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# ON THE INVERSE RESONANCE PROBLEM

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Abstract

A new technique is presented which gives conditions under which perturbations of certain base potentials are uniquely determined from the location of eigenvalues and resonances in the context of a Schrödinger operator on a half line. The method extends to complex-valued potentials and certain potentials whose first moment is not integrable.

## 1. *Introduction*

Marchenko [**14**] showed in 1955 that a real-valued potential $q$ on $[0,\infty)$ for which $(1+x)q(x)$ is integrable is uniquely determined from the scattering phase, the eigenvalues and their norming constants. The scattering phase is given in terms of the Jost function of the problem for real arguments. (The solution $\psi(z,\cdot)$ of $-y''+qy=z^2y$ which asymptotically equals $\exp(izx)$ is called the Jost solution of the problem; the function $\psi(\cdot,0)$ is then called the Jost function.) The eigenvalues are the squares of the zeros of the Jost function in the upper half plane and the norming constants can also be expressed in terms of the Jost function at least when it can be analytically extended to the entire complex $z$-plane. This is certainly the case when the potential has compact support. The Jost function is then an entire function of growth order one and thus the location of its zeros determines it uniquely up to a factor $\exp(az+b)$. This factor may also be determined since it is known that $\psi(z,0)$ tends to one as $z$ tends to infinity on any ray which emanates from zero and lies in the upper half plane. From a physical point of view these zeros represent (Dirichlet) eigenvalues or resonances (depending on whether they are in the upper or lower half plane). In short, one may say therefore that the location of eigenvalues and resonances determines a compactly supported real-valued potential. (The function $\psi(\cdot,0)$ cannot have real zeros except for zero. Whether or not this is the case is, of course, also required information.)

In this paper we are setting out to prove analogous statements in more general contexts. In particular, we want to treat complex-valued potentials as well as potentials with slower or perhaps no decay at infinity but where the concept of a resonance still makes sense. The starting point in this endeavour is the Weyl–Titchmarsh $m$-function which uniquely determines a potential $q$ even if $q$ is just locally integrable and which also holds for complex-valued potentials as was shown recently in [**7**]. Our interest in relating the $m$-function to eigenvalues and resonances stems from the fact that the former can not be obtained directly from laboratory measurements while the latter are fundamental objects in quantum physics with a

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long history, dating back to the early days of the theory when Weisskopf and Wigner [**19**] and others [**5**, **12**, **16**] studied the behavior of unstable particles. Physically, while eigenvalues represent real energy levels and states in which the particles are permanently localized, unless disturbed, resonances correspond to quasi-stationary (metastable) states that only exist for a finite time, proportional to the inverse of the imaginary part of the resonance, and have energy proportional to the real part of the resonance. Resonances are intimately connected with the dynamics of quantum particles, and in particular their scattering properties [**2**, **21**]; non-classical properties like quantum tunnelling relate directly to the finite lifetime of these quasi-stationary states. In many ways, resonances are most naturally considered in the context of the time-dependent Schrödinger equation, as much of the more recent activity in this field (see for example [**15**], and references therein), and the above heuristics, indicate. Resonances that are close to the real axis appear as bumps in the scattering cross section and are thus of great physical interest; in particular, they can be measured in the laboratory.

Our main results in the present paper (Theorems 1 and 2) provide (somewhat implicit) conditions under which a statement of the desired nature remains true. Our method allows for the potential to be complex-valued (in which case eigenvalues and resonances may have multiplicities larger than one). The proofs of these theorems are fairly simple and use essentially only the residue theorem, Hadamard's factorization theorem, and the fact that the $m$-function determines the potential. Theorem 1 works for sufficiently fast decaying perturbations of $q_0 = 0$ while Theorem 2 is designed for fast decaying perturbations of $q_0(x) = 2/(x + x_0)^2$. We emphasize here that, in the latter case, $xq_0(x)$ is not integrable.

These theorems have to be viewed as models for similar theorems which work for perturbations of some base potential $q_0$, whose Jost function is defined on some Riemann surface which is a twofold cover of the complex plane. The perturbations are such that this property is not destroyed. A potential should then be uniquely determined by the Riemann surface and the zeros of an analytic function on this Riemann surface. With such theorems in place one still has a hurdle to overcome, namely to find an explicit characterization of the class of potentials for which it holds. Technically the biggest problem there is to estimate the asymptotic behavior of the $m$-function on the unphysical sheet (cf. Section 4). A better understanding of the $m$-function on the unphysical sheet would perhaps be useful in other contexts, too. As an example of the intended applications of Theorem 1, we show that the hypotheses of the theorem are satisfied for certain classes of compactly supported potentials (cf. Theorem 3).

We mention here that recently both Korotyaev [**13**] and Zworski [**22**] have worked on related questions. For real compactly supported potentials on $[0, \infty)$, Korotyaev describes the set of all possible Jost functions. Zworski observes that compactly supported even potentials on $\mathbb{R}$ are uniquely determined by the scattering matrix, which he recovers from the location of its poles. These poles are the zeros of the product $\psi(\cdot, 0)\psi'(\cdot, 0)$ so that, in fact, he needs both the Dirichlet poles and the Neumann poles, in the language of the half line problem, rather than just the Dirichlet poles as in Marchenko's approach, even though the scattering matrix and hence the potential are not uniquely determined when $z = 0$ is a pole of $\psi(\cdot, 0)\psi'(\cdot, 0)$. (The scattering problem for an even potential on $\mathbb{R}$ is in one-to-one correspondence with the scattering problem on a half line.) The seeming contradiction is resolved when one realizes that there is information in knowing whether a scattering pole is a root of $\psi(\cdot,0)$ or of $\psi'(\cdot,0)$. One might mention that Zworski's theorem contains a slight inaccuracy when he says that there are precisely two distinct compactly supported even potentials with the same scattering poles. The case of $q=0$ is a counterexample to this statement, albeit the only one.

The paper is organized as follows. In Section 2 we provide some technical background concerning complex-valued potentials and a formal definition of the $m$-function and its extension $M$ to the two-sheeted Riemann surface mentioned above. In Section 3 we state and prove our main theorems. In Section 4 we treat compactly supported potentials as an example of an application of Theorem 1. Appendices A and B give some background on entire functions and asymptotics of $m$-functions, respectively.

## 2. *Preliminaries*

Let $\Sigma$ be a fixed open sector of the complex plane whose vertex is at the origin. If $S$ is a subset of the complex plane we denote its complement by $S^{\mathrm{c}}$ and its closed convex hull by $\overline{\mathrm{co}}(S)$. Then define $\mathcal{Q}_\Sigma$ to be the set of those complex-valued, locally integrable functions on $[0,\infty)$ for which there is an open half plane $\Lambda$ satisfying the following two requirements.

(1) $\Lambda^{\mathrm{c}} \cap \Sigma$ is bounded.

(2) The set $Q = \overline{\mathrm{co}}(\{q(x) + r : x, r \in [0,\infty)\})$ does not intersect $\Lambda$.

REMARK 1. (1) Conditions of this type were first introduced by Brown *et al.* [**6**].

(2) If $\Sigma$ contains the positive real axis then $\mathcal{Q}_\Sigma$ is empty.

(3) When one is interested in real-valued potentials only (so that the sets $Q$ are subsets of the real line), one may choose for $\Sigma$ any sector (with vertex zero) contained in the upper or lower half plane. When $q$ is real and bounded below, $\Sigma$ could be any sector (with vertex zero) not containing the positive real axis.

Given a function $q \in \mathcal{Q}_\Sigma$ we consider the differential expression $L = -d^2/dx^2 + q$ on $[0,\infty)$. We will say that $q$ is of Class I, if at most one (up to constant multiples) solution of $Ly = \lambda y$ is square integrable on $[0,\infty)$. Otherwise, if all solutions of $Ly = \lambda y$ are square integrable on $[0,\infty)$, we will say that $q$ is of Class II. This classification is independent of the choice of $\lambda$. For real-valued potentials it coincides with the classical limit-point and limit-circle distinction. However, for complex-valued potentials it does not coincide with Sims's distinction (cf. [**18**]) between the limit-point and limit-circle cases. See [**7**] for a discussion of this issue.

Throughout the paper, we will use the following notation for derivatives. If $f$ is a function of several variables, $\dot{f}$ and $f'$ denote the derivative of $f$ with respect to the first and last variable, respectively. If $f$ is a function of two variables, $f^{(j,k)}$ denotes the function obtained by differentiating $j$ times with respect to the first variable and $k$ times with respect to the second. Now let $\theta(\lambda,\cdot)$ and $\phi(\lambda,\cdot)$ be linearly independent solutions of $Ly = \lambda y$ satisfying the initial conditions

$$\theta(\lambda,0) = 1, \quad \phi(\lambda,0) = 0,$$
$$\theta'(\lambda,0) = 0, \quad \phi'(\lambda,0) = 1.$$

It is shown in Brown *et al.* [**6**] (see also [**7**]) that for every $\lambda \in \Lambda$ there is at least one square integrable solution of $Ly = \lambda y$ which is not a multiple of $\phi(\lambda, \cdot)$. Hence if $q$ is of Class I then there is precisely one square integrable solution (up to constant multiples) for those $\lambda$ and there is a unique number $m(\lambda)$ such that $\Psi(\lambda, \cdot) = \theta(\lambda, \cdot) + m(\lambda)\phi(\lambda, \cdot)$ is square integrable. This function $m : \Lambda \longrightarrow \mathbb{C} : \lambda \mapsto m(\lambda)$ is the generalization of the Titchmarsh–Weyl $m$-function to the case of complex-valued potentials. Note that

$$m(\lambda) = \frac{\Psi'(\lambda, 0)}{\Psi(\lambda, 0)}.$$

Just as in the self-adjoint case, $m$ is an analytic function (see [**6**]). It may well be possible to extend it analytically to a larger domain than $\Lambda$. Sometimes $m$ may even be extended to the Riemann surface of $\lambda \mapsto \sqrt{\lambda}$. This is the case we are interested in and therefore we introduce the function

$$M(z) = m(z^2),$$

putting the branch cut on the positive real axis (so that $\Im(z) > 0$ represents the so-called physical $\lambda$-sheet).

## 3. *The main theorems*

THEOREM 1. *Let $\mathscr{C}$ be the family of potentials $q \in \mathscr{Q}_\Sigma$ which are of Class I and for which there exist functions $\psi : \mathbb{C} \times [0, \infty) \longrightarrow \mathbb{C}$ satisfying the following conditions:*

(i) *For every complex number $z$ the functions $\psi(z, \cdot)$ and $\psi(-z, \cdot)$ are nontrivial solutions of the differential equation $-y'' + qy = z^2 y$.*

(ii) *The Wronskian of $\psi(z, \cdot)$ and $\psi(-z, \cdot)$ satisfies*

$$W(\psi(z, \cdot), \psi(-z, \cdot)) = \psi(z, \cdot)\psi'(-z, \cdot) - \psi(-z, \cdot)\psi'(z, \cdot) = -2iz.$$

(iii) *$\psi(z, \cdot)$ is square integrable for all $z$ in some open subset of $\mathbb{C}$.*

(iv) *$\psi(\cdot, 0)$ and $\psi'(\cdot, 0)$ are entire functions of finite growth order.*

(v) *There exists a ray such that $\psi(z, 0)$ tends to one as $z$ tends to infinity along the ray.*

(vi) *There is an integer $p$ and a sequence of circles $t \mapsto r_n \exp(it)$ such that $r_n$ tends to infinity and $|M(r_n \exp(it))| r_n^{-p-1}$ tends to zero uniformly for $t \in [0, 2\pi]$.*

*Then the zeros of $\psi(\cdot, 0)$ and their multiplicities determine $q$ uniquely among the elements of $\mathscr{C}$.*

REMARK 2. Conditions (i) through (v) are satisfied for all sufficiently fast decaying potentials and, in particular, for $q = 0$. We show the validity of condition (vi) for certain classes of compactly supported potentials in Section 4.

*Proof of Theorem* 1. It is well known that, in the self-adjoint case, the Titchmarsh–Weyl $m$-function determines the potential $q$. A rather concise proof of this fact was given by Bennewitz in [**4**] who, in fact, showed that $q$ is uniquely determined from knowing the $m$-function along some non-real ray. This proof has recently been extended to complex potentials of Class I in [**7**], the only difference being that the knowledge of $m$ on two non-real rays which are eventually in $\Lambda$ is needed to reach the conclusion. (The condition of the rays being non-real can be dropped when the boundary of $\Lambda$ is not parallel to the real axis.) Since, of course, $M$ determines $m$, we only have to show that the given information suffices to determine

$M$. In fact we will determine $M$ and hence $m$ everywhere so that there will be plenty of rays to choose from.

Note that

$$M(z) = \frac{\psi'(z,0)}{\psi(z,0)}$$

is meromorphic and that its poles are the zeros of $\psi(\cdot,0)$. We denote the nonzero poles of $M$ by the pairwise distinct numbers $z_1, z_2, \ldots$ and we use $n_1, n_2, \ldots$ for their respective multiplicities. The zeros are labelled such that $|z_1| \leqslant |z_2| \leqslant \ldots$. Assume for now that $\psi(0,0) \neq 0$.

Let $h_z(\mu) = (z/\mu)^{p+1}/(z-\mu)$. Also define $\gamma_n(t) = r_n \exp(it)$ for $t \in [0, 2\pi]$ and $B_n = \{z : |z| < r_n\}$. Then, by the residue theorem,

$$\frac{1}{2\pi i}\int_{\gamma_n} h_z(\mu)M(\mu)\,d\mu = -M(z) + \sum_{k=0}^{p} \frac{M^{(k)}(0)}{k!} z^k + \sum_{z_j \in B_n} \operatorname{res}_{z_j}(h_z M)$$

if $0 \neq |z| < r_n$ and if $z$ is none of the poles of $M$. According to our assumption on $M$ the integral on the left tends to zero as $n$ tends to infinity, proving firstly the convergence of the series and secondly that

$$M(z) = \sum_{k=0}^{p} \frac{M^{(k)}(0)}{k!} z^k + \sum_{j=1}^{\infty} \operatorname{res}_{z_j}(h_z M). \tag{3.1}$$

Suppose that we had already determined the infinite series on the right-hand side of equation (3.1). We can then find the polynomial $\sum_{k=0}^{p} M^{(k)}(0) z^k/k!$ from the asymptotic behavior of the $m$-function along some ray. It is well known that, in the self-adjoint case, $m(z^2) = iz + o(1)$ as $z$ tends to infinity in sectors contained in the upper half plane. Theorem 6 in Appendix B extends this result to the case at hand.

Thus the theorem is proved once we determine the residues of $h_z M$ at the poles of $M$. To do this, let

$$f_j(\mu) = \frac{(\mu - z_j)^{n_j}}{\psi(\mu,0)}.$$

Then

$$\begin{aligned}\operatorname{res}_{z_j}(h_z M) &= \frac{1}{(n_j-1)!}(\psi'(\cdot,0)h_z f_j)^{(n_j-1)}(z_j)\\ &= \frac{1}{(n_j-1)!}\sum_{r=0}^{n_j-1}\binom{n_j-1}{r}\psi^{(r,1)}(z_j,0)(h_z f_j)^{(n_j-1-r)}(z_j)\end{aligned}$$

and this quantity may be computed once we know the function $\psi(\cdot,0)$ (and hence the functions $f_j$) and the numbers $\psi^{(r,1)}(z_j,0)$ for $r = 0, \ldots, n_j - 1$. We will now show that this information can be obtained from the given data.

Firstly, $\psi(\cdot,0)$ is given through Hadamard's factorization theorem as

$$\psi(z,0) = z^k \exp(g(z)) \prod_{j=1}^{\infty} E_\rho(z/z_j)^{n_j}$$

where $k$ and $\rho$ are integers and where $g$ is a polynomial. The number $\rho$ is to be chosen such that $\sum_{j=1}^{\infty} n_j |z_j|^{-\rho+1}$ is finite. This is always possible since otherwise $\psi(\cdot,0)$ would not have finite growth order (cf. Appendix A). The polynomial $g$ may be determined from the given asymptotic behavior of $\psi(\cdot,0)$ and we have $k=0$ since $\psi(0,0) \neq 0$.

Secondly, taking $r$ derivatives of the equation $W(\psi(z,\cdot),\psi(-z,\cdot)) = -2iz$ with respect to $z$ and evaluating them at $z_j$ gives

$$\psi^{(r,1)}(z_j,0)\psi(-z_j,0) = -W^{(r)}(z_j) - \sum_{s=0}^{r-1} \frac{(-1)^{r-n} r!}{(r-n)!n!} \psi^{(s,1)}(z_j,0)\psi^{(r-s,0)}(-z_j,0)$$

as long as $r \leqslant n_j - 1$ since $z_j$ is a zero of $\psi(\cdot,0) = 0$ of order $n_j$. We know that $\psi(-z_j,0) \neq 0$ since $\psi(z_j,\cdot)$ and $\psi(-z_j,\cdot)$ are linearly independent. Hence the numbers $\psi^{(0,1)}(z_j,0),\ldots,\psi^{(n_j-1,1)}(z_j,0)$ may be recursively computed.

We still have to discuss the case when $z = 0$ happens to be a zero of $\psi(\cdot,0)$. We will show shortly that $z = 0$ is a simple zero of $\psi(\cdot,0)$. Therefore the residue theorem gives

$$M(z) = \sum_{k=0}^{p} \frac{g^{(k+1)}(0)}{(k+1)!} z^k + \frac{\operatorname{res}_0(M)}{z} + \sum_{j=1}^{\infty} \operatorname{res}_{z_j}(h_z M)$$

where $g$ is defined by $g(\mu) = \mu M(\mu)$. The series and the polynomial occurring here are determined in the same way as before but, since we know the asymptotics of $M$ only up to order $o(1)$, we can not determine $\operatorname{res}_0(M) = \psi'(0,0)/\dot{\psi}(0,0)$ from asymptotic considerations. Instead we differentiate the equation $W(\psi(z,\cdot),\psi(-z,\cdot)) = -2iz$ with respect to $z$ and evaluate at $(0,0)$ to find $\dot{\psi}(0,0)\psi'(0,0) = -i$. This proves firstly, as promised, that $\dot{\psi}(0,0) \neq 0$ and secondly that the residue of $M$ at $z = 0$ equals $-i/\dot{\psi}(0,0)^2$. □

THEOREM 2. *Let $\mathscr{C}$ be the family of potentials $q \in \mathscr{Q}_\Sigma$ which are of Class I and for which there exist functions $\psi : \mathbb{C} \times [0,\infty) \longrightarrow \mathbb{C}$ satisfying the following conditions:*

(i) *For every complex number $z$ the functions $\psi(z,\cdot)$ and $\psi(-z,\cdot)$ are nontrivial solutions of the differential equation $-y'' + qy = z^2 y$.*

(ii) *The Wronskian of $\psi(z,\cdot)$ and $\psi(-z,\cdot)$ satisfies*

$$W(\psi(z,\cdot),\psi(-z,\cdot)) = \psi(z,\cdot)\psi'(-z,\cdot) - \psi(-z,\cdot)\psi'(z,\cdot) = -2iz^3.$$

(iii) *$\psi(z,\cdot)$ is square integrable for all $z$ in some open subset of $\mathbb{C}$.*

(iv) *$\psi(\cdot,0)$ and $\psi'(\cdot,0)$ are entire functions of finite growth order.*

(v) *There exists a ray such that $\psi(z,0)/(iz)$ tends to one as $z$ tends to infinity along the ray.*

(vi) *There is an integer $p$ and a sequence of circles $t \mapsto r_n \exp(it)$ such that $r_n$ tends to infinity and $|M(r_n \exp(it))| r_n^{-p-1}$ tends to zero uniformly for $t \in [0,2\pi]$.*

*If $\psi(0,0) \neq 0$, then the zeros of $\psi(\cdot,0)$ together with their multiplicities determine $q$ uniquely among the elements of $\mathscr{C}$. If $\psi(0,0) = 0$, then $z = 0$ is a zero of order two or three which we denote by $r$. In this case the zeros of $\psi(\cdot,0)$ together with their multiplicities and the number $\psi^{(r-2,1)}(0,0)$ determine $q$ uniquely among the elements of $\mathscr{C}$.*

REMARK 3. Conditions (i) through (v) are satisfied for sufficiently fast decaying perturbations of the base potential $q_0(x) = 2/(x+x_0)^2$ (where $x_0$ is any complex number away from the closed negative real axis). The Jost functions (or rather their generalizations) for $q_0$ are given by

$$\psi(z,x) = \frac{iz(x+x_0)-1}{x+x_0} \exp(izx).$$

We have not studied whether condition (vi) holds, for instance, for compactly supported perturbations of $q_0$. Note that $xq_0(x)$ is not integrable and that therefore Marchenko's approach does not work. We also mention again that these theorems are model theorems which have analogues for different base potentials each associated with its own Riemann surface.

*Proof of Theorem* 2. The proof of Theorem 2 is nearly identical to the proof of Theorem 1 except when it comes to the case when $\psi(0,0)=0$. If $z=0$ is a zero of $\psi(\cdot,0)$ of order $r$ then the residue theorem gives

$$M(z)=\sum_{k=0}^{p}\frac{1}{(k+r)!}g^{(k+r)}(0)z^k+\sum_{k=0}^{r-1}\frac{1}{k!}g^{(k)}(0)z^{k-r}+\sum_{j=1}^{\infty}\mathrm{res}_{z_j}(h_zM)$$

where $g(\mu)=\mu^rM(\mu)$. Again we obtain the polynomial part by asymptotic considerations and also the infinite series is determined as before. However, we now need the numbers $g(0),\dots,g^{(r-1)}(0)$. These may be computed when $\psi^{(0,1)}(0,0),\dots,\psi^{(r-1,1)}(0,0)$ are known (recall that $\psi(\cdot,0)$ and therefore its derivatives with respect to the first variable are known).

Again we will obtain the necessary information by taking derivatives of

$$W(\psi(z,\cdot),\psi(-z,\cdot))=-2iz^3$$

and evaluating at $z=0$. Note that the equations obtained from even derivatives are always trivially satisfied. From the first derivative we obtain

$$\dot\psi(0,0)\psi'(0,0)=0$$

which implies that $\dot\psi(0,0)$ is necessarily zero so that $r\geqslant 2$. The third derivative gives

$$3\psi^{(2,0)}(0,0)\psi^{(1,1)}(0,0)-\psi^{(3,0)}(0,0)\psi^{(0,1)}(0,0)=6i \tag{3.2}$$

which shows that $r\leqslant 3$. If indeed $r=3$, the fifth derivative gives

$$-10\psi^{(3,0)}(0,0)\psi^{(2,1)}(0,0)+5\psi^{(4,0)}(0,0)\psi^{(1,1)}(0,0)-\psi^{(5,0)}(0,0)\psi^{(0,1)}(0,0)=0. \tag{3.3}$$

If $r=2$ we need to know $\psi^{(0,1)}(0,0)$ and $\psi^{(1,1)}(0,0)$. Our hypothesis gives us the first while the second may be obtained using equation (3.2). If $r=3$ then $\psi^{(0,1)}(0,0)$ is determined by equation (3.2), our hypothesis provides $\psi^{(1,1)}(0,0)$ while $\psi^{(2,1)}(0,0)$ is computed using equation (3.3).

It may be worth mentioning that $\dot\psi(0,\cdot)$ is also a solution of $-y''+qy=0$. $\square$

## 4. *Compactly supported potentials*

In this section we will apply Theorem 1 to prove that the resonances determine uniquely a potential $q:[0,\infty)\longrightarrow\mathbb{C}$ supported and absolutely continuous on $[0,R]$ for which $q(R)\neq 0$. However, some of our intermediate results hold under less restrictive conditions.

The approach we are taking in this section follows in part the one in Simon's paper [**17**].

Suppose that $q\in L^1([0,\infty))$. Consider the integral equation

$$y(x)=\exp(izx)+\int_x^{\infty}K(z,t,x)q(t)y(t)\,dt$$

where

$$K(z,t,x) = \frac{\sin(z(t-x))}{z} = \exp(-iz(t-x)) \int_0^{t-x} \exp(2izl)\, dl.$$

Define $\psi_0(z,x_0) = \exp(izx_0)$ and, recursively,

$$\begin{aligned}\psi_n(z,x_0) &= \int_{x_0}^{\infty} K(z,x_1,x_0) q(x_1) \psi_{n-1}(z,x_1)\, dx_1 \\ &= \int_{x_0<x_1<\ldots<x_n} \exp(izx_n) \prod_{j=1}^{n} [K(z,x_j,x_{j-1}) q(x_j)]\, dx_n \ldots dx_1.\end{aligned}$$

For $0 \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n$ and $\alpha \in \mathbb{R}$, define

$$R_n(x_0,\ldots,x_n;\alpha) = \int_0^{x_n-x_{n-1}} \ldots \int_0^{x_1-x_0} \delta(x_0/2 + l_1 + \ldots + l_n - \alpha)\, dl_1 \ldots dl_n$$

(where we consider the integrals to be taken over closed intervals). Furthermore, for $0 \leqslant x$ and $\alpha \in \mathbb{R}$, let

$$t_n(\alpha,x) = \int_{x<x_1<\ldots<x_n} q(x_1)\ldots q(x_n) R_n(x,x_1,\ldots,x_n;\alpha)\, dx_n \ldots dx_1.$$

With these definitions we have

$$\psi_n(z,x) = \int_{-\infty}^{\infty} t_n(\alpha,x) \exp(2iz\alpha)\, d\alpha.$$

We will now investigate the functions $R_n$. Let us first compute $R_1$ and $R_2$ explicitly. We find

$$R_1(x_0,x_1;\alpha) = \begin{cases} 1 & \text{if } x_0/2 \leqslant \alpha \leqslant x_1 - x_0/2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$R_2(x_0,x_1,x_2;\alpha) = \int_0^{x_2-x_1} R_1(x_0,x_1;\alpha-l_2)\, dl_2 = [\tau-\sigma]_+$$

where $\tau = \min\{\alpha - x_0/2, x_2 - x_1\}$ and $\sigma = \max\{0, \alpha + x_0/2 - x_1\}$.

LEMMA 1. *The functions $R_n$ have the following properties:*

(i) $0 \leqslant R_n(x_0,\ldots,x_n;\alpha)$ *for all* $\alpha \in \mathbb{R}$ *and* $0 \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n$.

(ii) $R_n(x_0,\ldots,x_n;\alpha) = \int_0^{x_n-x_{n-1}} R_{n-1}(x_0,\ldots,x_{n-1};\alpha-l_n)\, dl_n$.

(iii) $R_n(x_0,\ldots,x_n;\alpha) = R_n(x_0,\ldots,x_n;x_n-\alpha)$.

(iv) $R_n(x_0,\ldots,x_n;\alpha) = 0$ *unless* $x_0/2 \leqslant \alpha \leqslant x_n - x_0/2$.

(v) *Let* $R_n^{(k)}$ *denote the kth derivative of* $R_n$ *with respect to the last argument. Then* $R_n^{(k)}(x_0,\ldots,x_n;\cdot)$ *is continuous if* $k \leqslant n-2$ *and piecewise continuous if* $k = n-1$.

(vi) *The following estimate holds for* $k \leqslant n-1$

$$\left|R_n^{(k)}(x_0,\ldots,x_n;\alpha)\right| \leqslant \frac{2^k}{(n-1-k)!} \min\{[\alpha - x_0/2]_+, [x_n - x_0/2 - \alpha]_+\}^{n-1-k}.$$

*Proof.* The proof of the first statement is trivial. The representation (ii) is proven by induction and is the basis of the (inductive) proofs of the remaining statements. To prove (iii) the substitution $u(l_n) = x_n - x_{n-1} - l_n$ is useful. To prove (v) note that

$$R_n^{(k)}(x_0,\ldots,x_n;\alpha) = R_{n-1}^{(k-1)}(x_0,\ldots,x_{n-1};\alpha) - R_{n-1}^{(k-1)}(x_0,\ldots,x_{n-1};\alpha-(x_n-x_{n-1})).$$

Finally, for (vi) one proves the first estimate for the case $k = 0$ by induction over $n$ using the fact that

$$R_n(x_0, \ldots, x_n; \alpha) \leqslant \int_0^{\alpha - x_0/2} \frac{(\alpha - x_0/2 - t)^{n-2}}{(n-2)!}\, dt$$

since $R_{n-1}(x_0, \ldots, x_{n-1}; \alpha - t) = 0$ when $t > \alpha > \alpha - x_0/2$. Then induction over $k$ gives the first estimate for any $k \in \mathbb{N}$. The second estimate follows from the symmetry property of $R_n$. $\square$

We now turn to the properties of the functions $t_n$.

LEMMA 2. *Suppose that $q \in L^1([0,\infty))$. Then the functions $t_n$ have the following properties:*

(i) $t_1(\alpha, x) = \int_{\alpha + x/2}^{\infty} q(t)\, dt$ *if* $\alpha \geqslant x/2$.

(ii) *If $n \geqslant 2$ then $t_n(\cdot, x)$ is $n-2$ times differentiable and $t_n^{(n-2,0)}(\cdot, x)$ is absolutely continuous on $\mathbb{R}$.*

(iii) *If $k \leqslant n-2$ then*

$$\left|t_n^{(k,0)}(\alpha, x)\right| \leqslant \frac{2^k [\alpha - x/2]_+^{n-1-k}}{(n-1-k)!} \frac{\|q\|_1^{n-1}}{(n-1)!} \int_{\alpha + x/2}^{\infty} |q(t)|\, dt.$$

(iv) *If $q$ is supported on $[0, R]$ and $k \leqslant n-2$ then also*

$$\left|t_n^{(k,0)}(\alpha, x)\right| \leqslant \frac{2^k [R - x/2 - \alpha]_+^{n-1-k}}{(n-1-k)!} \frac{\|q\|_1^{n-1}}{(n-1)!} \int_{\alpha + x/2}^{R} |q(t)|\, dt.$$

(v) *For every $\alpha \in \mathbb{R}$ the series $\sum_{n=1}^{\infty} t_n(\alpha, \cdot)$ converges absolutely and uniformly to a function $t(\alpha, \cdot)$.*

*Proof.* Statement (i) is immediate. As for (ii) note that, due to the dominated convergence theorem, derivatives up to order $n-2$ can be placed inside the integral defining $t_n$. The absolute continuity follows since, according to Lemma 1(vi), $|R_n^{(n-1)}(x_0, \ldots, x_n, \cdot)|$ is bounded by $2^{n-1}$ so that

$$\left|R_n^{(n-2)}(x_0, \ldots, x_n, \beta) - R_n^{(n-2)}(x_0, \ldots, x_n, \alpha)\right| \leqslant 2^{n-1} |\beta - \alpha|.$$

We now prove (iii). If $\alpha + x_0/2 \leqslant x_{n-1}$ we have

$$\int_{x_{n-1}}^{\infty} |q(x_n)| \left|R_n^{(k)}(x_0, \ldots, x_n; \alpha)\right| dx_n \leqslant \frac{2^k [\alpha - x_0/2]_+^{n-1-k}}{(n-1-k)!} \int_{\alpha + x_0/2}^{\infty} |q(x_n)|\, dx_n.$$

If $\alpha + x_0/2 > x_{n-1}$ we obtain the same estimate since $R_n(x_0, \ldots, x_{n-1}, \cdot\,; \alpha)$ is identically equal to zero on the interval $[x_{n-1}, \alpha + x_0/2]$. Since

$$\int_{x_0 < x_1 < \ldots < x_{n-1}} |q(x_1)| \ldots |q(x_{n-1})|\, dx_{n-1} \ldots dx_1 = \frac{\|q\|_1^{n-1}}{(n-1)!}$$

we have proved (iii). Statement (iv) follows similarly using the inequality in Lemma 1(vi) and the fact that one may assume that $x_n \leqslant R$.

Finally, to prove the last statement, note that $t(\alpha, x) = 0$ if $\alpha < 0$ and that, for $\alpha \geqslant 0$,

$$\sum_{n=1}^{\infty} |t_n(\alpha, x)| \leqslant \sum_{n=1}^{\infty} \frac{(\alpha \|q\|_1)^{n-1}}{(n-1)!^2} \int_{\alpha}^{\infty} |q(s)|\, ds \leqslant e^{\alpha \|q\|_1} \int_{\alpha}^{\infty} |q(s)|\, ds$$

regardless of $x \in [0, \infty)$. $\square$

LEMMA 3. *Suppose that $q$ decays super-exponentially in the mean, that is, for every positive $r$ the integral $\int_0^\infty e^{rx}|q(x)|dx$ is finite. Then the following statements are true:*

(i) *$t(\cdot,x)$ decays super-exponentially in the mean for every $x \in [0,\infty)$.*

(ii) *$t(\cdot,x)$ is locally absolutely continuous on $[x/2,\infty)$ for every $x \in [0,\infty)$.*

(iii) *The integral $\varphi(z,x) = \int_{-\infty}^{\infty} t(\alpha,x)\exp(2iz\alpha)\,d\alpha$ exists for every $z \in \mathbb{C}$ and every $x \in [0,\infty)$.*

(iv) *$\psi(z,x) = \sum_{n=0}^\infty \psi_n(z,x) = \exp(izx) + \varphi(z,x)$. Moreover, $\psi(z,\cdot)$ and $\psi'(z,\cdot)$ are locally absolutely continuous. $\psi(z,\cdot)$ satisfies the integral equation $y(x) = \exp(izx) + \int_x^\infty K(z,t,x)q(t)\psi(z,t)\,dt$ and the differential equation $-y'' + qy = zy$. Finally, $\psi(\cdot,x)$ and $\psi'(\cdot,x)$ are entire.*

*Proof.* Since $q$ decays super-exponentially in the mean we find that

$$\int_0^\infty |t(\alpha,x)|e^{r\alpha}\,d\alpha \leqslant \int_0^\infty e^{\alpha(r+\|q\|_1)} \int_\alpha^\infty |q(s)|\,ds\,d\alpha = \int_0^\infty |q(s)| \int_0^s e^{\alpha(r+\|q\|_1)}\,d\alpha\,ds$$

is finite. This proves (i).

Obviously, $t_1(\cdot,x)$ is absolutely continuous on $[x/2,\infty)$ while $t_2(\cdot,x)$ is absolutely continuous on $\mathbb{R}$ by Lemma 2(ii). For $n \geqslant 3$, $t_n(\cdot,x)$ is continuously differentiable and

$$\left|t_n^{(1,0)}(\alpha,x)\right| \leqslant 2\|q\|_1 \frac{\alpha^{n-2}}{(n-2)!}\frac{\|q\|_1^{n-1}}{(n-1)!}.$$

Hence $\sum_{n=3}^\infty t_n(\cdot,x)$ is continuously differentiable. This proves (ii).

Only if $z$ is in the lower half plane, is the existence of $\varphi$ at all questionable. However, the function $\alpha \mapsto t(\alpha,x)\exp(2iz\alpha)$ is always integrable due to the fact that $q$ decays super-exponentially in the mean. This proves (iii). The proof of the last statement is standard. □

We want an estimate from above for $|M|$ on a sequence of circles whose radii tend to infinity. We will see that it is enough to estimate $|\psi(z,0)|$ from below. In various parts of the plane we need different hypotheses on $q$ to achieve this goal. Lemmas 4 and 5 provide estimates outside the sectors $-K|\Re(z)| \leqslant \Im(z) \leqslant 0$ which contain all sufficiently large zeros of $\psi(\cdot,0)$. Estimates in these sectors are therefore somewhat more delicate. Lemmas 6 and 7 are concerned with these.

LEMMA 4. *Suppose that $q$ decays super-exponentially in the mean and let $\varphi$ be the function defined in Lemma 3. Then the following statements hold:*

(i) *Let $v$ be a fixed real number. Then $\varphi(u+iv,x)$ tends to zero as $u \in \mathbb{R}$ tends to $\pm\infty$.*

(ii) *$\varphi(u+iv,x)$ tends to zero uniformly in $u \in \mathbb{R}$ as $v \geqslant 0$ tends to $\infty$.*

*Proof.* Statement (i) follows immediately from the Riemann–Lebesgue lemma. To prove (ii) note that, if $2v > \|q\|_1$,

$$|\varphi(u+iv,x)| \leqslant \int_0^\infty |t(\alpha,x)|e^{-2v\alpha}\,d\alpha \leqslant \|q\|_1 \int_0^\infty e^{(\|q\|_1-2v)\alpha}\,d\alpha = \frac{\|q\|_1}{2v-\|q\|_1}. \qquad \square$$

LEMMA 5. *Suppose that $q$ is supported on $[0,R]$ and that there exist numbers $v>0$, $c_1\neq 0$ and $c_2>0$ such that*

$$\lim_{s\downarrow 0} s^{-v}\int_{R-s}^{R} q(x)\,dx = c_1 \quad \text{and} \quad s^{-v}\int_{R-s}^{R}|q(x)|\,dx \leqslant c_2 \qquad \text{for } s\in[0,R].$$

*Let $K$ be a positive number. Then there are positive constants $C$ and $Z$ such that*

$$|\psi(z,0)| \geqslant C\frac{\mathrm{e}^{2R|\Im(z)|}}{|\Im(z)|^{v+1}}$$

*holds for all $z$ satisfying $\Im(z)\leqslant \min\{-Z,-K|\Re(z)|\}$.*

*Proof.* First note that

$$|\psi(z,0)| \geqslant |\psi_1(z,0)| - \left|1+\sum_{n=2}^{\infty}\psi_n(z,0)\right|.$$

According to our assumptions we obtain, using Lemma 2(i) and the substitution $u=R-\alpha$, that

$$\psi_1(z,0) = c_1\mathrm{e}^{2iRz}\int_0^R u^v(1+h(u))\mathrm{e}^{-2izu}\,du$$

where $h$ is some function such that $\lim_{u\to 0}h(u)=0$. Furthermore,

$$\left|\int_0^R u^v\mathrm{e}^{-2izu}\,du - \frac{\Gamma(v+1)}{(2iz)^{v+1}}\right| \leqslant \int_R^{\infty} u^v\mathrm{e}^{-2|\Im(z)|u}\,du \leqslant \frac{\mathrm{e}^{R(\mu-2|\Im(z)|)}}{|\Im(z)|}$$

if we choose $\mu$ such that $u^v\leqslant \mathrm{e}^{\mu u}$ for all $u\geqslant R$ and $\Im(z)<-\mu$. Widder [**20**, Theorem V.1] states that

$$\limsup_{s\to\infty}\left|s^{v+1}\int_0^{\infty}\mathrm{e}^{-ts}\,d\beta(t)\right| \leqslant \limsup_{t\downarrow 0}|\Gamma(v+2)\beta(t)t^{-v-1}|$$

if $v+1>0$. Now let $\varepsilon$ be given. Then $|h(x)|<\varepsilon$ for all sufficiently small $x$ and hence

$$\Gamma(v+2)t^{-v-1}\int_0^t x^v|h(x)|\,dx \leqslant \varepsilon\frac{\Gamma(v+2)}{v+1}$$

for all sufficiently small $t$. This shows, using $\beta(t)=\int_0^t x^v|h(x)|\,dx$, that

$$|2\Im(z)|^{v+1}\int_0^R t^v|h(t)|\mathrm{e}^{-2|\Im(z)|t}\,dt$$

tends to zero as $|\Im(z)|$ tends to infinity. Hence, for any positive $\varepsilon$ there is a positive number $Z$ ($\geqslant\mu$) such that

$$|\psi_1(z,0)| \geqslant |c_1\mathrm{e}^{2iRz}|\left\{\frac{\Gamma(v+1)}{(2|z|)^{v+1}} - \frac{\mathrm{e}^{R(\mu-2|\Im(z)|)}}{|\Im(z)|} - \frac{\varepsilon}{|2\Im(z)|^{v+1}}\right\}$$

if $\Im(z)\leqslant -Z$. Choose a proper $\varepsilon$ and note that $|z|\leqslant\sqrt{1+1/K^2}|\Im(z)|$ to obtain

$$|\psi_1(z,0)| \geqslant C'\frac{\mathrm{e}^{2R|\Im(z)|}}{|\Im(z)|^{v+1}} \tag{4.1}$$

for some $C'>0$ provided that $\Im(z)\leqslant -Z$.

From Lemma 2(iv) and our hypotheses on $q$ we obtain next that

$$|\psi_n(z,0)| \leqslant c_2\frac{\|q\|_1^{n-1}}{(n-1)!^2}\int_0^R(R-\alpha)^{v+n-1}\mathrm{e}^{2\alpha|\Im(z)|}\,d\alpha.$$

Since, for $n+v>0$ and $s>0$,

$$\int_0^\infty u^{n+v-1}\mathrm{e}^{-us}\,du = \frac{\Gamma(v+n)}{s^{v+n}} \leqslant \frac{(n-1)!(v+1)^{n-1}\Gamma(v+1)}{s^{v+n}}$$

this becomes

$$|\psi_n(z,0)| \leqslant c_2\frac{\Gamma(v+1)}{|2\Im(z)|^{v+1}}\frac{(v+1)^{n-1}\|q\|_1^{n-1}}{|2\Im(z)|^{n-1}(n-1)!}\,\mathrm{e}^{2R|\Im(z)|}.$$

Also since $\mathrm{e}^{(v+1)\|q\|_1/|2\Im(z)|}$ is bounded for those values of the variables we are interested in, there is a constant $C''$ such that

$$1+\sum_{n=2}^{\infty}|\psi_n(z,0)| \leqslant C''\frac{\mathrm{e}^{2R|\Im(z)|}}{|\Im(z)|^{v+2}}$$

for sufficiently large $|\Im(z)|$. Combining this with (4.1) gives the desired result. □

LEMMA 6. *Let $K$ and $v$ be positive numbers and $c_1$ a non-zero complex number. Suppose that*

$$\varphi(z,0)=\int_0^\infty t(\alpha,0)\mathrm{e}^{2iz\alpha}\,d\alpha = c_1\frac{\mathrm{e}^{2izR}}{z^v}(1+f_1(z))+f_2(z)$$

*where $|f_1(z)|\leqslant 1/48$ and $|f_2(z)|\leqslant 1/3$ for all sufficiently large $z$ in the sectors $-K|\Re(z)| \leqslant \Im(z) \leqslant 0$. Then there is a number $\tau$ such that $|\psi(z,0)| \geqslant 1/3$ for all $z$ on the circular arcs given by $|z|=(2n\pi+\tau)/(2R)$ and $-K|\Re(z)| \leqslant \Im(z) \leqslant 0$ and sufficiently large integers $n$.*

*Proof.* We write $x=\Re(z)$, $y=\Im(z)$, and $c_1=\mathrm{e}^{\sigma+i\kappa}$ where $\sigma,\kappa\in\mathbb{R}$. To prove the lemma we distinguish three cases.

*Case* 1: $-2Ry \leqslant v\log(n\pi/R)-\sigma-2$.
In this case $\varphi(z,0)$ is negligible since

$$\left|c_1\frac{\mathrm{e}^{2izR}}{z^v}\right| = \mathrm{e}^{\sigma-2Ry-2\log(n\pi/R)}\left(1+\frac{\tau}{2n\pi}\right)^{-v} \leqslant 1/4$$

which holds for sufficiently large $n$.

*Case* 2: $-2Ry \geqslant v\log(n\pi/R)-\sigma+2$.
Here the main contribution comes from the term $c_1\mathrm{e}^{2izR}/z^v$. In fact,

$$\left|c_1\frac{\mathrm{e}^{2izR}}{z^v}\right| \geqslant 4$$

when $n$ is sufficiently large.

*Case* 3: $v\log(n\pi/R)-\sigma-2 \leqslant -2Ry \leqslant v\log(n\pi/R)-\sigma+2$.
We obtain firstly that

$$|\psi(z,0)| \geqslant \left|1+c_1\frac{\mathrm{e}^{2izR}}{z^v}\right| - \frac{1}{3}-\frac{1}{6} \geqslant \frac{1}{2}+\Re\left(c_1\frac{\mathrm{e}^{2izR}}{z^v}\right)$$

since $|c_1\mathrm{e}^{2izR}/z^v| \leqslant 8$ when $n$ is sufficiently large.

Now let $\beta=\arg(c_1\mathrm{e}^{2izR}/z^v)=2Rx+\kappa-v\arg(z)$ and note that $\arg(z)=3\pi/2\pm\pi/2+\arctan(y/x)$ where one has to choose the positive sign for positive $x$ and the negative sign for negative $x$ (recall that $y$ is negative in any case). After a small calculation one finds that $\pm 2Rx = 2n\pi + \tau + r(n)$ where $r(n) = O((\log n)^2/n)$ as $n$ tends to infinity. This implies also that $\arctan(y/x) = O(\log(n)/n)$ as $n$ tends to infinity. Hence

$$\begin{aligned}\cos(\beta) &= \cos\left(\kappa + \frac{3v\pi}{2} \pm \left(\tau + \frac{v\pi}{2} + r(n)\right) - v\arctan(y/x)\right)\\ &\geqslant -|\sin(\pm r(n) - v\arctan(y/x))| \geqslant -\frac{1}{48}\end{aligned}$$

provided that $\tau$ is chosen in such a way that $\cos(\kappa + 3v\pi/2 \pm (\tau + v\pi/2))$ is nonnegative for either choice of the sign. This can be achieved by choosing $\tau$ such that $\tau + v\pi/2$ equals zero or $\pi$ depending on whether $\cos(\kappa + 3v\pi/2)$ is nonnegative or not. Therefore we arrive at the estimate

$$\Re\left(c_1 \frac{e^{2izR}}{z^v}\right) \geqslant -8|\sin(\pm r(n) - v\arctan(y/x))| \geqslant -\frac{1}{6}$$

which holds for sufficiently large $n$. $\square$

We will now show that the hypotheses of Lemma 6 can indeed be satisfied for certain classes of potentials.

LEMMA 7. *Suppose that $q$ is supported and absolutely continuous on* $[0, R]$. *Then for any positive K the hypothesis of Lemma 6 is satisfied in each of the following two cases:*

(i) $q(R) \neq 0$. *In this case* $c_1 = -q(R)/4$ *and* $v = 2$.

(ii) $q(R) = 0$ *but $q'$ is absolutely continuous on* $[0, R]$ *and* $q'(R) \neq 0$. *In this case* $c_1 = -iq'(R)/8$ *and* $v = 3$.

*Proof.* Suppose first that $q(R) \neq 0$. We know from Lemma 3 that $t(\cdot, 0)$ is absolutely continuous on $[0, R]$. We prove next that this is also true for $t^{(1,0)}(\cdot, 0)$. We have $t_1^{(1,0)}(\alpha, 0) = -q(\alpha)$ and

$$\begin{aligned}t_2^{(1,0)}(\alpha, 0) &= \int_{0<x_1<x_2} q(x_1)q(x_2)(R_1(0, x_1, \alpha) - R_1(0, x_1, \alpha - (x_2 - x_1)))\,dx_2\,dx_1\\ &= t_1(\alpha, 0)(t_1(\alpha, 0) - t_1(0, 0)) + \int_0^R q(x_1)t_1(x_1 + \alpha, 0)\,dx_1\end{aligned}$$

are absolutely continuous since $q$ is integrable. $t_3^{(1,0)}(\cdot, 0)$ is absolutely continuous by Lemma 2. Finally, since

$$\left|t_n^{(2,0)}(\alpha, x)\right| \leqslant 4\|q\|_1 \frac{\alpha^{n-3}}{(n-3)!}\frac{\|q\|_1^{n-1}}{(n-1)!}$$

we find that $\sum_{n=4}^{\infty} t_n(\cdot, x)$ is twice continuously differentiable.

We are now allowed to integrate by parts twice to obtain

$$\varphi(z, 0) = -\frac{t(0, 0)}{2iz} + \frac{t'(0, 0)}{(2iz)^2} + \frac{e^{2izR}}{(2iz)^2}\left(-t'(R, 0) + \int_0^R t''(R - u)e^{-2izu}\,du\right)$$

and we note that $-t'(R, 0) = q(R) \neq 0$.

The Riemann–Lebesgue lemma gives that $\int_0^R t''(R - u)e^{-2i(x+iy)u}\,du$ tends to zero as $x$ tends to infinity when $y$ is fixed. A closer look at its proof reveals that this is in fact true uniformly in $y$ as long as $y$ is bounded above. Hence there is a positive

$X$ such that

$$\left|\int_0^R t''(R-u)\mathrm{e}^{-2i(x+iy)u}\,du\right| \leqslant \frac{|q(R)|}{48}$$

as long as $\Im(z) \leqslant 0$ and $|\Re(z)| \geqslant X$. This can be seen as follows. If $t''(R-u) = \chi_{[a,b]}(u)$ then

$$\left|\int_0^R t''(R-u)\mathrm{e}^{-2izu}\,du\right| = \frac{1}{|2iz|}|\mathrm{e}^{-2izb} - \mathrm{e}^{-2iza}| \leqslant \frac{1}{|z|} \leqslant \frac{1}{x}.$$

Similarly, if $t''(R-u) = \sum_{j=1}^J c_j\chi_{[a_j,b_j]}(u)$ then

$$\left|\int_0^R t''(R-u)\mathrm{e}^{-2izu}\,du\right| \leqslant \frac{1}{x}\sum_{j=1}^J |c_j|.$$

Finally, if $t''$ is just integrable, then there are $a_j$, $b_j$, and $c_j$ such that $g(u) = \sum_{j=1}^J c_j\chi_{[a_j,b_j]}(u)$ and $\|t(R-\cdot)-g\|_1$ is arbitrarily small. The triangle inequality then gives the desired result. It is also obvious that

$$\left|-\frac{t(0,0)}{2iz} + \frac{t'(0,0)}{(2iz)^2}\right| \leqslant \frac{1}{3}$$

if $|z|$ is sufficiently large.

When $q'$ is also absolutely continuous we can integrate by parts once more to obtain

$$\varphi(z,0) = -\frac{t(0,0)}{2iz} + \frac{t'(0,0)}{(2iz)^2} - \frac{t''(0,0)}{(2iz)^3} + \frac{\mathrm{e}^{2izR}}{(2iz)^3}\left(t''(R,0) - \int_0^R t'''(R-u)\mathrm{e}^{-2izu}\,du\right)$$

and we note that $t''(R,0) = -q'(R) \neq 0$. □

The results of Lemmas 4–7 can be combined to provide an estimate on $M(z)$ for $z$ on certain large circles.

Theorem 3. *Suppose that $q \in L^1([0,\infty)$ is compactly supported and absolutely continuous. Furthermore, assume that $q$ satisfies one of the following two conditions:*

(i) *$q$ has a jump discontinuity at the right endpoint of its support.*

(ii) *$q$ is continuous on $[0,\infty)$ and $q'$ is absolutely continuous on its support with a jump discontinuity at the right endpoint of the support.*

*Then $q$ is uniquely determined once the zeros of the Jost function and their respective multiplicities are given.*

Recall that the zeros in the upper half plane are roots of eigenvalues and that the zeros in the lower half plane are roots of resonances.

*Proof of Theorem* 3. Lemma 3 proves the existence of a function $\psi$ satisfying conditions (i), (iv), and (v) of Theorem 1 except for the statement on the growth order of $\psi(\cdot,0)$ and $\psi'(\cdot,0)$. Note that $\psi(\cdot,x)$ has growth order one since $t(\cdot,x)$ is compactly supported. Now recall that $\psi'(z,0) = iz + \int_{\mathrm{supp}(q)} K'(z,t,0)q(t)\psi(z,t)\,dt$ which shows that $\psi'(\cdot,x)$ also has growth order one.

Conditions (ii) and (iii) of Theorem 1 are trivially satisfied.

We will now check condition (vi). Note that $K'(z,t,0) = -\mathrm{e}^{izt} + izK(z,t,0)$ and hence

$$\psi'(z,0) = iz\psi(z,0) - \int_0^\infty \mathrm{e}^{izt} q(t)\psi(z,t)\,dt. \tag{4.2}$$

Suppose first that $\Im(z) \geqslant 0$. From Lemma 4 we obtain

$$\tfrac{1}{2} \leqslant 1 - |\varphi(z,t)| \leqslant |\psi(z,t)| \leqslant 1 + |\varphi(z,t)| \leqslant \tfrac{3}{2}.$$

This and equation (4.2) gives

$$|M(z) - iz| \leqslant \frac{\int_0^\infty |\mathrm{e}^{izt} q(t)\psi(z,t)|\,dt}{|\psi(z,0)|} \leqslant 3\|q\|_1$$

when $z$ is sufficiently large.

To estimate $M(z)$ for $z$ in the lower half plane, note that the Wronskian of $\psi(z,\cdot)$ and $\psi(-z,\cdot)$ satisfies

$$W(\psi(z,\cdot),\psi(-z,\cdot)) = -2iz.$$

Therefore

$$M(z) = M(-z) + \frac{2iz}{\psi(z,0)\psi(-z,0)}$$

and since $|\psi(-z,0)| \geqslant 1/2$ we need a lower bound on $\psi(z,0)$.

In the sector $\Im(z) \leqslant -K|\Re(z)|$ this lower bound is provided by Lemma 5 which applies either with $v = 1$, $c_1 = q(R)$, and $c_2 = 2|q(R)|$ or with $v = 2$, $c_1 = q'(R)/4$, and $c_2 = |q'(R)|/2$. Hence

$$|M(z) + iz| \leqslant |M(-z) + iz| + \frac{2|z|}{|\psi(z,0)\psi(-z,0)|} \leqslant 3\|q\|_1 + C'\frac{|z|\Im(z)^{v+1}}{\mathrm{e}^{2R|\Im(z)|}} \leqslant C''$$

for appropriate constants $C'$ and $C''$.

Finally, if $-K\Re(z) \leqslant \Im(z) \leqslant 0$ and $z$ is on a circle of radius $(2n\pi + \tau)/(2R)$ then $|\psi(z,0)|$ is bounded below by $1/3$ so that $|M(z)| \leqslant 20|z|$. □

## *Appendix* A. *Entire functions*

For proofs of the statements, see any textbook on complex analysis, for example, Conway **[8]**.

An entire function $f$ is of finite order if there are positive constants $a$ and $r$ such that

$$|f(z)| < \exp(|z|^a)$$

whenever $|z| > r$. If $f$ is not of finite order then $f$ is said to be of infinite order. If $f$ is of finite order then the number

$$\lambda = \inf\{a : \exists\, r : |z| > r \Rightarrow |f(z)| < \exp(|z|^a)\}$$

is called the order of $f$.

Let $p$ be a nonnegative integer. The functions $E_p$ defined by

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right)$$

are called canonical factors.

LEMMA 8. *Let $a_n$ be a sequence of complex numbers which satisfies*

$$0 < |a_1| \leqslant |a_2| \leqslant \dots \quad \text{and} \quad a_n \to \infty.$$

*Furthermore, let $p_n$ be a sequence of nonnegative integers and assume that, for all $r > 0$,*

$$\sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n+1} < \infty. \tag{A.1}$$

*Then the function $P$ defined by*

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$$

*converges uniformly on compact subsets of the plane and hence defines an entire function.*

Note that equation (A.1) may always be satisfied by the choice $p_n = n - 1$.

THEOREM 4 (Weierstrass factorization theorem). *Suppose that $f$ is an entire function, that $z = 0$ is a zero of $f$ of multiplicity $m$, and that the nonzero zeros of $f$ are given by the sequence $a_n$ (This sequence can be finite or infinite. In our notation we assume that it is infinite. Otherwise the statements remain true when the notation is suitably adapted.) which takes into account possible repetitions and which satisfies $0 < |a_1| \leqslant |a_2| \leqslant \dots$. Then there is an entire function $g$ and a sequence $p_n$ of nonnegative integers such that*

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} E_{p_n}(z/a_n).$$

An entire function $f$ whose nonzero zeros are given by the sequence $a_n$ which takes into account possible repetitions and which satisfies $0 < |a_1| \leqslant |a_2| \leqslant \dots$ is said to have finite rank if there exists an integer $p$ such that

$$\sum_{n=1}^{\infty} |a_n|^{-p-1} < \infty.$$

The smallest such integer is called the rank of $f$. Note that any entire function with finitely many zeros has rank zero. According to Weierstrass's theorem an entire function $f$ of finite rank $p$ can be written as

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} E_p(z/a_n)$$

where $g$ is entire. The function

$$P(z) = \prod_{n=1}^{\infty} E_p(z/a_n)$$

is then called the canonical product associated with $f$.

An entire function $f$ of finite rank $p$ is said to have finite genus if the function $g$ in the Weierstrass factorization is a polynomial. The number

$$\mu = \max\{p, \deg(g)\}$$

is then called the genus of $f$.

THEOREM 5 (Hadamard factorization theorem). *If $f$ is an entire function of finite order $\lambda$ then $f$ has finite genus $\mu$ and $\mu$ does not exceed $\lambda$.*

## *Appendix* B. *Asymptotics of the m-function*

The asymptotics of the $m$-function for a real potential on $[0,\infty)$ has been investigated by Everitt [**9**], Atkinson [**1**], Harris [**10**], Kaper and Kwong [**11**], and Bennewitz [**3**] amongst others. At least, Bennewitz's proof extends to complex potentials with hardly any change. We repeat it here for easy reference.

Suppose that $q \in L^1([0,\infty))$ is compactly supported so that it is of Class I (that is, $-y'' + qy = \lambda y$ never has two square integrable solutions). For $\Im(z) \geqslant 0$ and $x \in [0,\infty)$ define

$$q_1(z,x) = \int_x^\infty e^{2iz(t-x)} q(t)\, dt$$

and

$$a(z,x) = \sup\{|q_1(z,t)| : t \geqslant x\}.$$

Furthermore, for $j \in \mathbb{N}$, let

$$q_{j+1}(z,x) = \int_x^\infty e^{2iz(t-x)-2f_j(z,t,x)} q_j(z,t)^2\, dt$$

where

$$f_j(z,t,x) = \sum_{n=1}^{j} \int_x^t q_n(z,y)\, dy.$$

The Riemann–Lebesgue lemma shows that $q_1(z,x)$ and $a(z,x)$ tend to zero uniformly in $x$ as $z$ tends to infinity in the closed upper half plane. Also, the support of $a(z,\cdot)$ is contained in the support of $q$.

Next one proves by induction that

$$|q_j(z,x)| \leqslant \Im(z) \left(\frac{a(z,x)}{\Im(z)}\right)^{2^{j-1}} \tag{B.1}$$

provided that $a(z,x)/\Im(z) \leqslant 1/3$, a condition which is satisfied whenever $\Im(z) \geqslant \varepsilon > 0$ and $|z|$ is bigger than a certain constant (depending on $\varepsilon$). This induction proof uses the fact that

$$|f_{n-1}(z,t,x)| \leqslant (t-x)\Im(z) \sum_{k=1}^{n-1} 3^{-2^{k-1}} \leqslant \tfrac{1}{2}\Im(z)(t-x).$$

Note that equation (B.1) implies that

$$\left|\sum_{j=1}^{\infty} q_j(z,x)\right| \leqslant \frac{a(z,x)}{1 - a(z,x)/\Im(z)} \leqslant \frac{3}{2} a(z,x).$$

Now define $\mu(z,x) = iz - \sum_{j=1}^{\infty} q_j(z,x)$. One shows that this series can be differentiated with respect to $x$ term by term and that $\mu(z,\cdot)$ satisfies the Riccati equation

$$\mu'(z,x) + \mu(z,x)^2 = q(x) - z^2.$$

Therefore $\psi(z,x) = \exp(\int_0^x \mu(z,t)\,dt)$ satisfies the differential equation $-y'' + qy = z^2 y$. Also $\psi(z,\cdot)$ is square integrable since $\mu(z,t) = iz$ when $t$ is outside the support of $q$. Hence $m(z^2) = \psi'(z,0)/\psi(z,0) = \mu(z,0)$.

Now suppose that $q \in \mathcal{Q}_\Sigma$ is only locally integrable but still of Class I. Let $\Lambda$ be a half plane establishing that fact. Fix a positive number $a$ and define $\tilde{q}$ by $\tilde{q}(x) = q(x)\chi_{[0,a]}(x)$. A moments thought reveals that $\tilde{q}$ is also in $\mathcal{Q}_\Sigma$ since $\tilde{\Lambda}$ may be chosen as a subset of $\Lambda$ which does not intersect $[0,\infty)$. The following statement on the associated $m$-functions $m$ and $\tilde{m}$ was shown in [**7**]. There is a constant $C$ such that

$$|m(\lambda) - \tilde{m}(\lambda)| \leqslant C\exp(-a\Im(\sqrt{\lambda}))$$

whenever $\lambda$ tends to infinity along a ray which eventually lies in $\Lambda$ but is not parallel to the boundary of $\Lambda$ and where the branch of the root is chosen so that $\Im(\sqrt{\lambda})$ is positive.

Combining this with the previous result, we finally arrive at the following theorem.

THEOREM 6. *Suppose that $q \in \mathcal{Q}_\Sigma$ is of Class I and $\mathcal{R}$ is a ray which eventually lies in $\Lambda$ but is not parallel to the boundary of $\Lambda$. Then*

$$m(z^2) = iz + o(1), \qquad \Im(z) > 0$$

*as $z^2$ tends to infinity along $\mathcal{R}$.*

## *References*


1. F. V. ATKINSON, 'On the location of the Weyl circles', *Proc. Roy. Soc. Edinburgh Sect.* A 88 (1981) 345–356.
2. A. I. BAZ, 'Some topics in the theory of quasi-stationary states', *Razlozheniya po sobstvennym funktsiyam, svyazannye s differentsialnymi uravneniyami vtorogo poryadka. Chast* I (Izdat. Inostr. Lit., Moscow, 1960) Appendix IX (translation from the English by V. B. Lidskĭ of the book by E. C. Titchmarsh, translation edited by B. M. Levitan).
3. C. BENNEWITZ, 'A note on the Titchmarsh–Weyl *m*-function', Preprint ANL-87-26, Argonne National Laboratory, IL, USA, 1988.
4. C. BENNEWITZ, 'A proof of the local Borg–Marchenko theorem', *Comm. Math. Phys.* 218 (2001) 131–132.
5. W. BREIT and E. WIGNER, 'Capture of slow neutrons', *Phys. Rev.* 49 (1936) 519–531.
6. B. M. BROWN, D. K. R. MCCORMACK, W. D. EVANS and M. PLUM, 'On the spectrum of second-order differential operators with complex coefficients', *Proc. Roy. Soc. London Ser.* A 455 (1999) 1235–1257.
7. B. M. BROWN, R. PEACOCK and R. WEIKARD, 'A local Borg–Marchenko theorem for complex potentials', *J. Comput. Appl. Math.* 148 (2002) 115–131.
8. J. B. CONWAY, *Functions of one complex variable*, 2nd edn (Springer, New York, 1978).
9. W. N. EVERITT, 'On a property of the *m*-coefficient of a second-order linear differential equation', *J. London Math. Soc.* (2) 4 (1971/72) 443–457.
10. B. J. HARRIS, 'An exact method for the calculation of certain Titchmarsh–Weyl *m*-functions', *Proc. Roy. Soc. Edinburgh Sect.* A 106 (1987) 137–142.
11. H. G. KAPER and M. K. KWONG, 'Asymptotics of the Titchmarsh–Weyl *m*-coefficient for integrable potentials', *Proc. Roy. Soc. Edinburgh Sect.* A 103 (1986) 347–358.
12. P. L. KAPUR and R. PEIERLS, 'The dispersion formula for nuclear reactions', *Proc. Roy. Soc. London Ser.* A 166 (1938) 277–295.
13. E. KOROTYAEV, 'Inverse resonance scattering on the half line', Preprint, University of Potsdam, 2000.
14. V. A. MARCHENKO, 'Reconstruction of the potential energy from the phases of scattered waves', *Dokl. Akad. Nauk SSSR* 104 (1955) 695–698 (Russian).
15. M. MERKLI and I. M. SIGAL, 'A time-dependent theory of quantum resonances', *Comm. Math. Phys.* 201 (1999) 549–576.
16. J. SCHWINGER, 'Field theory of unstable particles', *Ann. Physics* 9 (1960) 169–193.
17. B. SIMON, 'Resonances in one dimension and Fredholm determinants', *J. Funct. Anal.* 178 (2000) 396–420.
18. A. R. SIMS, 'Secondary conditions for linear differential operators of the second order', *J. Math. Mech.* 6 (1957) 247–285.

**19.** V. Weisskopf and E. Wigner, 'Berechnung der natülichen Linienbreite auf Grund de Diracshen Lichttheorie', *Z. Phys.* 63 (1930) 54–73.
**20.** D. V. Widder, *The Laplace transform* (Princeton University Press, Princeton, NJ, 1941).
**21.** M. Zworski, 'Resonances in physics and geometry', *Notices Amer. Math. Soc.* 46 (1999) 319–328.
**22.** M. Zworski, 'A remark on isopolar potentials', *SIAM J. Math. Anal.* 32 (2001) 1324–1326 (electronic).

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