorname{dom}(H_0)$.

Proof. Suppose $u\in\ker(H_{\max})$ and $u_0\in\operatorname{dom}(H_0)$. Let $f_0=H_0u_0$. Then Green's formula (2.55) yields $(f_0,u)_{L^2((a,b);dx;\mathscr{H})}=0$ so that $\operatorname{ran}(H_0)\subseteq[\ker(H_{\max})]^{\perp}$.

Next, assume that $f_0\in[\ker(H_{\max})]^{\perp}$. Since f_0 is integrable, there is a solution u_0 of the initial value problem $\tau u_0=f_0$, $u_0(b)=u_0'(b)=0$. If $u_1\in\ker(H_{\max})$, one has

$$0=(f_0,u_1)_{L^2((a,b);dx;\mathscr{H})}=-(u_0(a),u_1'(a))_{\mathscr{H}}+(u_0'(a),u_1(a))_{\mathscr{H}}, \tag{3.11}$$

using Green's formula (2.55) once more. Since one can choose u_1 so that $u_1'(a)=0$ and $u_1(a)$ is an arbitrary vector in $\mathscr{H}$, one necessarily concludes that $u_0'(a)=0$. Similarly, choosing $u_1(a)=0$ and $u_1'(a)$ arbitrarily shows that $u_0(a)=0$. Hence $u_0\in\operatorname{dom}(H_0)$ and $f_0\in\operatorname{ran}(H_0)$.

We have now shown that $\operatorname{ran}(H_0)=[\ker(H_{\max})]^{\perp}$. Taking orthogonal complements and recalling from Lemma 3.1 that $\ker(H_{\max})$ is closed, concludes the proof of Lemma 3.3. $\square$

THEOREM 3.4. *Assume Hypothesis 2.7. Then the operator $\dot{H}_{\min}$ is densely defined. Moreover, $H_{\max}$ is the adjoint of $\dot{H}_{\min}$,*

$$H_{\max}=(\dot{H}_{\min})^*. \tag{3.12}$$

In particular, $H_{\max}$ is closed. In addition, $\dot{H}_{\min}$ is symmetric and $H_{\max}^*$ is the closure of $\dot{H}_{\min}$, that is,

$$H_{\max}^*=\overline{\dot{H}_{\min}}=H_{\min}. \tag{3.13}$$

Proof. Suppose f_1 is perpendicular to $\operatorname{dom}(\dot{H}_{\min})$ and let u_1 be a solution of $\tau u_1=f_1$. Let $[\tilde{a},\tilde{b}]$ be a compact interval contained in (a,b) and introduce the operators $\widetilde{H_{\max}}$ and $\widetilde{\dot{H}_{\min}}$ associated with that interval and acting in the Hilbert space $L^2(\widetilde{(a,b)};dx;\mathscr{H})=L^2\big((\tilde{a},\tilde{b});dx;\mathscr{H}\big)$ with inner product $(\cdot,\cdot)_{L^2(\widetilde{(a,b)};dx;\mathscr{H})}$. We extend any function $u_0\in\operatorname{dom}\Big(\widetilde{\dot{H}_{\min}}\Big)$ by zero outside the interval $[\tilde{a},\tilde{b}]$ to get an element of $\mathrm{dom}(\dot{H}_{\min})$, also denoted by u_0. Similarly, we consider the restriction of f_1 to $[\tilde{a},\tilde{b}]$, and for simplicity, also denote it by f_1. Thus, setting $f_0 = \tau u_0$, we get via Green's formula (2.55)

$$0 = (u_0, f_1)_{L^2((a,b);dx;\mathscr{H})} = (u_0, f_1)_{L^2(\widetilde{(a,b)};dx;\mathscr{H})} = (f_0, u_1)_{L^2(\widetilde{(a,b)};dx;\mathscr{H})}. \tag{3.14}$$

Lemma 3.3 then implies that $u_1 \in \ker\big(\widetilde{H_{\max}}\big)$ and hence that f_1 is zero almost everywhere in $[\tilde{a},\tilde{b}]$. Since we may choose $\tilde{a}$ arbitrarily close to a, and $\tilde{b}$ arbitrarily close to b, we get $f_1 = 0$ a.e., proving that $\dot{H}_{\min}$ is densely defined.

To show that $H_{\max}$ is the adjoint of $\dot{H}_{\min}$ (and hence a closed operator), we first recall that the domain of $(\dot{H}_{\min})^*$ is given by

$$\begin{aligned} \mathrm{dom}\big((\dot{H}_{\min})^*\big) = \{u \in L^2((a,b);dx;\mathscr{H}) \,|\, &\text{there exists } u^* \in L^2((a,b);dx;\mathscr{H}), \text{ such} \\ &\text{that for all } u_0 \in \mathrm{dom}(\dot{H}_{\min}), (\dot{H}_{\min}u_0, u)_{L^2((a,b);dx;\mathscr{H})} = (u_0, u^*)_{L^2((a,b);dx;\mathscr{H})}\}. \end{aligned} \tag{3.15}$$

The inclusion $\mathrm{dom}(H_{\max}) \subseteq \mathrm{dom}\big((\dot{H}_{\min})^*\big)$ then follows immediately from Green's formula (2.55) because we can choose u^* to be τu whenever $u \in \mathrm{dom}(H_{\max})$.

For proving the reverse inclusion, let $u \in \mathrm{dom}((\dot{H}_{\min})^*)$, note that $u^* = (\dot{H}_{\min})^* u$ is locally integrable, and let h be a solution of the differential equation $\tau h = u^*$. As a consequence of Green's formula (2.55) one obtains that

$$\begin{aligned} \int_a^b dx\,(\tau v, u-h)_{\mathscr{H}} &= (\dot{H}_{\min} v, u)_{L^2((a,b);dx;\mathscr{H})} - \int_a^b dx\,(\tau v, h)_{\mathscr{H}} \\ &= (v, u^*)_{L^2((a,b);dx;\mathscr{H})} - \int_a^b dx\,(v, \tau h)_{\mathscr{H}} = 0, \end{aligned} \tag{3.16}$$

whenever $v \in \mathrm{dom}(\dot{H}_{\min})$. Thus, the restriction of $u-h$ to any interval $[\tilde{a},\tilde{b}] \supseteq \mathrm{supp}(v)$ is orthogonal to $\mathrm{ran}\big(\widetilde{H_{\min}}\big)$ and hence lies in $\ker\big(\widetilde{H_{\max}}\big)$. This shows that u and u' are locally absolutely continuous and that $\tau u = u^* \in L^2((a,b);dx;\mathscr{H})$, that is, $u \in \mathrm{dom}(H_{\max})$.

Since

$$\dot{H}_{\min} \subseteq H_{\max} = (\dot{H}_{\min})^*, \tag{3.17}$$

$\dot{H}_{\min}$ is symmetric in $L^2((a,b);dx;\mathscr{H})$. Hence $H_{\max}^*$ is a restriction of $H_{\max}$ and thus an extension of $\dot{H}_{\min}$. Finally, (3.13) is an immediate consequence of (3.12). $\square$

Lemmas 3.1, 3.3, and Theorem 3.4, under additional hypotheses on V (typically involving continuity assumptions) are of course well-known and go back to Rofe-Beketov [88], [89] (see also [50, Sect. 3.4], [91, Ch. 5]).

REMARK 3.5. In the special case where a and b are regular endpoints for τ, the operator H_0 introduced in (3.9) coincides with the minimal operator $H_{\min}$.

Using the dominated convergence theorem and Green's formula (2.55) one can show that $\lim_{x\to a} W_*(u,v)(x)$ and $\lim_{x\to b} W_*(u,v)(x)$ both exist whenever $u,v \in \operatorname{dom}(H_{\max})$. We will denote these limits by $W_*(u,v)(a)$ and $W_*(u,v)(b)$, respectively. Thus Green's formula also holds for $x_1 = a$ and $x_2 = b$ if u and v are in $\operatorname{dom}(H_{\max})$, that is,

$$(H_{\max}u,v)_{L^2((a,b);dx;\mathscr{H})} - (u,H_{\max}v)_{L^2((a,b);dx;\mathscr{H})} = W_*(u,v)(b) - W_*(u,v)(a). \quad (3.18)$$

This relation and the fact that $H_{\min} = H_{\max}^*$ is a restriction of $H_{\max}$ show that

$$\begin{aligned}\operatorname{dom}(H_{\min}) = \{u \in \operatorname{dom}(H_{\max}) \,|\, W_*(u,v)(b) = W_*(u,v)(a) = 0 \\ \text{for all } v \in \operatorname{dom}(H_{\max})\}.\end{aligned} \quad (3.19)$$

DEFINITION 3.6. Assume Hypothesis 2.7. Then the endpoint a (resp., b) is said to be of *limit-point type for* τ if $W_*(u,v)(a) = 0$ (resp., $W_*(u,v)(b) = 0$) for all $u,v \in \operatorname{dom}(H_{\max})$.

By using the term "limit-point type" one recognizes Weyl's contribution to the subject in his celebrated paper [108].

Next, we introduce the subspaces

$$\mathscr{D}_z = \{u \in \operatorname{dom}(H_{\max}) \,|\, H_{\max}u = zu\}, \quad z \in \mathbb{C}. \quad (3.20)$$

For $z \in \mathbb{C}\backslash\mathbb{R}$, $\mathscr{D}_z$ represent the deficiency subspaces of $H_{\min}$. Von Neumann's theory of extensions of symmetric operators implies that

$$\operatorname{dom}(H_{\max}) = \operatorname{dom}(H_{\min}) \dotplus \mathscr{D}_i \dotplus \mathscr{D}_{-i} \quad (3.21)$$

where $\dotplus$ indicates the direct (but not necessarily orthogonal direct) sum.

LEMMA 3.7. *Assume Hypothesis 2.7. Suppose a is a regular endpoint for τ, let $f_1 \in \mathscr{H}$, $f_2 \in \mathscr{H}$. Then there are elements $u \in \operatorname{dom}(H_{\max})$ such that $u(a) = f_1$, $u'(a) = f_2$, and u vanishes on $[c,b)$ for some $c \in (a,b)$. The analogous statements hold with the roles of a and b interchanged.*

Proof. Let $h = [\theta_0(0,\cdot,a)g_1 + \phi_0(0,\cdot,a)g_2]\chi_{[a,c]}$, where $g_1 \in \mathscr{H}$, $g_2 \in \mathscr{H}$, and $c \in (a,b)$ are as yet undetermined. Then $h \in L^2((a,b);dx;\mathscr{H})$. Solving the initial value problem $\tau u = h$, $u(c) = u'(c) = 0$, implies that $u \in \operatorname{dom}(H_{\max})$ and that u is zero on $[c,b)$. Moreover, Green's formula (2.56) shows that

$$\int_a^c dx'\, \theta_0(0,x',a)^* h(x') = \int_a^c dx'\, \theta_0(0,x',a)^*(-u'' + Vu) = u'(a) \quad (3.22)$$

and

$$\int_a^c dx'\, \phi_0(0,x',a)^* h(x') = \int_a^c dx'\, \phi_0(0,x',a)^*(-u'' + Vu) = -u(a). \quad (3.23)$$

We want to choose g_1 and g_2 so that $u(a) = f_1$ and $u'(a) = f_2$, that is, $A_c(g_1,g_2)^\top = (f_2,-f_1)^\top$, where $A_c : \mathscr{H}\oplus\mathscr{H} \to \mathscr{H}\oplus\mathscr{H}$ is given by

$$A_c = \begin{pmatrix} \int_a^c dx'\, \theta_0(0,x',a)^*\theta_0(0,x',a) & \int_a^c dx'\, \theta_0(0,x',a)^*\phi_0(0,x',a) \\ \int_a^c dx'\, \phi_0(0,x',a)^*\theta_0(0,x',a) & \int_a^c dx'\, \phi_0(0,x',a)^*\phi_0(0,x',a) \end{pmatrix}. \tag{3.24}$$

Hence the proof will be complete if we can show that A_c is invertible for a proper choice of c. Let $F = (g_1,g_2)^\top \in \mathscr{H}\oplus\mathscr{H}$. Since

$$(F, A_cF)_{\mathscr{H}\oplus\mathscr{H}} = \int_a^c dx'\, \|\theta_0(0,x',a)g_1 + \phi_0(0,x',a)g_2\|_{\mathscr{H}}^2, \tag{3.25}$$

and since $\theta_0(0,x',a)g_1 + \phi_0(0,x',a)g_2 = 0$ only if $g_1 = g_2 = 0$, it follows that A_c is positive definite and hence injective. To show that A_c is also surjective we will prove that $(F,A_cF)_{\mathscr{H}\oplus\mathscr{H}} \geqslant \gamma\|F\|^2_{\mathscr{H}\oplus\mathscr{H}}$ for some constant $\gamma > 0$ since this implies that zero cannot be in the approximate point spectrum of A_c (we recall that the spectrum and approximate point spectrum coincide for self-adjoint operators and refer for additional comments to the paragraph preceding Lemma 3.12).

By Lemma 2.10,

$$(F, A_cF)_{\mathscr{H}\oplus\mathscr{H}} = \int_a^c dx'\, \|\theta_0(0,x',a)g_1 + \phi_0(0,x',a)g_2\|_{\mathscr{H}}^2 \geqslant c_0^2(c-a)^3\|F\|^2_{\mathscr{H}\oplus\mathscr{H}} \tag{3.26}$$

provided $c-a$ is sufficiently small. Thus, γ can be chosen as $c_0^2(c-a)^3$. □

We now set out to determine the self-adjoint restrictions of $H_{\max}$ assuming that a is a regular endpoint for τ and b is of limit-point type for τ. To this end we first briefly recall the concept of a Hermitian relation. For more information the reader may consult, for instance, [91, Appendix A].

A subset $\mathscr{M}$ of $\mathscr{H}\oplus\mathscr{H}$ is called a *Hermitian relation* in the Hilbert space $\mathscr{H}$ if it has the following two properties:

1. If (f_1,f_2) and (g_1,g_2) are in $\mathscr{M}$, then $(f_1,g_2)_{\mathscr{H}} = (f_2,g_1)_{\mathscr{H}}$.
2. If $(f_1,f_2) \in \mathscr{H}\oplus\mathscr{H}$ and $(f_1,g_2)_{\mathscr{H}} = (f_2,g_1)_{\mathscr{H}}$ for all $(g_1,g_2) \in \mathscr{M}$, then $(f_1,f_2) \in \mathscr{M}$.

Thus, a Hermitian relation is a linear subspace of $\mathscr{H}\oplus\mathscr{H}$ and one can show that $\mathscr{M} = \widetilde{\mathscr{M}}$ if $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ are Hermitian relations such that $\mathscr{M} \subseteq \widetilde{\mathscr{M}}$. Moreover, the following lemma holds:

LEMMA 3.8. *The maps $\pi_\pm : \mathscr{M} \to \mathscr{H} : (f_1,f_2) \mapsto f_\pm = f_2 \pm if_1$ are linear bijections and $U = \pi_- \circ \pi_+^{-1} : \mathscr{H} \to \mathscr{H}$ is unitary*[1].

Proof. It is clear that $\pi_\pm$ are linear. If $(f_1,f_2) \in \mathscr{M}$, a straightforward calculation yields

$$\|f_\pm\|^2_{\mathscr{H}} = \|f_1\|^2_{\mathscr{H}} + \|f_2\|^2_{\mathscr{H}} \tag{3.27}$$

[1]We note that U is called the Cayley transform of $\mathscr{M}$.

and so proves injectivity of $\pi_\pm$ and that U is a partial isometry. The proof will be finished when we show that $\pi_\pm$ are also surjective.

We begin by showing that the range of π_+ is dense in $\mathscr{H}$. To do so assume that $g\in\mathscr{H}$ is orthogonal to f_2+if_1, that is, $0=(g,f_2+if_1)_{\mathscr{H}}=(g,f_2)_{\mathscr{H}}-(ig,f_1)_{\mathscr{H}}$ for all $(f_1,f_2)\in\mathscr{M}$. This implies that $(g,ig)\in\mathscr{M}$. Then $\pi_-(g,ig)=0$ and, using (3.27), we have $g=0$. Now let $f_+\in\mathscr{H}$. Then there is a sequence $(f_{1,n},f_{2,n})\in\mathscr{M}$, $n\in\mathbb{N}$, such that $f_{2,n}+if_{1,n}$ converges to f_+. Thus $f_{2,n}+if_{1,n}$ is Cauchy in $\mathscr{H}$ and (3.27) entails that $f_{1,n}$ and $f_{2,n}$, $n\in\mathbb{N}$, are separately Cauchy and hence convergent in $\mathscr{H}$. Denote the limit of $(f_{1,n},f_{2,n})$ as $n\to\infty$ by (f_1,f_2). In view of the continuity of scalar products one finds that

$$(f_1,g_2)_{\mathscr{H}}=\lim_{n\to\infty}(f_{1,n},g_2)_{\mathscr{H}}=\lim_{n\to\infty}(f_{2,n},g_1)_{\mathscr{H}}=(f_2,g_1)_{\mathscr{H}},\quad (g_1,g_2)\in\mathscr{M}. \tag{3.28}$$

This implies that $(f_1,f_2)\in\mathscr{M}$ and $f_+=f_2+if_1\in\operatorname{ran}(\pi_+)$. Surjectivity of π_- is shown in the same manner. $\square$

Next, suppose that α is a (bounded or unbounded) self-adjoint operator in $\mathscr{H}$. Then

$$\mathscr{M}_\alpha=\{(f_1,f_2)\in\mathscr{H}\oplus\mathscr{H}\,|\,\sin(\alpha)f_2+\cos(\alpha)f_1=0\} \tag{3.29}$$

is a Hermitian relation. This follows since $\sin(\alpha)f_2+\cos(\alpha)f_1=0$ if and only if there is an $h\in\mathscr{H}$ such that $f_1=-\sin(\alpha)h$ and $f_2=\cos(\alpha)h$. In fact, $h=\cos(\alpha)f_2-\sin(\alpha)f_1$, if $(f_1,f_2)\in\mathscr{M}_\alpha$ is given.

We now use the theory of Hermitian relations to characterize all self-adjoint restrictions of $H_{\max}$ under the following set of assumptions:

HYPOTHESIS 3.9. *In addition to Hypothesis 2.7 suppose that a is a regular endpoint for τ and b is of limit-point type for τ.*

THEOREM 3.10. *Assume Hypothesis 3.9. If H is a self-adjoint restriction of $H_{\max}$, then there is a bounded and self-adjoint operator $\alpha\in\mathscr{B}(\mathscr{H})$ such that*

$$\operatorname{dom}(H)=\{u\in\operatorname{dom}(H_{\max})\,|\,\sin(\alpha)u'(a)+\cos(\alpha)u(a)=0\}. \tag{3.30}$$

Conversely, for every $\alpha\in\mathscr{B}(\mathscr{H})$, (3.30) gives rise to a self-adjoint restriction of $H_{\max}$ in $L^2((a,b);dx;\mathscr{H})$.

Proof. Suppose $H=H^*\subseteq H_{\max}$ and define

$$\mathscr{M}=\{(f_1,f_2)\in\mathscr{H}\oplus\mathscr{H}\,|\,\text{there exists } u\in\operatorname{dom}(H)\text{ such that } f=u(a), f'=u'(a)\}. \tag{3.31}$$

We show first that $\mathscr{M}$ is a Hermitian relation: For $(f_1,f_2),(g_1,g_2)\in\mathscr{M}$ let $u,v\in\operatorname{dom}(H)$ be such that $u(a)=f_1$, $u'(a)=f_2$, $v(a)=g_1$, and $v'(a)=g_2$. Since H is self-adjoint one infers from Green's formula (2.55) that

$$\begin{aligned}0&=(Hu,v)_{L^2((a,b);dx;\mathscr{H})}-(u,Hv)_{L^2((a,b);dx;\mathscr{H})}\\&=-W_*(u,v)(a)=(u'(a),v(a))_{\mathscr{H}}-(u(a),v'(a))_{\mathscr{H}}.\end{aligned} \tag{3.32}$$

Next assume $(f_1,f_2)\in\mathcal{H}\oplus\mathcal{H}$ and that $(f_1,v'(a))_{\mathcal{H}}=(f_2,v(a))_{\mathcal{H}}$ for all $v\in\operatorname{dom}(H)$. By Lemma 3.7 there is a $u\in\operatorname{dom}(H_{\max})$ with initial values (f_1,f_2) and hence,

$$\begin{aligned}(H_{\max}u,v)_{L^2((a,b);dx;\mathcal{H})}&-(u,Hv)_{L^2((a,b);dx;\mathcal{H})}\\&=-W_*(u,v)(a)=(f_2,v(a))_{\mathcal{H}}-(f_1,v'(a))_{\mathcal{H}}=0.\end{aligned}\tag{3.33}$$

This implies that $u\in\operatorname{dom}(H^*)=\operatorname{dom}(H)$ (with $H^*u=H_{\max}u$) and hence that $(f_1,f_2)\in\mathcal{M}$. Thus $\mathcal{M}$ is indeed a Hermitian relation. Denote its Cayley transform by U and the family of strongly right-continuous spectral projections associated with U by $\{F_U(t)\}_{t\in[0,2\pi]}$, implying[2],

$$(f,Ug)_{\mathcal{H}}=\int_{[0,2\pi]}e^{it}d(f,F_U(t)g)_{\mathcal{H}},\quad F(0)=0.\tag{3.34}$$

Additionally, let α be the bounded self-adjoint operator defined by

$$(f,\alpha g)_{\mathcal{H}}=\frac{1}{2}\int_{[0,2\pi]}t\,d(f,F_U(t)g)_{\mathcal{H}}.\tag{3.35}$$

Since U is the Cayley transform of $\mathcal{M}$, we have $U(f_2+if_1)=f_2-if_1$, or equivalently, $(U-I_{\mathcal{H}})f_2+i(U+I_{\mathcal{H}})f_1=0$. Since $U=e^{2i\alpha}$, the latter relation implies that $\sin(\alpha)f_2+\cos(\alpha)f_1=0$. Thus, $\mathcal{M}\subseteq\mathcal{M}_\alpha$, implying (as shown in the paragraph preceding Lemma 3.8), that $\mathcal{M}=\mathcal{M}_\alpha$. Thus the first part of Theorem 3.10 follows.

For the converse part, assume $\alpha=\alpha^*\in\mathcal{B}(\mathcal{H})$ is given, and let H denote the restriction of $H_{\max}$ to those functions satisfying $\sin(\alpha)u'(a)+\cos(\alpha)u(a)=0$, that is, $u\in\operatorname{dom}(H)$ if and only if $(u(a),u'(a))\in\mathcal{M}_\alpha$. Therefore, if $u,v\in\operatorname{dom}(H)$, then $W_*(u,v)(a)=W_*(u,v)(b)=0$ so that $(Hu,v)_{L^2((a,b);dx;\mathcal{H})}=(u,Hv)_{L^2((a,b);dx;\mathcal{H})}$, implying $\operatorname{dom}(H)\subseteq\operatorname{dom}(H^*)$. To show the opposite inclusion one first notes that $\operatorname{dom}(H^*)\subseteq\operatorname{dom}(H_{\max})$ since $\operatorname{dom}(H_{\max}^*)\subseteq\operatorname{dom}(H)$. Now assume that $u\in\operatorname{dom}(H^*)$ and $v\in\operatorname{dom}(H)$. Then $H^*u=H_{\max}u$ so that $W_*(u,v)(a)=0$ for all $(v(a),v'(a))\in\mathcal{M}_\alpha$. This implies that $(u(a),u'(a))\in\mathcal{M}_\alpha$, that is, $\operatorname{dom}(H^*)\subseteq\operatorname{dom}(H)$. $\square$

Henceforth, under the assumptions of Theorem 3.10, we denote the operator H in $L^2((a,b);dx;\mathcal{H})$ associated with the boundary condition induced by $\alpha=\alpha^*\in\mathcal{B}(\mathcal{H})$, that is, the restriction of $H_{\max}$ to the set

$$\operatorname{dom}(H_\alpha)=\{u\in\operatorname{dom}(H_{\max})\,|\,\sin(\alpha)u'(a)+\cos(\alpha)u(a)=0\}\tag{3.36}$$

by H_α. For a discussion of boundary conditions at infinity, see, for instance, [68], [75], and [90].

Our next goal is to construct the square integrable solutions $Y(z,\cdot)\in\mathcal{B}(\mathcal{H})$ of $\tau Y=zY$, $z\in\mathbb{C}\backslash\mathbb{R}$, the $\mathcal{B}(\mathcal{H})$-valued Weyl–Titchmarsh solutions, under the assumptions that a is a regular endpoint for τ and b is of limit-point type for τ.

For ease of notation, we denote in the following the resolvent of H_α by $R_{z,\alpha}$, that is, $R_{z,\alpha}=(H_\alpha-zI_{L^2((a,b);dx;\mathcal{H})})^{-1}$.

[2]We employ the standard slight abuse of notation where $F_U(t)=F_U([0,t))$, $t\in[0,2\pi]$, and use the normalization $\text{s-lim}_{\varepsilon\downarrow0}F_U(-\varepsilon)=0$, $F_U(2\pi)=\text{s-lim}_{\varepsilon\downarrow0}F_U(2\pi+\varepsilon)=I_{\mathcal{H}}$.

One recalls that the graph of H_α, given by

$$\Gamma = \{(f, H_\alpha f) \in L^2((a,b);dx;\mathscr{H}) \oplus L^2((a,b);dx;\mathscr{H}) \,|\, f \in \operatorname{dom}(H_\alpha)\}, \tag{3.37}$$

is a Hilbert subspace of $L^2((a,b);dx;\mathscr{H}) \oplus L^2((a,b);dx;\mathscr{H})$. Equivalently, one can consider $\operatorname{dom}(H_\alpha)$ as a Hilbert space with scalar product

$$\begin{aligned}(f,g)_\Gamma = \int_a^b dx\, (f(x), g(x))_{\mathscr{H}} + \int_a^b dx\, ((H_\alpha f)(x), (H_\alpha g)(x))_{\mathscr{H}},& \\ f,g \in \operatorname{dom}(H_\alpha),&\end{aligned} \tag{3.38}$$

and the corresponding norm $\|f\|_\Gamma = (f,f)_\Gamma^{1/2}$, $f \in \operatorname{dom}(H_\alpha)$. Given a compact interval $J \subset [a,b)$ we know that $\operatorname{dom}(H_\alpha)$ is contained in the Banach space $C^1(J;\mathscr{H})$ of continuously differentiable functions on J with values in $\mathscr{H}$ and norm given by $\|f\|_J = \sup_{x\in J} \|f(x)\|_{\mathscr{H}} + \sup_{x \in J} \|f'(x)\|_{\mathscr{H}}$. In fact, the following lemma holds.

LEMMA 3.11. *Assume Hypothesis 3.9 and suppose that $\alpha \in \mathscr{B}(\mathscr{H})$ is self-adjoint. For each compact interval $J \subset [a,b)$ there is a constant C_J such that $\|y\|_J \leqslant C_J \|y\|_\Gamma$ for every $y \in \operatorname{dom}(H_\alpha)$.*

Proof. Suppose $\{y_n\}_{n\in\mathbb{N}} \subset \operatorname{dom}(H_\alpha)$ is a sequence converging to $y \in \operatorname{dom}(H_\alpha)$ with respect to the norm $\|\cdot\|_\Gamma$ and that $y_n|_J$ converges in $C^1(J;\mathscr{H})$ to $\tilde y$ as $n \to \infty$. It follows that

$$\|y_n - y\|_{L^2((a,b);dx;\mathscr{H})} + \|y_n - y\|_{L^2(J;dx;\mathscr{H})} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.39}$$

On account of the uniform convergence in $C^1(J;\mathscr{H})$ one also concludes that $\|y_n - \tilde y\|_{L^2(J;dx;\mathscr{H})} \to 0$ as $n \to \infty$. Thus, $y|_J = \tilde y$ so that the restriction map $y \mapsto y|_J$ defined on $\operatorname{dom}(H_\alpha)$ is closed and hence bounded by the closed graph theorem. □

We recall that a point $\lambda \in \mathbb{C}$ is said to be in the *approximate point spectrum* of a closed operator $T \in \mathscr{B}(\mathscr{H})$ if there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ such that $\|x_n\|_{\mathscr{H}} = 1$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|(T - \lambda I_{\mathscr{H}})x_n\|_{\mathscr{H}} = 0$. If λ is an eigenvalue, then it is, of course, in the approximate point spectrum. λ is also in the approximate point spectrum, if $T - \lambda I_{\mathscr{H}}$ is injective and its image is dense in $\mathscr{H}$ but not closed, a fact that can be seen as follows: In this case $(T - \lambda I_{\mathscr{H}})^{-1}$ is a densely defined unbounded operator, that is, there is a sequence f_n such that $\|f_n\|_{\mathscr{H}} = 1$ and $\|(T - \lambda I_{\mathscr{H}})^{-1} f_n\|_{\mathscr{H}} > n$, $n \in \mathbb{N}$. This is equivalent to the existence of a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ (namely $y_n = (T-\lambda)^{-1} f_n / \|(T-\lambda)^{-1} f_n\|$, $n \in \mathbb{N}$) such that $\|y_n\|_{\mathscr{H}} = 1$ and $\|(T - \lambda I_{\mathscr{H}})y_n\|_{\mathscr{H}} < 1/n$, $n \in \mathbb{N}$, so that λ is in the approximate point spectrum. If T has no residual spectrum, in particular, if T is self-adjoint, its spectrum coincides with its approximate point spectrum.

LEMMA 3.12. *Suppose $\alpha \in \mathscr{B}(\mathscr{H})$ is self-adjoint. If $c_j \in \mathbb{C}$, $j = 1,2$, with $c_1/c_2 \in \mathbb{C}\backslash\mathbb{R}$, then $0 \in \rho(c_1 \sin(\alpha) + c_2 \cos(\alpha))$.*

Proof. Let $A = \sin(\alpha)$, $B = \cos(\alpha)$, and assume that $c_j \in \mathbb{C}$, $j = 1,2$, with $c_1/c_2 \in \mathbb{C}\backslash\mathbb{R}$. The spectral theorem implies that the spectra of A and B are contained in $[-1,1]$ and that the spectra of A^2 and B^2 are contained in $[0,1]$. By way of contradiction, assume that 0 is in the approximate point spectrum of c_1A+c_2B. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ such that $\|x_n\|_{\mathscr{H}} = 1$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|(c_1A+c_2B)x_n\|_{\mathscr{H}} = 0$. Accordingly, also

$$\|(c_1^2A^2 + c_1c_2AB)x_n\|_{\mathscr{H}} \to 0 \text{ and } \|(c_1c_2BA + c_2^2B^2)x_n\|_{\mathscr{H}} \to 0 \tag{3.40}$$

as $n \to \infty$. Hence, $(c_1^2A^2 - c_2^2B^2)x_n = (c_1^2+c_2^2)A^2x_n - c_2^2x_n$ tends to zero as $n\to\infty$, so that $c_2^2/(c_1^2+c_2^2)$ is in the approximate point spectrum of A^2. This implies that c_1/c_2 is real, a contradiction. Thus, 0 is not in the approximate point spectrum of c_1A+c_2B. Hence, for 0 to be in the spectrum of c_1A+c_2B would require that its image not be dense in $\mathscr{H}$, that is, that $\ker(\overline{c_1}A+\overline{c_2}B) = \ker((c_1A+c_2B)^*) = \operatorname{ran}(c_1A+c_2B)^\perp \supsetneqq \{0\}$. But this is impossible as we have just shown. $\square$

Fix $c \in (a,b)$ and $z \in \rho(H_\alpha)$. For any $f_0 \in \mathscr{H}$ let $f = f_0\chi_{[a,c]} \in L^2((a,b);dx;\mathscr{H})$ and $u(f_0,z,\cdot) = R_{z,\alpha}f \in \operatorname{dom}(H_\alpha)$. By the variation of constants formula,

$$\begin{aligned} u(f_0,z,x) = \theta_\alpha(z,x,a)\bigg(g(z) + \int_x^c dx'\, \phi_\alpha(\overline{z},x',a)^*f_0\bigg)& \\ + \phi_\alpha(z,x,a)\bigg(h(z) - \int_x^c dx'\, \theta_\alpha(\overline{z},x',a)^*f_0\bigg)& \end{aligned} \tag{3.41}$$

for suitable vectors $g(z) \in \mathscr{H}$, $h(z) \in \mathscr{H}$. Since $u(f_0,z,\cdot) \in \operatorname{dom}(H_\alpha)$, one infers that

$$g(z) = -\int_a^c dx'\, \phi_\alpha(\overline{z},x',a)^*f_0, \quad z \in \rho(H_\alpha), \tag{3.42}$$

and that

$$h(z) = \cos(\alpha)u'(f_0,z,a) - \sin(\alpha)u(f_0,z,a) + \int_a^c dx'\, \theta_\alpha(\overline{z},x',a)^*f_0, \quad z \in \rho(H_\alpha). \tag{3.43}$$

LEMMA 3.13. *Assume Hypothesis 3.9 and suppose that $\alpha \in \mathscr{B}(\mathscr{H})$ is self-adjoint. In addition, choose $c \in (a,b)$ and introduce $g(\cdot)$ and $h(\cdot)$ as in (3.42) and (3.43). Then the maps*

$$C_{1,\alpha}(c,z) : \begin{cases} \mathscr{H} \to \mathscr{H}, \\ f_0 \mapsto g(z), \end{cases} \qquad C_{2,\alpha}(c,z) : \begin{cases} \mathscr{H} \to \mathscr{H}, \\ f_0 \mapsto h(z), \end{cases} \qquad z \in \rho(H_\alpha), \tag{3.44}$$

are linear and bounded. Moreover, $C_{1,\alpha}(c,\cdot)$ is entire and $C_{2,\alpha}(c,\cdot)$ is analytic on $\rho(H_\alpha)$. In addition, $C_{1,\alpha}(c,z)$ is boundedly invertible if $z \in \mathbb{C}\backslash\mathbb{R}$ and c is chosen appropriately.

Proof. According to equation (3.42) one has

$$C_{1,\alpha}(c,z) = -\int_a^c dx'\, \phi_\alpha(\overline{z},x',a)^*. \tag{3.45}$$

By Corollary 2.5 (iii), $C_{1,\alpha}(c,\cdot)$ is entire.

Next, one observes that $\rho(H_\alpha) \ni z \mapsto u(f_0,z,x) = (R_{z,\alpha}f)(x)$ is analytic and its derivative at z_0 is given by $(R_{z_0,\alpha}^2 f)(x)$. This follows from Lemma 3.11 and the first resolvent identity since

$$\begin{aligned} &\left\| \frac{(R_{z,\alpha}f)(x) - (R_{z_0,\alpha}f)(x)}{z-z_0} - (R_{z_0,\alpha}^2 f)(x) \right\|_{\mathscr{H}} \leqslant \left\| \frac{R_{z,\alpha}f - R_{z_0,\alpha}f}{z-z_0} - R_{z_0,\alpha}^2 f \right\|_J \\ &\quad \leqslant C_J \left\| \frac{R_{z,\alpha}f - R_{z_0,\alpha}f}{z-z_0} - R_{z_0,\alpha}^2 f \right\|_\Gamma \leqslant C_J \left\| (R_{z,\alpha} - R_{z_0,\alpha}) R_{z_0,\alpha} f \right\|_\Gamma, \end{aligned} \tag{3.46}$$

as long as $x \in J$, with $J \subset (a,b)$ a compact interval, noting in addition that

$$H_\alpha (R_{z,\alpha} - R_{z_0,\alpha}) = zR_{z,\alpha} - z_0 R_{z_0,\alpha}. \tag{3.47}$$

Similarly, $z \mapsto u'(f_0,z,x) = (R_{z,\alpha}f)'(x)$ is analytic, proving that $C_{2,\alpha}(c,\cdot)$ is analytic on $\rho(H_\alpha)$.

It remains to show the bounded invertibility of $C_{1,\alpha}(c,z)$ for $z \in \mathbb{C}\backslash\mathbb{R}$ and appropriate $c \in (a,b)$. In order for the expression

$$\frac{\tan(\mu)}{\mu} = \frac{1-\cos(2\mu)}{\mu\sin(2\mu)}, \quad \mu \in \mathbb{C}, \tag{3.48}$$

to be real-valued it is necessary that μ be either real or purely imaginary. Hence, using Lemma 3.12, one finds that the operator

$$\begin{aligned} S &= \sin(\alpha)\frac{\sin(k(c-a))}{k} + \cos(\alpha)\frac{\cos(k(c-a))-1}{k^2} \\ &= \int_a^c dx' \left[\sin(\alpha)\cos(k(x'-a)) - \cos(\alpha)\frac{\sin(k(x'-a))}{k} \right] \end{aligned} \tag{3.49}$$

is boundedly invertible unless $k^2 \in \mathbb{R}$. A proof similar to that of Lemma 2.10 then shows that

$$\left\| C_{1,\alpha}(c,k^2) - S \right\|_{\mathscr{B}(\mathscr{H})} \tag{3.50}$$

is arbitrarily small for $c-a$ is sufficiently small. This proves that $C_{1,\alpha}(c,z)$ is boundedly invertible if $z \in \mathbb{C}\backslash\mathbb{R}$ and c is chosen appropriately. $\square$

Using the bounded invertibility of $C_{1,\alpha}(c,z)$ we now define

$$\psi_\alpha(z,x) = \theta_\alpha(z,x,a) + \phi_\alpha(z,x,a)C_{2,\alpha}(c,z)C_{1,\alpha}(c,z)^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R},\ x \in [a,b), \tag{3.51}$$

still assuming Hypothesis 3.9 and $\alpha = \alpha^* \in \mathscr{B}(\mathscr{H})$. By Lemma 3.13, $\psi_\alpha(\cdot,x)$ is analytic on $z \in \mathbb{C}\backslash\mathbb{R}$ for fixed $x \in [a,b]$.

Since $\psi_\alpha(z,\cdot)f_0$ is the solution of the initial value problem

$$\tau y = zy, \quad y(c) = u(f_0,z,c),\ y'(c) = u'(f_0,z,c), \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{3.52}$$

the function $\psi_\alpha(z,x)C_{1,\alpha}(z,c)f_0$ equals $u(f_0,z,x)$ for $x\geqslant c$, and thus is square integrable for every choice of $f_0\in\mathscr{H}$. In particular, choosing $c\in(a,b)$ such that $C_{1,\alpha}(z,c)^{-1}\in\mathscr{B}(\mathscr{H})$, one infers that

$$\int_a^b dx\,\|\psi_\alpha(z,x)f\|_{\mathscr{H}}^2<\infty, \quad f\in\mathscr{H},\ z\in\mathbb{C}\backslash\mathbb{R}. \tag{3.53}$$

Every $\mathscr{H}$-valued solution of $\tau y = zy$ may be written as

$$y = \theta_\alpha(z,\cdot,a)f_{\alpha,a}+\phi_\alpha(z,\cdot,a)g_{\alpha,a}, \tag{3.54}$$

with

$$f_{\alpha,a} = (\cos\alpha)y(a)+(\sin\alpha)y'(a), \quad g_{\alpha,a} = -(\sin\alpha)y(a)+(\cos\alpha)y'(a). \tag{3.55}$$

Hence we can define the maps

$$\mathscr{C}_{1,\alpha,z}:\begin{cases}\mathscr{D}_z\to\mathscr{H},\\ \theta_\alpha(z,\cdot,a)f_{\alpha,a}+\phi_\alpha(z,\cdot,a)g_{\alpha,a}\mapsto f_{\alpha,a},\end{cases} \tag{3.56}$$

$$\mathscr{C}_{2,\alpha,z}:\begin{cases}\mathscr{D}_z\to\mathscr{H},\\ \theta_\alpha(z,\cdot,a)f_{\alpha,a}+\phi_\alpha(z,\cdot,a)g_{\alpha,a}\mapsto g_{\alpha,a}.\end{cases} \tag{3.57}$$

LEMMA 3.14. *Assume Hypothesis 3.9, suppose that $\alpha\in\mathscr{B}(\mathscr{H})$ is self-adjoint, and let $z\in\mathbb{C}\backslash\mathbb{R}$. Then the operators $\mathscr{C}_{1,\alpha,z}$ and $\mathscr{C}_{2,\alpha,z}$ are linear bijections and hence*

$$\mathscr{C}_{1,\alpha,z},\mathscr{C}_{1,\alpha,z}^{-1},\mathscr{C}_{2,\alpha,z},\mathscr{C}_{2,\alpha,z}^{-1}\in\mathscr{B}(\mathscr{H}). \tag{3.58}$$

Proof. It is clear that $\mathscr{C}_{1,\alpha,z}$ and $\mathscr{C}_{2,\alpha,z}$ are linear. Given $f\in\mathscr{H}$ one concludes that $u=\psi_\alpha(z,\cdot)f$ and $v=\psi_{\alpha+\pi/2}(z,\cdot)f$ are in $\mathscr{D}_z$ and $\mathscr{C}_{1,\alpha,z}u=\mathscr{C}_{2,\alpha,z}v=f$. This proves surjectivity of $\mathscr{C}_{1,\alpha,z}$ and $\mathscr{C}_{2,\alpha,z}$.

Next, let $u=\theta_\alpha f+\phi_\alpha g\in\mathscr{D}_z$ and $f=0$ or $g=0$. Then $W_*(u,u)(a)=0$. Moreover, since b is of limit-point type for τ, $W_*(u,u)(b)=0$. Hence, by (3.18),

$$\begin{aligned} 0 &= (H_{\max}u,u)_{L^2((a,b);dx;\mathscr{H})}-(u,H_{\max}u)_{L^2((a,b);dx;\mathscr{H})}\\ &= (zu,u)_{L^2((a,b);dx;\mathscr{H})}-(u,zu)_{L^2((a,b);dx;\mathscr{H})} = (\overline{z}-z)\|u\|^2_{L^2((a,b);dx;\mathscr{H})},\end{aligned} \tag{3.59}$$

implying $u=0$ and injectivity of $\mathscr{C}_{1,\alpha,z}$ and $\mathscr{C}_{2,\alpha,z}$. Since for any invertible operator T in $\mathscr{H}$ one has that T^{-1} is closed if and only if T is (cf. [59, Sect. III.5.2]), the closed graph theorem (see, [59, Sect. III.5.4]) yields (3.58). $\square$

At this point we are finally in the position to define the Weyl–Titchmarsh m-function for $z\in\mathbb{C}\backslash\mathbb{R}$ by setting

$$m_\alpha(z)=\mathscr{C}_{2,\alpha,z}\mathscr{C}_{1,\alpha,z}^{-1}, \quad z\in\mathbb{C}\backslash\mathbb{R}. \tag{3.60}$$

THEOREM 3.15. *Assume Hypothesis 3.9 and suppose that* $\alpha \in \mathscr{B}(\mathscr{H})$ *is self-adjoint. Then*

$$m_\alpha(z) \in \mathscr{B}(\mathscr{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.61}$$

and $m_\alpha(\cdot)$ *is analytic on* $\mathbb{C}\backslash\mathbb{R}$. *Moreover,*

$$m_\alpha(z) = m_\alpha(\overline{z})^*, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.62}$$

Proof. The boundedness relation (3.61) follows from (3.58) and (3.60). To prove analyticity we first show that $m_\alpha(z) = C_{2,\alpha}(c,z)C_{1,\alpha}^{-1}(c,z)$ where $C_{1,\alpha}$, $C_{2,\alpha}$ and c are as in Lemma 3.13. To this end let h be an arbitrary element of $\mathscr{H}$. Then

$$\begin{aligned}\mathscr{C}_{2,\alpha,z}\mathscr{C}_{1,\alpha,z}^{-1}h &= \mathscr{C}_{2,\alpha,z}\psi_\alpha(z,\cdot)h\\ &= \mathscr{C}_{2,\alpha,z}(\theta_\alpha(z,\cdot,a)h + \phi_\alpha(z,\cdot,a)C_{2,\alpha}(c,z)C_{1,\alpha}(c,z)^{-1}h)\\ &= C_{2,\alpha}(c,z)C_{1,\alpha}(c,z)^{-1}h \end{aligned} \tag{3.63}$$

establishing the claimed identity. The analyticity of m_α on $\mathbb{C}\backslash\mathbb{R}$ now follows from Lemma 3.13.

To prove (3.62) one first observes that (2.38)–(2.41) yield

$$W(\psi_\alpha(\overline{z},\cdot)^*, \psi_\alpha(z,\cdot))(x) = m_\alpha(z) - m_\alpha(\overline{z})^*. \tag{3.64}$$

Fixing arbitrary $f, g \in \mathscr{H}$, then yields

$$(f, (m_\alpha(z) - m_\alpha(\overline{z})^*)g)_{\mathscr{H}} = W(\psi_\alpha(\overline{z},\cdot)^* f, \psi_\alpha(z,\cdot)g)(x) \underset{x\uparrow b}{\longrightarrow} 0, \tag{3.65}$$

since both $\psi_\alpha(\overline{z},\cdot)f$ and $\psi_\alpha(z,\cdot)g$ are in $\mathrm{dom}(H_{\max})$ and since b is of limit-point-type for τ. $\square$

As a consequence of (3.63), the $\mathscr{B}(\mathscr{H})$-valued function $\psi_\alpha(z,\cdot)$ in (3.51) can be rewritten in the form

$$\psi_\alpha(z,x) = \theta_\alpha(z,x,a) + \phi_\alpha(z,x,a)m_\alpha(z), \quad z \in \mathbb{C}\backslash\mathbb{R},\ x \in [a,b). \tag{3.66}$$

In particular, this implies that $\psi_\alpha(z,\cdot)$ is independent of the choice of the parameter $c \in (a,b)$ in (3.51). Following the tradition in the scalar case ($\dim(\mathscr{H}) = 1$), we will call $\psi_\alpha(z,\cdot)$ the *Weyl–Titchmarsh* solution associated with $\tau Y = zY$.

We remark that, given a function $u \in \mathscr{D}_z$, the operator $m_0(z)$ assigns the Neumann boundary data $u'(a)$ to the Dirichlet boundary data $u(a)$, that is, $m_0(z)$ is the (z-dependent) Dirichlet-to-Neumann map.

With the aid of the Weyl–Titchmarsh solutions we can now give a detailed description of the resolvent $R_{z,\alpha} = (H_\alpha - zI_{L^2((a,b);dx;\mathscr{H})})^{-1}$ of H_α.

THEOREM 3.16. *Assume Hypothesis 3.9 and suppose that* $\alpha \in \mathscr{B}(\mathscr{H})$ *is self-adjoint. Then the resolvent of* H_α *is an integral operator of the type*

$$\begin{gathered}\big((H_\alpha - zI_{L^2((a,b);dx;\mathscr{H})})^{-1}u\big)(x) = \int_a^b dx'\, G_\alpha(z,x,x')u(x'),\\ u \in L^2((a,b);dx;\mathscr{H}),\ z \in \rho(H_\alpha),\ x \in [a,b),\end{gathered} \tag{3.67}$$

with the $\mathscr{B}(\mathscr{H})$-valued Green's function $G_\alpha(z,\cdot,\cdot)$ given by

$$G_\alpha(z,x,x') = \begin{cases} \phi_\alpha(z,x,a)\psi_\alpha(\overline{z},x')^*, & a \leqslant x \leqslant x' < b, \\ \psi_\alpha(z,x)\phi_\alpha(\overline{z},x',a)^*, & a \leqslant x' \leqslant x < b, \end{cases} \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.68}$$

Proof. First assume that $u \in L^2((a,b);dx;\mathscr{H})$ is compactly supported and let

$$v(x) = \psi_\alpha(z,x)\int_a^x \phi_\alpha(\overline{z},x',a)^* u(x')dx' + \phi_\alpha(z,x,a)\int_x^b \psi_\alpha(\overline{z},x')^* u(x')dx'. \tag{3.69}$$

We need to show that $v = R_{z,\alpha}u$. To this end one notes that both v and v' are in $W^{(1,1)}_{\rm loc}((a,b),dx;\mathscr{H})$. Near the endpoints v is a multiple of either $\phi_\alpha(z,\cdot,a)$ or $\psi_\alpha(z,\cdot)$. Hence it satisfies the boundary condition at a and is square integrable. Differentiating once more shows that $\tau v = u$ so that $v \in L^2((a,b);dx;\mathscr{H})$ and $v = R_{z,\alpha}u$. The fact that compactly supported functions are dense in $L^2((a,b);dx;\mathscr{H})$ completes the proof. □

One recalls from Definition A.1 that a nonconstant function $N : \mathbb{C}_+ \to \mathscr{B}(\mathscr{H})$ is called a (bounded) operator-valued Herglotz function, if $z \mapsto (u,N(z)u)_{\mathscr{H}}$ is analytic and has a non-negative imaginary part for all $u \in \mathscr{H}$.

THEOREM 3.17. *Assume Hypothesis 3.9 and suppose that $\alpha \in \mathscr{B}(\mathscr{H})$ and $\beta \in \mathscr{B}(\mathscr{H})$ are self-adjoint. Then the $\mathscr{B}(\mathscr{H})$-valued function $m_\alpha(\cdot)$ is an operator-valued Herglotz function and explicitly determined by the Green's function for H_α as follows,*

$$m_\alpha(z) = \big(-\sin(\alpha),\cos(\alpha)\big)\begin{pmatrix} G_\alpha(z,a,a) & G_{\alpha,x'}(z,a,a) \\ G_{\alpha,x}(z,a,a) & G_{\alpha,x,x'}(z,a,a)\end{pmatrix}\begin{pmatrix}-\sin(\alpha) \\ \cos(\alpha)\end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.70}$$

where we denoted

$$\begin{aligned} G_{\alpha,x}(z,a,a) &= \operatorname*{s-lim}_{\substack{x'\to a\\ a<x<x'}} \frac{\partial}{\partial x} G_\alpha(z,x,x'), \\ G_{\alpha,x'}(z,a,a) &= \operatorname*{s-lim}_{\substack{x'\to a\\ a<x<x'}} \frac{\partial}{\partial x'} G_\alpha(z,x,x'), \\ G_{\alpha,x,x'}(z,a,a) &= \operatorname*{s-lim}_{\substack{x'\to a\\ a<x<x'}} \frac{\partial}{\partial x}\frac{\partial}{\partial x'} G_\alpha(z,x,x') \end{aligned} \tag{3.71}$$

(the strong limits referring to the strong operator topology in $\mathscr{H}$). In addition, $m_\alpha(\cdot)$ extends analytically to the resolvent set of H_α.

Moreover, $m_\alpha(\cdot)$ and $m_\beta(\cdot)$ are related by the following linear fractional transformation,

$$m_\beta = (C + Dm_\alpha)(A + Bm_\alpha)^{-1}, \tag{3.72}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta)\end{pmatrix}\begin{pmatrix}\cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha)\end{pmatrix}. \tag{3.73}$$

Proof. Pick $z \in \mathbb{C}\backslash\mathbb{R}$ throughout this proof. We begin by establishing the validity of the linear fractional transformation. Let ψ be any $\mathscr{H}$-valued square integrable solution of $\tau\psi = z\psi$. Since

$$\psi(x) = \theta_\alpha(z,\cdot,a)f + \phi_\alpha(z,\cdot,a)g = \theta_\beta(z,\cdot,a)u + \phi_\beta(z,\cdot,a)v \tag{3.74}$$

for appropriate $f,g,u,v \in \mathscr{H}$, one gets

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}. \tag{3.75}$$

Since $v = m_\beta u$, $g = m_\alpha f$, and since $A + Bm_\alpha(z) = \mathscr{C}_{1,\beta,z}\mathscr{C}_{1,\alpha,z}^{-1}$ is invertible, one obtains (3.72).

In view of this relationship between m-operators for different boundary conditions we prove the first part of the theorem first for a specific boundary condition, namely $\alpha_0 = \frac{\pi}{2}I_{\mathscr{H}}$ so that $\sin(\alpha_0) = I_{\mathscr{H}}$ and $\cos(\alpha_0) = 0$. Then, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\theta_{\pi/2}(z,x,a)\|_{\mathscr{B}(\mathscr{H})}$ and $\|\phi_{\pi/2}(z,x,a) - I_{\mathscr{H}}\|_{\mathscr{B}(\mathscr{H})}$ are smaller than ε provided $x - a < \delta$. Next, for any fixed $u_0 \in \mathscr{H}$ let $u_\delta = u_0\chi_{[a,a+\delta]}/\delta^{1/2}$. Using Theorems 3.15 and 3.16, one obtains

$$\begin{aligned}
&(u_\delta, R_{z,\pi/2}u_\delta)_{L^2((a,b);dx;\mathscr{H})} \\
&\quad = \int_a^{a+\delta} dx \bigg\{ \bigg(u_\delta(x), \theta_{\pi/2}(z,x,a)\int_a^x dx'\, \phi_{\pi/2}(\overline{z},x',a)^* u_\delta(x')\bigg)_{\mathscr{H}} \\
&\qquad + \bigg(u_\delta(x), \phi_{\pi/2}(z,x,a)\int_x^b dx'\, \theta_{\pi/2}(\overline{z},x',a)^* u_\delta(x')\bigg)_{\mathscr{H}} \\
&\qquad + \bigg(u_\delta(x), [\phi_{\pi/2}(z,x,a) - I_{\mathscr{H}}]m_{\pi/2}(z)\int_a^b dx'\, \phi_{\pi/2}(\overline{z},x',a)^* u_\delta(x')\bigg)_{\mathscr{H}} \\
&\qquad + \bigg(u_\delta(x), m_{\pi/2}(z)\int_a^b dx'\, [\phi_{\pi/2}(\overline{z},x',a)^* - I_{\mathscr{H}}]u_\delta(x')\bigg)_{\mathscr{H}} \bigg\} \\
&\qquad + (u_0, m_{\pi/2}(z)u_0)_{\mathscr{H}}.
\end{aligned} \tag{3.76}$$

Hence,

$$\begin{aligned}
&|(u_0, m_{\pi/2}(z)u_0)_{\mathscr{H}} - (u_\delta, R_{z,\pi/2}u_\delta)_{L^2((a,b);dx;\mathscr{H})}| \\
&\quad \leqslant (\varepsilon(1 + 2\|m_{\pi/2}(z)\|) + \varepsilon^2(1 + \|m_{\pi/2}(z)\|))\|u_0\|^2.
\end{aligned} \tag{3.77}$$

Since δ goes to zero with ε one gets

$$\begin{aligned}
\operatorname{Im}\big((u_0, m_{\pi/2}(z)u_0)_{\mathscr{H}}\big) &= \lim_{\delta\downarrow 0} \operatorname{Im}\big((u_\delta, R_{z,\pi/2}u_\delta)_{L^2((a,b);dx;\mathscr{H})}\big) \\
&= \operatorname{Im}(z) \lim_{\delta\downarrow 0} \int_{\mathbb{R}} \frac{d(u_\delta, E_{H_{\pi/2}}((-\infty,t])u_\delta)_{L^2((a,b);dx;\mathscr{H})}}{|t-z|^2} \geqslant 0,
\end{aligned} \tag{3.78}$$

where $E_{H_{\pi/2}}(\cdot)$ denotes the strongly right-continuous family of spectral projections associated with $H_{\pi/2}$. Since we already showed that $m_{\pi/2}$ is analytic away from the real axis, it follows that it is an operator-valued Herglotz function.

It remains to show that m_β possesses the Herglotz property for general β. Using (3.72) for $\alpha = \pi/2$ and setting $v_0 = (A + Bm_{\pi/2})^{-1}u_0$ for an arbitrary element u_0 of $\mathscr{H}$ one finds

$$\begin{aligned} 2i\,\mathrm{Im}\big((u_0, m_\beta u_0)_{\mathscr{H}}\big) &= (u_0, m_\beta u_0)_{\mathscr{H}} - (m_\beta u_0, u_0)_{\mathscr{H}} \\ &= (v_0, m_{\pi/2} v_0)_{\mathscr{H}} - (m_{\pi/2} v_0, v_0)_{\mathscr{H}} \\ &= 2i\,\mathrm{Im}\big((v_0, m_{\pi/2} v_0)_{\mathscr{H}}\big) \geqslant 0, \end{aligned} \tag{3.79}$$

proving that m_β is Herglotz.

Finally, (3.70) follows by a simple calculation. $\square$

We also mention that $G_\alpha(\cdot,x,x)$ is a bounded Herglotz operator in $\mathscr{H}$ for each $x \in (a,b)$, as is clear from (2.47), (3.66), (3.68), and the Herglotz property of m_α.

REMARK 3.18. The Weyl–Titchmarsh theory established in this section is modeled after right half-lines $(a,b) = (0,\infty)$. Of course precisely the analogous theory applies to left half-lines $(-\infty,0)$. Given the two half-line results, one then establishes the full-line result on $\mathbb{R}$ in the usual fashion with $x = 0$ a reference point and a 2×2 block operator formalism as in the well-known scalar or matrix-valued cases; we omit further details at this point.

A. Basic facts on operator-valued Herglotz functions

In this appendix we review some basic facts on (bounded) operator-valued Herglotz functions, applicable to m_α and $G_\alpha(\cdot,x,x)$, $x \in (a,b)$, discussed in the bulk of this paper.

In the remainder of this appendix, let $\mathscr{H}$ be a separable, complex Hilbert space with inner product denoted by $(\cdot,\cdot)_{\mathscr{H}}$.

DEFINITION A.1. The map $M : \mathbb{C}_+ \to \mathscr{B}(\mathscr{H})$ is called a bounded operator-valued Herglotz function in $\mathscr{H}$ (in short, a bounded Herglotz operator in $\mathscr{H}$) if M is analytic on $\mathbb{C}_+$ and $\mathrm{Im}(M(z)) \geqslant 0$ for all $z \in \mathbb{C}_+$.

Here we follow the standard notation

$$\mathrm{Im}(M) = (M - M^*)/(2i), \quad \mathrm{Re}(M) = (M + M^*)/2, \quad M \in \mathscr{B}(\mathscr{H}). \tag{A.1}$$

Note that M is a bounded Herglotz operator if and only if the scalar-valued functions $(u, Mu)_{\mathscr{H}}$ are Herglotz for all $u \in \mathscr{H}$.

As in the scalar case one usually extends M to $\mathbb{C}_-$ by reflection, that is, by defining

$$M(z) = M(\overline{z})^*, \quad z \in \mathbb{C}_-. \tag{A.2}$$

Hence M is analytic on $\mathbb{C}\backslash\mathbb{R}$, but $M\big|_{\mathbb{C}_-}$ and $M\big|_{\mathbb{C}_+}$, in general, are not analytic continuations of each other.

Of course, one can also consider unbounded operator-valued Herglotz functions, but they will not be used in this paper.

In contrast to the scalar case, one cannot generally expect strict inequality in $\operatorname{Im}(M(\cdot)) \geqslant 0$. However, the kernel of $\operatorname{Im}(M(\cdot))$ has simple properties:

LEMMA A.2. *Let $M(\cdot)$ be a bounded operator-valued Herglotz function in $\mathcal{H}$. Then the kernel $\mathcal{H}_0 = \ker(\operatorname{Im}(M(z)))$ is independent of $z \in \mathbb{C}_+$. Consequently, upon decomposing $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, $\mathcal{H}_1 = \mathcal{H}_0^{\perp}$, $\operatorname{Im}(M(\cdot))$ takes on the form*

$$\operatorname{Im}(M(z)) = \begin{pmatrix} 0 & 0 \\ 0 & N_1(z) \end{pmatrix}, \quad z \in \mathbb{C}_+, \tag{A.3}$$

where $N_1(\cdot) \in \mathcal{B}(\mathcal{H}_1)$ satisfies

$$N_1(z) > 0, \quad z \in \mathbb{C}_+. \tag{A.4}$$

For a proof of Lemma A.2 see, for instance, [38, Proposition 1.2 (ii)] (alternatively, the proof of [45, Lemma 5.3] in the matrix-valued context extends to the present infinite-dimensional situation).

Next we recall the definition of a bounded operator-valued measure (see, also [25, p. 319], [67], [85]):

DEFINITION A.3. Let $\mathcal{H}$ be a separable, complex Hilbert space. A map $\Sigma : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$, with $\mathfrak{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$, is called a *bounded, nonnegative, operator-valued measure* if the following conditions (i) and (ii) hold:

(i) $\Sigma(\emptyset) = 0$ and $0 \leqslant \Sigma(B) \in \mathcal{B}(\mathcal{H})$ for all $B \in \mathfrak{B}(\mathbb{R})$.

(ii) $\Sigma(\cdot)$ is strongly countably additive (i.e., with respect to the strong operator topology in $\mathcal{H}$), that is,

$$\Sigma(B) = \operatorname*{s-lim}_{N\to\infty} \sum_{j=1}^{N} \Sigma(B_j) \tag{A.5}$$

$$\text{whenever } B = \bigcup_{j\in\mathbb{N}} B_j, \text{ with } B_k \cap B_\ell = \emptyset \text{ for } k \neq \ell,\ B_k \in \mathfrak{B}(\mathbb{R}),\ k,\ell \in \mathbb{N}.$$

In addition, $\Sigma(\cdot)$ is called an (*operator-valued*) *spectral measure* (or an *orthogonal operator-valued measure*) if the following condition (iii) holds:

(iii) $\Sigma(\cdot)$ is projection-valued (i.e., $\Sigma(B)^2 = \Sigma(B)$, $B \in \mathfrak{B}(\mathbb{R})$) and $\Sigma(\mathbb{R}) = I_{\mathcal{H}}$.

(iv) Let $f \in \mathcal{H}$ and $B \in \mathfrak{B}(\mathbb{R})$. Then the vector-valued measure $\Sigma(\cdot)f$ has *finite variation on B*, denoted by $V(\Sigma f; B)$, if

$$V(\Sigma f; B) = \sup\left\{ \sum_{j=1}^{N} \|\Sigma(B_j)f\|_{\mathcal{H}} \right\} < \infty, \tag{A.6}$$

where the supremum is taken over all finite sequences $\{B_j\}_{1\leqslant j\leqslant N}$ of pairwise disjoint subsets on $\mathbb{R}$ with $B_j \subseteq B$, $1 \leqslant j \leqslant N$. In particular, $\Sigma(\cdot)f$ has *finite total variation* if $V(\Sigma f; \mathbb{R}) < \infty$.

We recall that due to monotonicity considerations (cf. (A.17)), taking the limit in the strong operator topology in (A.5) is equivalent to taking the limit with respect to the weak operator topology in $\mathcal{H}$.

We also note that integrals of the type (A.7)–(A.10) below are now taken with respect to an operator-valued measure, as opposed to the Bochner integrals we used in the bulk of this paper, Sections 2 and 3.

For relevant material in connection with the following result we refer the reader, for instance, to [1], [9], [10], [22], [25, Sect. VI.5,], [29, Sect. I.4], [30], [31], [34], [36]–[38], [41, Sects. XIII.5–XIII.7], [55], [61], [62], [65]–[67], [80, Ch. VI], [81]–[83], [102], [104], [107, Sects. 8–10].

THEOREM A.4. ([10], [29, Sect. I.4], [102].) Let M be a bounded operator-valued Herglotz function in $\mathcal{H}$. Then the following assertions hold:
(i) For each $f \in \mathcal{H}$, $(f, M(\cdot)f)_{\mathcal{H}}$ is a (scalar) Herglotz function.
(ii) Suppose that $\{e_j\}_{j\in\mathbb{N}}$ is a complete orthonormal system in $\mathcal{H}$ and that for some subset of $\mathbb{R}$ having positive Lebesgue measure, and for all $j \in \mathbb{N}$, $(e_j, M(\cdot)e_j)_{\mathcal{H}}$ has zero normal limits. Then $M \equiv 0$.
(iii) There exists a bounded, nonnegative $\mathcal{B}(\mathcal{H})$-valued measure Ω on $\mathbb{R}$ such that the Nevanlinna representation

$$M(z) = C + Dz + \int_{\mathbb{R}} \frac{d\Omega(\lambda)}{1+\lambda^2}\, \frac{1+\lambda z}{\lambda - z} \tag{A.7}$$

$$= C + Dz + \int_{\mathbb{R}} d\Omega(\lambda) \left[\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right], \quad z \in \mathbb{C}_+, \tag{A.8}$$

$$\widetilde{\Omega}((-\infty,\lambda]) = \operatorname*{s-lim}_{\varepsilon\downarrow 0} \int_{-\infty}^{\lambda+\varepsilon} \frac{d\Omega(t)}{1+t^2}, \quad \lambda \in \mathbb{R}, \tag{A.9}$$

$$\widetilde{\Omega}(\mathbb{R}) = \mathrm{Im}(M(i)) = \int_{\mathbb{R}} \frac{d\Omega(\lambda)}{1+\lambda^2} \in \mathcal{B}(\mathcal{H}), \tag{A.10}$$

$$C = \mathrm{Re}(M(i)), \quad D = \operatorname*{s-lim}_{\eta\uparrow\infty} \frac{1}{i\eta} M(i\eta) \geqslant 0, \tag{A.11}$$

holds in the strong sense in $\mathcal{H}$. Here $\widetilde{\Omega}(B) = \int_B \left(1+\lambda^2\right)^{-1} d\Omega(\lambda)$, $B \in \mathfrak{B}(\mathbb{R})$.
(iv) Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$. Then the Stieltjes inversion formula for Ω reads

$$\Omega((\lambda_1,\lambda_2])f = \pi^{-1} \operatorname*{s-lim}_{\delta\downarrow 0} \operatorname*{s-lim}_{\varepsilon\downarrow 0} \int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\, \mathrm{Im}(M(\lambda + i\varepsilon))f, \quad f \in \mathcal{H}. \tag{A.12}$$

(v) Any isolated poles of M are simple and located on the real axis, the residues at poles being nonpositive bounded operators in $\mathcal{B}(\mathcal{H})$.
(vi) For all $\lambda \in \mathbb{R}$,

$$\operatorname*{s-lim}_{\varepsilon\downarrow 0} \varepsilon\, \mathrm{Re}(M(\lambda + i\varepsilon)) = 0, \tag{A.13}$$

$$\Omega(\{\lambda\}) = \operatorname*{s-lim}_{\varepsilon\downarrow 0} \varepsilon\, \mathrm{Im}(M(\lambda + i\varepsilon)) = -i \operatorname*{s-lim}_{\varepsilon\downarrow 0} \varepsilon M(\lambda + i\varepsilon). \tag{A.14}$$

(vii) If in addition $M(z) \in \mathscr{B}_\infty(\mathscr{H})$, $z \in \mathbb{C}_+$, then the measure Ω in (A.7) is countably additive with respect to the $\mathscr{B}(\mathscr{H})$-norm, and the Nevanlinna representation (A.7), (A.8) and the Stieltjes inversion formula (A.12) as well as (A.13), (A.14) hold with the limits taken with respect to the $\|\cdot\|_{\mathscr{B}(\mathscr{H})}$-norm.
$(viii)$ Let $f \in \mathscr{H}$ and assume in addition that $\Omega(\cdot)f$ is of finite total variation. Then for a.e. $\lambda \in \mathbb{R}$, the normal limits $M(\lambda + i0)f$ exist in the strong sense and

$$\operatorname*{s-lim}_{\varepsilon \downarrow 0} M(\lambda + i\varepsilon)f = M(\lambda + i0)f = H(\Omega(\cdot)f)(\lambda) + i\pi\Omega'(\lambda)f, \tag{A.15}$$

where $H(\Omega(\cdot)f)$ denotes the $\mathscr{H}$-valued Hilbert transform

$$H(\Omega(\cdot)f)(\lambda) = \text{p.v.}\int_{-\infty}^{\infty} d\Omega(t)f\,\frac{1}{t-\lambda} = \operatorname*{s-lim}_{\delta \downarrow 0}\int_{|t-\lambda|\geqslant\delta} d\Omega(t)f\,\frac{1}{t-\lambda}. \tag{A.16}$$

Sketch of proof. Item (i) is clear and it implies items (ii) together with the fact that $\sum_{j\in\mathbb{N}} 2^{-j}(e_j, \Omega(\cdot)e_j)_{\mathscr{H}}$ represents a (scalar) control measure for $\Omega(\cdot)$.

That equations (A.7)–(A.11) hold in the strong sense in $\mathscr{H}$ and the validity of the Stieltjes inversion formula (A.12) were proved by Allen and Narcowich [10]. Their proofs rely on the polarization identity and the one-to-one correspondence between bounded, symmetric sesquilinear forms on $\mathscr{H}$ and the set of bounded self-adjoint operators on $\mathscr{H}$. We also note that the proof of Theorem A.4 in the case where strong convergence is replaced by weak convergence readily follows from the corresponding scalar version (see also the matrix-valued case studied, e.g., in [45, Theorems 5.4 and 5.5]). The various extensions from weak convergence to strong convergence in Theorem A.4 then repeatedly use a standard result on monotonic sequences of bounded, nonnegative operators in $\mathscr{H}$ (called Vigier's theorem in [86, p. 263]):

$$\begin{gathered}\text{If } 0 \leqslant B_1 \leqslant B_2 \leqslant \cdots \leqslant B_\infty, \text{ with } B_n, B_\infty \in \mathscr{B}(\mathscr{H}),\ n \in \mathbb{N},\\ \text{then } \operatorname*{s-lim}_{n\to\infty} B_n = B \text{ for some } B \in \mathscr{B}(\mathscr{H}).\end{gathered} \tag{A.17}$$

Similarly, recalling the extension of this convergence result to compact operators (cf. [10, Lemma 2.1]):

$$\begin{gathered}\text{If } 0 \leqslant C_1 \leqslant C_2 \leqslant \cdots \leqslant C_\infty, \text{ with } C_n, C_\infty \in \mathscr{B}_\infty(\mathscr{H}),\ n \in \mathbb{N},\\ \text{then } \lim_{n\to\infty} \|C_n - C\|_{\mathscr{B}(\mathscr{H})} = 0 \text{ for some } C \in \mathscr{B}_\infty(\mathscr{H}),\end{gathered} \tag{A.18}$$

repeated applications of this fact yield the extensions to $\mathscr{B}(\mathscr{H})$-norm convergence in item (vii). Of course, the monotonically increasing and uniformly bounded families $\{B_n\}_{n\in\mathbb{N}}$ and $\{C_n\}_{n\in\mathbb{N}}$ in (A.17) and (A.18) can be replaced by monotonically decreasing families of uniformly bounded operators in $\mathscr{H}$. (For variations of (A.17) and (A.18) we also refer to [59, Theorems VIII.3.3 and VIII.3.5, Remark VIII.3.4].)

In the special case of scalar Herglotz functions m (cf. [17] and [57] for detailed treatments), isolated zeros of m are well-known to be necessarily simple and located on $\mathbb{R}$. This can be inferred from the fact that $-1/m$ is a Herglotz function whenever m is one, and hence isolated poles of $1/m$ are also necessarily simple with a negative residue. Studying $(f, M(z)f)_{\mathscr{H}}$ for all $f \in \mathscr{H}$ then illustrates item (v).

That item (vi) holds, in fact, with s-lim$_{\varepsilon\downarrow 0}$ rather than w-lim$_{\varepsilon\downarrow 0}$ follows again from monotonicity considerations: First, (choosing $D=0$ in (A.8) without loss of generality) one notes that the expression on the left-hand side in (A.19) below

$$\varepsilon\,\mathrm{Im}\left(\frac{1}{t-(\lambda+i\varepsilon)}\right)=\frac{\varepsilon^2}{(t-\lambda)^2+\varepsilon^2}\in[0,1],\quad (t,\lambda)\in\mathbb{R}^2,\ \varepsilon>0, \tag{A.19}$$

is nonnegative, uniformly bounded by 1, and monotonically decreasing with respect to ε as $\varepsilon\downarrow 0$. Moreover,

$$\lim_{\varepsilon\downarrow 0}\varepsilon\,\mathrm{Im}\left(\frac{1}{t-(\lambda+i\varepsilon)}\right)=\begin{cases}0, & t\in\mathbb{R}\backslash\{\lambda\},\\ 1, & t=\lambda.\end{cases} \tag{A.20}$$

Combining this with the analog of the monotonicity result (A.17) in the decreasing case proves the first equality in (A.14). In the remainder of the proof of item (vi) we make the simplifying assumption that M is of the form $M(z)=\int_{\mathbb{R}}d\Omega(\lambda)(\lambda-z)^{-1}$, $z\in\mathbb{C}_+$, which is permitted, without loss of generality, as only local considerations are at stake. Since

$$\varepsilon\,\mathrm{Re}\left(\frac{1}{t-(\lambda+i\varepsilon)}\right)=\frac{\varepsilon(t-\lambda)}{(t-\lambda)^2+\varepsilon^2}\in[-1/2,1/2],\quad (t,\lambda)\in\mathbb{R}^2,\ \varepsilon>0, \tag{A.21}$$

is not monotonic with respect to ε as $\varepsilon\downarrow 0$, we decompose it into three monotonic pieces as follows,

$$\varepsilon\,\mathrm{Re}\left(\frac{1}{t-(\lambda+i\varepsilon)}\right)=\psi_1(t-\lambda,\varepsilon)+\psi_2(t-\lambda,\varepsilon)-2^{-1}, \tag{A.22}$$

where

$$\psi_1(x,\varepsilon)=\begin{cases}\varepsilon x\big[x^2+\varepsilon^2\big]^{-1}, & |x|\geqslant\varepsilon,\\ 1/2, & |x|\leqslant\varepsilon,\end{cases}\qquad \psi_2(x,\varepsilon)=\begin{cases}\varepsilon x\big[x^2+\varepsilon^2\big]^{-1}, & |x|\leqslant\varepsilon,\\ 1/2, & |x|\geqslant\varepsilon.\end{cases} \tag{A.23}$$

By monotonicity of each of the three terms with respect to ε, one obtains that

$$\begin{aligned}\operatorname*{s-lim}_{\varepsilon\downarrow 0}\varepsilon\,\mathrm{Re}(M(\lambda+i\varepsilon)&=\operatorname*{s-lim}_{\varepsilon\downarrow 0}\int_{\mathbb{R}}d\Omega(t)\frac{\varepsilon(t-\lambda)}{(t-\lambda)^2+\varepsilon^2}\\ &=\operatorname*{s-lim}_{\varepsilon\downarrow 0}\int_{\mathbb{R}}d\Omega(t)\big[\psi_1(t-\lambda,\varepsilon)+\psi_2(t-\lambda,\varepsilon)-2^{-1}\big]=0,\end{aligned} \tag{A.24}$$

because the corresponding weak limits equal zero by the following well-known arguments: Let $f\in\mathcal{H}$, then

$$\begin{aligned}&\left|\varepsilon\int_{|t-\lambda|\geqslant 1}d(f,\Omega(t)f)_{\mathcal{H}}\frac{(t-\lambda)}{(t-\lambda)^2+\varepsilon^2}\right|\\ &\quad\leqslant\varepsilon\int_{|t-\lambda|\geqslant 1}d(f,\Omega(t)f)_{\mathcal{H}}|t-\lambda|^{-1}\underset{\varepsilon\downarrow 0}{\longrightarrow}0.\end{aligned} \tag{A.25}$$

By polarization, also

$$\lim_{\varepsilon\downarrow 0}\left|\varepsilon\int_{|t-\lambda|\geqslant 1} d(f,\Omega(t)g)_{\mathcal{H}}\frac{(t-\lambda)}{(t-\lambda)^2+\varepsilon^2}\right| = 0, \quad f,g\in\mathcal{H}. \tag{A.26}$$

Next, for $f\in\mathcal{H}$,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left|\int_{|t-\lambda|\leqslant 1} d(f,\Omega(t)f)_{\mathcal{H}}\frac{\varepsilon(t-\lambda)}{(t-\lambda)^2+\varepsilon^2}\right| \\
&\quad\leqslant \lim_{\varepsilon\downarrow 0}\int_{|t-\lambda|\leqslant 1} d(f,\Omega(t)f)_{\mathcal{H}}\frac{\varepsilon|t-\lambda|}{(t-\lambda)^2+\varepsilon^2} = 0,
\end{aligned} \tag{A.27}$$

applying the dominated convergence theorem, as

$$\frac{\varepsilon|t-\lambda|}{(t-\lambda)^2+\varepsilon^2}\leqslant\frac{1}{2}, \quad t\in\mathbb{R},\ \varepsilon>0. \tag{A.28}$$

Again by polarization,

$$\lim_{\varepsilon\downarrow 0}\left|\varepsilon\int_{|t-\lambda|\leqslant 1} d(f,\Omega(t)g)_{\mathcal{H}}\frac{(t-\lambda)}{(t-\lambda)^2+\varepsilon^2}\right| = 0, \quad f,g\in\mathcal{H}, \tag{A.29}$$

completing the proof of

$$\operatorname*{w-lim}_{\varepsilon\downarrow 0}\varepsilon\operatorname{Re}(M(\lambda+i\varepsilon)) = 0. \tag{A.30}$$

Thus, (A.24) together with the first equality in (A.14), then also prove the second equality in (A.14) and hence completes the proof of item (vi).

Item $(viii)$ is a consequence of [21, Subsections 1.2.4 and 1.2.5] (which in turn are based on [18]). $\square$

As usual, the normal limits in Theorem A.4 can be replaced by nontangential ones.

The nature of the boundary values of $M(\cdot+i0)$ when for some $p>0$, $M(z)\in\mathcal{B}_p(\mathcal{H})$, $z\in\mathbb{C}_+$, was clarified in detail in [26], [77], [78], [79].

Using an approach based on operator-valued Stieltjes integrals, a special case of Theorem A.4 was proved by Brodskii [29, Sect. I.4]. In particular, he proved the analog of the Herglotz representation for operator-valued Caratheodory functions. More precisely, if F is analytic on $\mathbb{D}$ (the open unit disk in $\mathbb{C}$) with nonnegative real part $\operatorname{Re}(F(w))\geqslant 0$, $w\in\mathbb{D}$, then F is of the form

$$\begin{aligned}
&F(w) = i\operatorname{Im}(F(0)) + \oint_{\partial\mathbb{D}} d\Upsilon(\zeta)\,\frac{\zeta+w}{\zeta-w}, \quad w\in\mathbb{D}, \\
&\operatorname{Re}(F(0)) = \Upsilon(\partial\mathbb{D}),
\end{aligned} \tag{A.31}$$

with Υ a bounded, nonnegative $\mathcal{B}(\mathcal{H})$-valued measure on $\partial\mathbb{D}$. The result (A.31) can also be derived by an application of Naimark's dilation theory (cf. [10] and [43, p. 68]), and it can also be used to derive the Nevanlinna representation (A.7), (A.8) (cf. [10], and in a special case also [29, Sect. I.4]). Finally, we also mention that Shmuly'an [102] discusses the Nevanlinna representation (A.7), (A.8); moreover, certain special classes of Nevanlinna functions, isolated by Kac and Krein [57] in the scalar context, are studied by Brodskii [29, Sect. I.4] and Shmuly'an [102].

For a variety of applications of operator-valued Herglotz functions, see, for instance, [1], [4], [16], [28], [31], [36]–[38], [44], [66]–[69], [102], and the literature cited therein.

Acknowledgments

We are indebted to Mark Malamud for a critical reading of our manuscript and for numerous helpful suggestions. We are also grateful to Joe Diestel for his help in clarifying the role of the Radon-Nikodym property in connection with vector-valued absolutely contionuous functions.

REFERENCES

[1] V. M. ADAMJAN AND H. LANGER, *Spectral properties of a class of rational operator valued functions*, J. Operator Th. **33** (1995), 259–277.

[2] S. AGMON, J. CRUZ-SAMPEDRO, AND I. HERBST, *Generalized Fourier transform for Schrödinger operators with potentials of order zero*, J. Funct. Anal. **167** (1999), 345–369.

[3] Z. S. AGRANOVICH AND V. A. MARCHENKO, *The Inverse Problem of Scattering Theory*, Gordon and Breach, New York, 1963.

[4] S. ALBEVERIO, J. F. BRASCHE, M. M. MALAMUD, AND H. NEIDHARDT, *Inverse spectral theory for symmetric operators with several gaps: scalar-type Weyl functions*, J. Funct. Anal. **228** (2005), 144–188.

[5] B. A. ALIEV, *Asymptotic behavior of the eigenvalues of a boundary-value problem for a second-order elliptic operator-differential equation*, Ukrain. Math. J. **58** (2006), 1298–1306.

[6] A. R. ALIEV, *On the generalized solution of the boundary-value problem for the operator-differential equations of the second order with variable coefficients*, J. Math. Phys. Anal. Geom. **2** (2006), 87–93.

[7] B. A. ALIEV, *Solvability of the boundary-value problem for the second-order elliptic differential-operator equation with spectral parameter in the equation and boundary conditions*, Ukrain. Math. J. **62** (2010), 1–14.

[8] A. R. ALIEV AND S. S. MIRZOEV, *On boundary value problem solvability theory for a class of high-order operator-differential equations*, Funct. Anal. Appl. **44** (2010), 209–211.

[9] G. D. ALLEN AND F. J. NARCOWICH, *On the representation and approximation of a class of operator-valued analytic functions*, Bull. Amer. Math. Soc. **81** (1975), 410–412.

[10] G. D. ALLEN AND F. J. NARCOWICH, *R-operators I. Representation Theory and Applications*, Indiana Univ. Math. J. **25** (1976), 945–963.

[11] M. S. ALMAMEDOV, M. BAĬRAMOGLY, AND V. I. KATANOVA, *Trace formulas for a differential equation of even order with unbounded operator coefficient*, Sov. Math. Dokl. **43** (1991), 406–410.

[12] M. S. ALMAMEDOV, M. BAĬRAMOGLY, AND V. I. KATANOVA, *On regularized traces of even-order differential equations with unbounded operator coefficients*, Diff. Eq. **29** (1993), 1–9.

[13] M. S. ALMAMEDOV AND V. I. KATANOVA, *A regularized trace of a high-order differential operator with a bounded operational coefficient*, Diff. Eqs. **28** (1992), 1–15.

[14] H. AMANN, *Linear and Quasilinear Parabolic Problems*, Monographs in Mathematics, Vol. 89, Birkhäuser, Basel, 1995.

[15] W. ARENDT, C. K. BATTY, M. HIEBER, F. NEUBRANDER, *Vector-Valued Laplace Transforms and Cauchy Transforms*, Monographs in Mathematics, Vol. 96, Birkhäuser, Basel, 2001.

[16] YU. ARLINSKII, S. BELYI, AND E. TSEKANOVSKII, *Conservative Realizations of Herglotz–Nevanlinna Functions*, Operator Theory advances and Applications, Vol. 217, Birkhäuser, Springer, Basel, 2011.

[17] N. ARONSZAJN AND W. F. DONOGHUE, *On exponential representations of analytic functions in the upper half-plane with positive imaginary part*, J. Analyse Math. **5** (1956–57), 321–388.

[18] K. ASANO, *Notes on Hilbert transforms of vector valued functions in the complex plane and their boundary values*, Proc. Japan Acad. **43** (1967), 572–577.

[19] N. M. ASLANOVA, *A trace formula of a boundary value problem for the operator Sturm–Liouville equation*, Sib. Math. J. **49** (2008), 959–967.

[20] M. BAIRAMOGLY AND N. M. ASLANOVA, *Distribution of eigenvalues and trace formula for the Sturm–Liouville operator equation*, Ukrain. Math. J. **62** (2010), 1005–1017.

[21] H. BAUMGÄRTEL AND M. WOLLENBERG, *Mathematical Scattering Theory*, Operator Theory: Advances and Applications, Vol. 9, Birkhäuser, Boston, 1983.

[22] S. V. BELYI AND E. R. TSEKANOVSKII, *Classes of operator R-functions and their realization by conservative systems*, Sov. Math. Dokl. **44** (1992), 692–696.

[23] R. BENGURIA AND M. LOSS, *A simple proof of a theorem of Laptev and Weidl*, Math. Res. Lett. **7** (2000), 195–203.

[24] C. BENNEWITZ, *Spectral theory in Hilbert space*, Lecture Notes, 2008.

[25] JU. BEREZANSKIĬ, *Expansions in Eigenfunctions of Selfadjoint Operators*, Transl. Math. Mongraphs, Vol. 17, Amer. Math. Soc., Providence, R.I., 1968.

[26] M. Š. BIRMAN, S. B. ÈNTINA, *The stationary method in the abstract theory of scattering theory*, Math. SSSR Izv. **1** (1967), 391–420.

[27] M. S. BIRMAN AND M. Z. SOLOMJAK, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Reidel, Dordrecht, 1987.

[28] J. F. BRASCHE, M. MALAMUD, AND H. NEIDHARDT, *Weyl function and spectral properties of self-adjoint extensions*, Integr. Eq. Oper. Th. **43** (2002), 264–289.

[29] M. S. BRODSKII, *Triangular and Jordan Representations of Linear Operators*, Transl. Math. Mongraphs, Vol. 32, Amer. Math. Soc., Providence, RI, 1971.

[30] D. BUSCHMANN, *Spektraltheorie verallgemeinerter Differentialausdrücke - Ein neuer Zugang*, Ph.D. Thesis, University of Frankfurt, Germany, 1997.

[31] R. W. CAREY, *A unitary invariant for pairs of self-adjoint operators*, J. Reine Angew. Math. **283** (1976), 294–312.

[32] J. CRUZ-SAMPEDRO, *Exact asymptotic behavior at infinity of solutions to abstract second-order differential inequalities in Hilbert spaces*, Math. Z. **237** (2001), 727–735.

[33] Ju. L. Daleckiĭ AND M. G. KREĬN, *Stability of Solutions of Differential Equations in Banach Space*, Transl. Math. Monographs, Vol. 43, Amer. Math. Soc., Providence, RI, 1974.

[34] L. DE BRANGES, *Perturbations of self-adjoint transformations*, Amer. J. Math. **84** (1962), 543–560.

[35] S. A. DENISOV, *Schrödinger operators and associated hyperbolic pencils*, J. Funct. Anal. **254** (2008), 2186–2226.

[36] V. A. DERKACH AND M. M. MALAMUD, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), 1–95.

[37] V. A. DERKACH AND M. M. MALAMUD, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. **73** (1995), 141–242.

[38] V. A. DERKACH AND M. M. MALAMUD, *On some classes of holomorphic operator functions with nonnegative imaginary part*, in *Operator Algebras and Related Topics*, 16th International Conference on Operator Theory, A. Gheondea, R. N. Gologan, and T. Timotin (eds.), The Theta Foundation, Bucharest, 1997, pp. 113–147.

[39] J. DIESTEL AND J. J. UHL, *Vector Measures*, Mathematical Surveys, Vol. 15, Amer. Math. Soc., Providence, RI, 1977.

[40] J. DIEUDONNÉ, *Foundations of Modern Analysis*, Pure and Appl. Math., Vol. 10, Academic Press, New York, 1960.

[41] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators Part II: Spectral Theory*, Interscience, New York, 1988.

[42] D. E. EDMUNDS AND W. D. EVANS, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989.

[43] P. FILLMORE, *Notes on Operator Theory*, Math. Studies, Vol. 30, Van Nostrand–Reinhold, New York, 1970.

[44] F. GESZTESY, N.J. KALTON, K. A. MAKAROV, AND E. TSEKANOVSKII, *Some applications of operator-valued Herglotz functions*, in "Operator Theory, System Theory and Related Topics," Oper. Theory Adv. Appl., Vol. 123, Birkhäuser, Basel, 2001, pp. 271–321.

[45] F. GESZTESY AND E. TSEKANOVSKII, *On matrix-valued Herglotz functions*, Math. Nachr. **218** (2000), 61–138.

[46] M. L. GORBAČUK, *On spectral functions of a second order differential operator with operator coefficients*, Ukrain. Math. J. **18**, No. 2, 3–21 (1966). (Russian.) Engl. transl. in Amer. Math. Soc. Transl. (2), **72** (1968), 177–202.

[47] V. I. GORBAČUK AND M. L. GORBAČUK, *Expansion in eigenfunctions of a second-order differential equation with operator coefficients*, Sov. Math. Dokl. **10** (1969), 158–162.

[48] M. L. GORBACHUK, *Self-adjoint boundary problems for a second-order differential equation with unbounded operator coefficient*, Funct. Anal. Appl. **5** (1971), 9–18.

[49] V. I. GORBACHUK AND M. L. GORBACHUK, *The spectrum of self-adjoint extensions of the minimal operator generated by a Sturm-Liouville equation with operator potential*, Ukrain. Math. J. **24** (1973), 582–588.

[50] V. I. GORBACHUK AND M. L. GORBACHUK, *Boundary Value Problems for Operator Differential Equations*, Kluwer, Dordrecht, 1991.

[51] M. L. GORBACHUK AND V. A. MIHAĬLEC, *Semibounded selfadjoint extensions of symmetric operators*, Sov. Math. Dokl. **17** (1976), 185–187.

[52] E. GÜL, *On the regularized trace of a second order differential operator*, Appl. Math. Comp. **198** (2008), 471–480.

[53] E. HILLE, *Lectures on Ordinary Differential Equations*, Addison–Wesley, Reading, MA, 1969.

[54] E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semi-Groups*, Colloquium Publications, Vol. 31, rev. ed., Amer. Math. Soc., Providence, RI, 1985.

[55] D. HINTON AND A. SCHNEIDER, *On the spectral representation for singular selfadjoint boundary eigenvalue problems*, in *Contributions to Operator Theory in Spaces with an Indefinite Metric*, A. Dijksma, I. Gohberg, M. A. Kaashoek, R. Mennicken (eds.), Operator Theory: Advances and Applications, Vol. **106** (1998), pp. 217–251.

[56] W. JÄGER, *Ein gewöhnlicher Differentialoperator zweiter Ordnung für Funktionen mit Werten in einem Hilbertraum*, Math. Z. **113** (1970), 68–98.

[57] I. S. KAC AND M. G. KREIN, *R-functions–analytic functions mapping the upper halfplane into itself*, Amer. Math. Soc. Transl. (2) **103** (1974), 1–18.

[58] T. KATO, *Growth properties of solutions of the reduced wave equation with a variable coefficient*, Commun. Pure Appl. Math. **12** (1959), 403–425.

[59] T. KATO, *Perturbation Theory for Linear Operators*, corr. printing of the 2nd ed., Springer, Berlin, 1980.

[60] A. G. KOSTYUCHENKO AND B. M. LEVITAN, *Asymptotic behavior of the eigenvalues of the Sturm–Liouville operator problem*, Funct. Anal. Appl. **1** (1967), 75–83.

[61] M. G. KREIN AND I. E. OVČARENKO, *Q-functions and sc-resolvents of nondensely defined Hermitian contractions*, Sib. Math. J. **18** (1977), 728–746.

[62] M. G. KREIN AND I. E. OVČARENKO, *Inverse problems for Q-functions and resolvent matrices of positive Hermitian operators*, Sov. Math. Dokl. **19** (1978), 1131–1134.

[63] A. LAPTEV AND T. WEIDL, *Sharp Lieb–Thirring inequalities in high dimensions*, Acta Math. **184** (2000), 87–111.

[64] A. LAPTEV, S. NABOKO, AND O. SAFRONOV, *Absolutely continuous spectrum of Schrödinger operators with slowly decaying and oscillating potentials*, Commun. Math. Phys. **253** (2005), 611–631.

[65] J. S. MAC NERNEY, *Hermitian moment sequences*, Trans. Amer. Math. Soc. **103** (1962), 45–81.

[66] M. M. MALAMUD AND S. M. MALAMUD, *On the spectral theory of operator measures*, Funct. Anal. Appl. **36** (2002), 154–158.

[67] M. M. MALAMUD AND S. M. MALAMUD, *On the spectral theory of operator measures in Hilbert space*, St. Petersburg Math. J. **15** (2004), 323–373.

[68] M. MALAMUD AND H. NEIDHARDT, *On the unitary equivalence of absolutely continuous parts of self-adjoint extensions*, J. Funct. Anal. **260** (2011), 613–638.

[69] M. MALAMUD AND H. NEIDHARDT, *Sturm–Liouville boundary value problems with operator potentials and unitary equivalence*, J. Diff. Eq. **252** (2012), 5875–5922.

[70] J. MIKUSIŃSKI, *The Bochner Integral*, Academic Press, New York, 1978.

[71] P. A. MISHNAEVSKII, *On the spectral theory for the Sturm–Liouville equation with operator coefficient*, Math. USSR Izv. **10** (1976), 145–180.

[72] P. A. MISHNAEVSKII, *An investigation of a second-order elliptic operator in regions with infinite boundaries, by means of an operational Sturm–Liouville equation. I*, Diff. Eq. **19** (1983), 334–344.

[73] P. A. MISHNAEVSKII, *An investigation of a second-order elliptic operator in regions with infinite boundaries, by means of an operational Sturm–Liouville equation. II*, Diff. Eq. **19** (1983), 475–483.

[74] V. I. MOGILEVSKII, *Description of spectral functions of differential operators with arbitrary deficiency indices*, Math. Notes **81** (2007), 553–559.

[75] V. MOGILEVSKII, *Boundary triplets and Titchmarsh–Weyl functions of differential operators with arbitrary deficiency indices*, Meth. Funct. Anal. Topology **15** (2009), 280–300.

[76] V. MOGILEVSKII, *Minimal spectral functions of an ordinary differential operator*, arXiv:1010.1117.

[77] S. N. NABOKO, *Boundary values of analytic operator functions with a positive imaginary part*, J. Soviet Math. **44** (1989), 786–795.

[78] S. N. NABOKO, *Nontangential boundary values of operator-valued R-functions in a half-plane*, Leningrad Math. J. **1** (1990), 1255–1278.

[79] S. N. NABOKO, *The boundary behavior of $\mathfrak{S}_p$-valued functions analytic in the half-plane with nonnegative imaginary part*, Functional Analysis and Operator Theory, Banach Center Publications, Vol. **30**, Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1994, pp. 277–285.

[80] M. A. NAIMARK, *Linear Differential Operators, Part II*, Ungar, New York, 1968.

[81] F. J. NARCOWICH, *Mathematical theory of the R matrix. II. The R matrix and its properties*, J. Math. Phys. **15** (1974), 1635–1642.

[82] F. J. NARCOWICH, *R-operators II. On the approximation of certain operator-valued analytic functions and the Hermitian moment problem*, Indiana Univ. Math. J. **26** (1977), 483–513.

[83] F. J. NARCOWICH AND G. D. ALLEN, *Convergence of the diagonal operator-valued Padé approximants to the Dyson expansion*, Commun. Math. Phys. **45** (1975), 153–157.

[84] B. J. PETTIS, *On integration in vector spaces*, Trans. Am. Math. Soc. **44** (1938), 277–304.

[85] A. I. PLESNER AND V. A. ROHLIN, *Spectral theory of linear operators*, Uspehi Matem. Nauk (N. S.) **1(11)**, No. 1, 71–191 (1946). (Russian.) Engl. transl. in Amer. Math. Soc. Transl. (2), **62** (1967), 29–175.

[86] F. RIESZ AND B. SZ.-NAGY, *Functional Analysis*, Dover, New York, 1990.

[87] F. S. ROFE-BEKETOV, *Expansions in eigenfunctions of infinite systems of differential equations in the non-self-adjoint and self-adjoint cases*, Mat. Sb. **51** (1960), 293–342. (Russian.)

[88] F. S. ROFE-BEKETOV, *Selfadjoint extensions of differential operators in a space of vector functions*, Sov. Math. Dokl. **10** (1969), 188–192.

[89] F. S. ROFE-BEKETOV, *Selfadjoint extensions of differential operators in a space of vector-valued functions*, Teor. Funkciĭ Funkcional. Anal. i Priložen. Vyp. **8** (1969), 3–24. (Russian.)

[90] F. S. ROFE-BEKETOV AND A. M. KHOLKIN, *A fundamental system of solutions of for an operator differential equation with a boundary condition at infinity*, Math. Notes **36** (1985), 846–853.

[91] F. S. ROFE-BEKETOV AND A. M. KHOLKIN, *Spectral Analysis of Differential Operators. Interplay Between Spectral and Oscillatory Properties*, Monograph Series in Mathematics, Vol. 7, World Scientific, Singapore, 2005.

[92] O. SAFRONOV, *Absolutely continuous spectrum of one random elliptic operator*, J. Funct. Anal. **255** (2008), 755–767.

[93] O. SAFRONOV AND G. STOLZ, *Absolutely continuous spectrum of Schrödinger operators with potentials slowly decaying inside a cone*, J. Math. Anal. Appl. **326** (2007), 192–208.

[94] Y. SAITŌ, *Eigenfunction expansions associated with second-order differential equations for Hilbert space-valued functions*, Publ. RIMS, Kyoto Univ. **7** (1971/72), 1–55.

[95] Y. SAITŌ, *The principle of limiting absorption for second-order differential equations with operator-valued coefficients*, Publ. RIMS, Kyoto Univ. **7** (1971/72), 581–619.

[96] Y. SAITŌ, *Spectral and scattering theory for second-order differential operators with operator-valued coefficients*, Osaka J. Math. **9** (1972), 463–498.

[97] Y. SAITŌ, *Spectral theory for second-order differential operators with long-range operator-valued coefficients II. Eigenfunction expansions and the Schrödinger operators with long-range potentials*, Japan. J. Math. **1** (1975), 311–349.
[98] Y. SAITŌ, *Spectral theory forsecond-order differential operators with long-range operator-valued coefficients I. Limiting absorption principle*, Japan. J. Math. **1** (1975), 351–382.
[99] Y. SAITŌ, *On the asymptotic behavior of solutions of the Schrödinger equation* $(-\Delta+Q(y)-k^2)V=F$, Osaka J. Math. **14** (1977), 11–35.
[100] Y. SAITŌ, *Eigenfunction expansions for the Schrödinger operators with long-range ptentials* $(Q(y)=O(|y|^{-\varepsilon})$, $\varepsilon>0$, Osaka J. Math. **14** (1977), 37–53.
[101] Y. SAITŌ, *Spectral Representations for Schrödinger Operators with Long-Range Potentials*, Lecture Notes in Mathematics, Vol. 727, Springer, Berlin, 1979.
[102] YU. L. SHMUL'YAN, *On operator R-functions*, Siberian Math. J. **12** (1971), 315–322.
[103] I. TROOSHIN, *Asymptotics for the spectral and Weyl functions of the operator-valued Sturm–Liouville problem*, in *Inverse Problems and Related Topics*, G. Nakamura, S. Saitoh, J. K. Seo, and M. Yamamoto (eds.), Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 189–208.
[104] E. R. TSEKANOVSKII, *Accretive extensions and problems on the Stieltjes operator-valued functions realizations*, in *Operator Theory and Complex Analysis*, T. Ando and I. Gohberg (eds.), Operator Theory: Advances and Applications, Vol. 59, Birkhäuser, Basel, 1992, pp. 328–347.
[105] L. I. VAINERMAN AND M. L. GORBACHUK, *On self-adjoint semibounded abstract differential operators*, Ukrain. Math. J. **22** (1970), 694–696.
[106] J. WEIDMANN, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Vol. 68, Springer, New York, 1980.
[107] J. WEIDMANN, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Math. **1258**, Springer, Berlin, 1987.
[108] H. WEYL, *"Uber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann. **68** (1910), 220–269.
[109] K. YOSIDA, *Functional Analysis*, 6th ed., Springer, Berlin, 1980.

(Received September 8, 2011)

Fritz Gesztesy
Department of Mathematics
University of Missouri
Columbia, MO 65211, USA
e-mail: gesztesyf@missouri.edu
http://www.math.missouri.edu/\unskip\penalty-\@Mpersonnel/facul

Rudi Weikard
Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294, USA
e-mail: rudi@math.uab.edu
http://www.math.uab.edu/~{}rudi

Maxim Zinchenko
Department of Mathematics
University of Central Florida
Orlando, FL 32816, USA
e-mail: maxim@math.ucf.edu
http://www.math.ucf.edu/~{}maxim