THETA FUNCTIONS ON SINGULAR HYPERELLIPTIC SURFACES

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Abstract. We investigate limits of (renormalized) hyperelliptic Riemann theta functions as two branch points of the underlying curve approach each other rendering a singular curve. Singular Riemann theta functions have applications in completely integrable systems, in particular the KdV hierarchy.

1. Introduction

The integration of the Korteweg-de Vries (KdV) equation

$$q_t = \frac{1}{4} q_{xxx} + \frac{3}{2} q q_x$$

is closely related to the spectral theory of the operator $L = d^2/dx^2 + q$ by way of the Lax representation of the expression $q_{xxx}/4 + 3qq_x/2$ as commutator $[P, L]$ where P is a third order ordinary differential expression whose coefficients are certain polynomials in q and its x-derivatives (Lax [10]). In the case of rapidly decreasing initial data the inverse scattering method produces the famous N-soliton solutions. For the Cauchy problem with periodic initial data a special role is played by stationary solutions of higher order KdV equations, i.e., equations of the form $q_t = [P, L]$ where P is now a suitable odd order differential expression. Novikov [12] showed in 1974 that a real-valued stationary solution of a higher order KdV equation is a finite-band potential, i.e., a potential for which the spectrum of the L^2-operator associated with L consists of a finite number of closed intervals. Shortly thereafter Dubrovin [3] proved also the converse, i.e., that a real-valued finite-band potential is a stationary solution of some higher order KdV equation. Already in 1961 Akhiezer [1] had reduced the description of a one-band potential to the Jacobi inversion problem. This inspired Dubrovin [3], [4] and Its and Matveev [8] in the mid 1970's to treat the case of a general real-valued finite-band potential. Their result may be summarized as follows: suppose we are given a periodic initial condition $q_0 = q(\cdot, 0)$ of the KdV equation for which the spectrum of $d^2/dx^2 + q_0$ is given by

$$(\infty, E_{2g}] \cup ... \cup [E_1, E_0]$$

where $E_{2g} < ... < E_0$. Let θ denote the Riemann theta function associated with the nonsingular hyperelliptic curve

$$M = \{(\lambda, \mu) : \mu^2 = \prod_{j=0}^{2g} (\lambda - E_j)\}.$$

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Then one may determine $U, V, D_0 \in \mathbb{C}^g$ and $c \in \mathbb{R}$ such that

$$q(x) = c + 2\frac{d^2}{dx^2} \log \theta(Ux + Vt + D_0) \tag{1}$$

satisfies the KdV equation and the initial condition $q(x, 0) = q_0(x)$. Also important is the so called Baker-Akhiezer function

$$\phi(Q, x) = \frac{\theta(\int_{Q_0}^{Q} \omega + Ux + D)}{\theta(Ux + D)} \exp(x\Omega(Q)) \tag{2}$$

where $Q = (\mu, \lambda)$, $D = D_0 + Vt$, ω is the vector of normalized holomorphic differentials on M, and Ω is a certain abelian integral of the second kind on M. Indeed, $\phi((\mu, \lambda), x)$ and $\phi((-\mu, \lambda), x)$ form a fundamental system of solutions of the differential equation $y'' + qy = \lambda y$ provided λ is none of the branch points of μ.

The simplest and most famous examples of finite-band potentials are given by the Lamé potentials

$$q(x) = -g(g + 1)\wp(x + \omega')$$

where g is a natural number and $\wp$ denotes Weierstrass' elliptic function with fundamental half periods $\omega \in \mathbb{R}$ and $\omega' \in i\mathbb{R}$ or, equivalently, with real invariants g_2 and g_3 satisfying $g_2^3 - 27g_3^2 > 0$. (These conditions guarantee that q is real-valued.) It is now fairly obvious that q satisfies a higher KdV equation even if these conditions are not satisfied. Since q ceases to be real-valued the spectrum of the associated operator need not be a subset of the real line anymore. However, the spectrum consists of a finite number of analytic arcs (Rofe-Beketov [13]) and one may still talk about finite-band potentials. A particularly interesting example is given by

$$q(x) = -6\wp(x + \omega').$$

If $0 < b < \sqrt{3}a$ then $g_2 = 12a^2 - 4b^2$ and $g_3 = 8a(a^2 + b^2)$ are positive but $g_2^3 - 27g_3^2$ is negative. In this case the spectrum of L is given by the union of the intervals $(-\infty, -6a]$, $[-\sqrt{3g_2}, \sqrt{3g_2}]$ and an arc joining the points $-a \pm ib$ and intersecting the interval $[-\sqrt{3g_2}, \sqrt{3g_2}]$. The results of Its and Matveev generalize immediately to this case, i.e., q can be written as the second logarithmic derivative of a theta function associated with the hyperelliptic curve

$$\mu^2 = (\lambda^2 - 3g_2)(\lambda - 2a)(\lambda + a - ib)(\lambda + a + ib)$$

and the solutions of the differential equation $y'' + qy = \lambda y$ are given by the branches of the Baker-Akhiezer function. Now, if b tends to $\sqrt{3}a$, i.e., if g_2 tends to zero the spectrum of L will consists only of two arcs namely the interval $(-\infty, -6a]$ and an arc joining the points $-a \pm ib$ and passing through zero. However q can not be represented as the second logarithmic derivative of a theta function associated with the elliptic curve

$$\mu^2 = (\lambda - 2a)(\lambda + a - ib)(\lambda + a + ib).$$

Hence an immediate generalization of the Its-Matveev theorem to the case of complex-valued potentials is not possible.

If q is a locally integrable periodic function with period p the band edges of $L = d^2/dx^2 + q$ are determined with the aid of Floquet theory. Let $c(E, x_0, \cdot)$ denote the solution of the equation $Ly = Ey$ under the initial conditions $y(x_0) = 1$, $y'(x_0) = 0$ and $s(E, x_0, \cdot)$ the solution for $y(x_0) = 0$, $y'(x_0) = 1$. Then define the Floquet discriminant

$$D(E) = c(E, x_0, x_0 + p) + s'(E, x_0, x_0 + p).$$

When q is real-valued all zeros of D^2-4 are either simple or double and the number of linearly independent Floquet solutions of $Ly = Ey$ is one or two, respectively. Away from the double zeros Floquet solutions vary continuously with E. When E_0 is a double zero then each of the two linearly independent Floquet solutions of $Ly = E_0 y$ which may be obtained as limits of certain Floquet solutions nearby. Such solutions have been called regular Floquet solutions in [15]. Hence, for real-valued q the set of regular Floquet solutions of $Ly = Ey$ is a line bundle over the curve obtained from a complete desingularization of D^2-4. In particular, if D^2-4 has only finitely many simple zeros this curve is hyperelliptic. The situation is somewhat different for complex-valued potentials. First of all D^2-4 may have zeros of any order. Secondly, even if E is a higher order zero of D^2-4 there might not be two linearly independent Floquet solutions of $Ly = Ey$. Finally, even if every solution of $Ly = Ey$ is Floquet it might happen that only one of them (and its multiples) is a regular Floquet solution (see [15] for more details). Therefore the set of regular Floquet solutions of a complex-valued finite-band potential is not a line bundle over a nonsingular curve but only over a singular one. This is precisely what happens in the case of $q = -6\wp$ when $g_2 = 0$, $g_3 = 8a(a^2 + b^2)$, and $b^2 = 3a^2$. At the point $E = 0$ which is not a band edge there exists only one Floquet solution and therefore the set of regular Floquet solutions is a line bundle over the singular curve

$$\{(\lambda, \mu) : \mu^2 = \lambda^2(\lambda - 2a)(\lambda + a - ib)(\lambda + a + ib)\}.$$

If q is an algebro-geometric potential associated with a nonsingular surface the Its-Matveev theorem shows that for all complex values of E all solutions of $Ly = Ey$ are meromorphic functions of the independent variable. It was shown in [6] that the converse is also true at least for elliptic potentials. More precisely, if q is an elliptic function and if, for all complex values of E, all solutions of $Ly = Ey$ are meromorphic functions of the independent variable then q is algebro-geometric. This result was extended in [16] to rational and simply periodic meromorphic potentials under certain boundedness conditions at infinity. Also, in [7] an analogous result was obtained for the AKNS-hierarchy with elliptic potentials.

In order to generalize the Its-Matveev theorem to the case of singular curves a notion of theta functions on singular surfaces is needed. McKean [11] did this in the case of a real-valued periodic potential for the case by shrinking simultaneously the gaps $[E_{2j}, E_{2j-1}]$ to points. The goal of the present paper is to treat the case of any complex curve where two points converge to form a double point.

In Section 2 we will investigate the period matrices associated with holomorphic differentials on M. We turn to the theta functions themselves in Section 3.

2. The period matrices

Consider the one parameter family of hyperelliptic curves of genus $g \geq 1$

$$M_\epsilon = \left\{ (\mu, \lambda) : \mu^2 = \prod_{i=1}^{2g+1} (\lambda - e_i(\epsilon)) \right\},$$

where each $e_i(\epsilon)$ is a continuous function on $[0,1]$ such that $e_j(s) \neq e_k(t)$ when $j \neq k$ and $s, t \in [0,1]$ except that $e_{2g+1}(0) = e_{2g}(0)$.

There is a δ small enough such that the closed disks of radius δ about the points $e_1(0)$, ..., $e_{2g}(0)$ do not intersect but large enough such that for all suitably small ϵ

$$|e_j(\epsilon) - e_j(0)| < \delta/2$$

for $j = 1, ..., 2g+1$. On each circle $|\lambda - e_j(0)| = \delta$ choose a point η_j. Then, for $1 \le j \le 2g$, let $\tilde{\gamma}_{j,\epsilon}$ be a straight line connecting $e_j(\epsilon)$ and η_j, i.e.,

$$\tilde{\gamma}_{j,\epsilon}(t) = e_j(\epsilon) + t(\eta_j - e_j(\epsilon)),$$

and, for $1 \le j \le 2g-1$, let γ_j be a piecewise continuously differentiable curve connecting η_j to η_{j+1} taking its course outside all of the disks $|\lambda - e_\ell(0)| < \delta$. The curves γ_j may be chosen in such a way that they do not intersect each other. Next define

$$\begin{aligned} \alpha_{j,\epsilon} &= \tilde{\gamma}_{2j,\epsilon} + \gamma_{2j} - \tilde{\gamma}_{2j+1,\epsilon} \quad \text{for } j = 1, ..., g-1, \\ \beta_{j,\epsilon} &= \tilde{\gamma}_{2j,\epsilon} - \gamma_{2j-1} - \tilde{\gamma}_{2j-1,\epsilon} \quad \text{for } j = 1, ..., g. \end{aligned}$$

Finally let $\alpha_{g,\epsilon}$ be given by

$$\alpha_{g,\epsilon}(t) = e_{2g}(\epsilon) + t(e_{2g+1}(\epsilon) - e_{2g}(\epsilon)).$$

Then the collection of cycles

$$a_{j,\epsilon} = \sum_{k=j}^{g} (\alpha_k - \Gamma\alpha_k), \quad b_{j,\epsilon} = \beta_j - \Gamma\beta_j, \tag{3}$$

where Γ is the hyperelliptic involution $\Gamma(\mu,\lambda) = (-\mu,\lambda)$, forms a canonical homology basis on M_ϵ.

The differentials

$$\omega_j(\epsilon) = \frac{(\lambda - e_{2g}(\epsilon))^{g-j}}{\mu_\epsilon(\lambda)} d\lambda, \quad j = 1, ..., g$$

where

$$\mu_\epsilon(\lambda) = \prod_{\ell=1}^{2g+1} \sqrt{\lambda - e_\ell(\epsilon)}$$

are a basis of the g-dimensional space of holomorphic differentials on M_ϵ.

The curve

$$\hat{M} = \left\{ (\mu,\lambda) : \mu^2 = \prod_{i=1}^{2g-1} (\lambda - e_i(0)) \right\},$$

is the desingularization of M_0. The cycles $a_{1,0}$, ..., $a_{g-1,0}$, $b_{1,0}$, ..., $b_{g-1,0}$ form a canonical homology basis on $\hat{M}$. A basis of the $g-1$-dimensional space of holomorphic differentials on $\hat{M}$ is given by

$$\hat{\omega}_j = \frac{(\lambda - e_{2g}(0))^{g-1-j}}{\hat{\mu}(\lambda)} d\lambda = \omega_j(0), \quad j = 1, ..., g-1$$

where

$$\hat{\mu}(\lambda) = \prod_{\ell=1}^{2g-1} \sqrt{\lambda - e_\ell(0)}.$$

We will prove the following

Theorem 1. *The matrix $A(\epsilon)$ of a-periods $A_{j,k}(\epsilon) = \int_{a_{k,\epsilon}} \omega_j(\epsilon)$ and the matrix $B(\epsilon)$ of b-periods $B_{j,k}(\epsilon) = \int_{b_{k,\epsilon}} \omega_j(\epsilon)$ of the holomorphic differentials $\omega_j(\epsilon)$ have* *finite limits as ϵ tends to zero with the exception of $B_{g,g}(\epsilon)$ which becomes infinite. In particular,*

$$\lim_{\epsilon\to 0} A_{j,k}(\epsilon) = \int_{a_{k,0}} \omega_j(0) = \int_{a_{k,0}} \hat{\omega}_j, \quad \text{if } 1 \le j,k \le g-1, \tag{4}$$

$$\lim_{\epsilon\to 0} A_{j,g}(\epsilon) = 0, \quad \text{if } 1 \le j \le g-1, \tag{5}$$

$$\lim_{\epsilon\to 0} A_{g,k}(\epsilon) = \sum_{n=k}^{g-1} \int_{\alpha_{n,0}} \omega_g(0) - \frac{2i\pi}{\hat{\mu}(e_{2g}(0))} \quad \text{if } 1 \le k \le g, \tag{6}$$

$$\lim_{\epsilon\to 0} B_{j,k}(\epsilon) = \int_{b_{k,0}} \omega_j(0), \quad \text{if } (j,k) \ne (g,g). \tag{7}$$

Moreover, if

$$r(\epsilon) = \frac{e_{2g}(\epsilon) - e_{2g+1}(\epsilon)}{\eta_{2g} - e_{2g}(\epsilon)}$$

then, as ϵ tends to zero,

$$A_{j,g}(\epsilon) = O(r(\epsilon)^{g-j}), \tag{8}$$

$$A_{g,g}(\epsilon) = \frac{-2\pi i}{\prod_{\ell=1}^{2g-1} \sqrt{e_{2g}(\epsilon) - e_\ell(\epsilon)}} + O(r(\epsilon)), \tag{9}$$

$$B_{g,g}(\epsilon) = -2\frac{\log r(\epsilon)}{\prod_{\ell=1}^{2g-1} \sqrt{e_{2g}(\epsilon) - e_\ell(\epsilon)}} + O(1). \tag{10}$$

Proof. If λ is a point on any of the curves γ_n then $|\lambda - e_\ell(\epsilon)| > \delta/2$. Since all curves remain in a bounded set and $|\gamma_n'(t)|$ is bounded, too, we get that

$$\left| \frac{(\gamma_n(t) - e_{2g}(\epsilon))^{g-j}\gamma_n'(t)}{\mu_\epsilon(\gamma_n(t))} \right|$$

is bounded by a constant independent of t and ϵ. Hence the dominated convergence theorem implies that for $n \in \{1, ..., 2g-1\}$,

$$\lim_{\epsilon\to 0} \int_{\gamma_n} \omega_j(\epsilon) = \int_0^1 \frac{(\gamma_n(t) - e_{2g}(\epsilon))^{g-j}\gamma_n'(t)}{\mu_\epsilon(\gamma_n(t))} dt = \int_{\gamma_n} \omega_j(0).$$

If λ is a point on the curve $\tilde{\gamma}_{n,\epsilon}$ where $1 \le n \le 2g-1$ then $|\lambda - e_\ell(\epsilon)| > \delta/2$ if $n \ne \ell$. Also $|\tilde{\gamma}_{n,\epsilon}(t) - e_n(\epsilon)| = t|\eta_n - e_n(\epsilon)| \ge t\delta/2$. Since $t^{-1/2}$ is integrable on $[0,1]$,

$$\lim_{\epsilon\to 0} \int_{\tilde{\gamma}_{n,\epsilon}} \omega_j(\epsilon) = \int_0^1 \frac{(\tilde{\gamma}_{n,\epsilon}(t) - e_{2g}(\epsilon))^{g-j}(\eta_n - e_n(\epsilon))}{\mu_\epsilon(\tilde{\gamma}_{n,\epsilon}(t))} dt = \int_{\tilde{\gamma}_{n,0}} \omega_j(0).$$

For the curve $\alpha_{g,\epsilon}$ we have the estimate

$$|\mu_\epsilon(\alpha_{g,\epsilon}(t))| = \prod_{\ell=1}^{2g+1} \sqrt{|\alpha_{g,\epsilon}(t) - e_\ell(\epsilon)|} \ge |e_{2g+1}(\epsilon) - e_{2g}(\epsilon)|\sqrt{t(t-1)} \left(\frac{\delta}{2}\right)^{g-1/2}.$$

Because $(t(t-1))^{-1/2}$ is integrable on $[0,1]$ we obtain

$$\left|\int_{\alpha_{g,\epsilon}} \omega_j(\epsilon)\right| \le \int_0^1 \left|\frac{(e_{2g+1}(\epsilon)-e_{2g}(\epsilon))^{g-j+1}t^{g-j}}{\mu_\epsilon(\alpha_{g,\epsilon}(t))}\right| dt$$
$$= \left(\frac{2}{\delta}\right)^{g-1/2} |e_{2g+1}(\epsilon)-e_{2g}(\epsilon)|^{g-j}\int_0^1 \frac{t^{g-j}}{\sqrt{t(t-1)}}dt = O(r(\epsilon)^{g-j})$$

which proves (8).

Since

$$\int_{a_{k,\epsilon}} \omega_j(\epsilon) = 2\sum_{n=k}^{g}\int_{\alpha_{n,\epsilon}} \omega_j(\epsilon) \text{ and } \int_{b_{k,\epsilon}} \omega_j(\epsilon) = 2\int_{\beta_{k,\epsilon}} \omega_j(\epsilon)$$

we have also proven (4), (5), and, as long as $1 \le k \le g-1$, (7).

To treat $A_{g,k}$ consider

$$\Delta_1(\epsilon) = \left|\int_{\alpha_{g,\epsilon}} \omega_g(\epsilon) - \int_{\alpha_{g,\epsilon}} \frac{d\lambda}{F_1(\epsilon,0)\sqrt{(\lambda-e_{2g}(\epsilon))(\lambda-e_{2g+1}(\epsilon))}}\right|$$

where

$$F_1(\epsilon,t)^2 = \prod_{\ell=1}^{2g-1}(\alpha_{g,\epsilon}(t)-e_\ell(\epsilon)).$$

This yields

$$\Delta_1(\epsilon) \le \int_0^1 \left|\frac{e_{2g+1}(\epsilon)-e_{2g}(\epsilon)}{\sqrt{(\alpha_{g,\epsilon}(t)-e_{2g}(\epsilon))(\alpha_{g,\epsilon}(t)-e_{2g+1}(\epsilon))}}\left(\frac{1}{F_1(\epsilon,t)}-\frac{1}{F_1(\epsilon,0)}\right)\right| dt$$
$$= \int_0^1 \frac{1}{\sqrt{t(t-1)}}\left|\frac{1}{F_1(\epsilon,t)}-\frac{1}{F_1(\epsilon,0)}\right| dt$$

The function $1/F_1(\epsilon,\cdot)$ is continuous on $[0,1]$ and differentiable on $(0,1)$. Therefore (see Theorem 5.19 of Rudin [14])

$$\left|\frac{1}{F(\epsilon,t)}-\frac{1}{F(\epsilon,0)}\right| \le tS_1(\epsilon)$$

where

$$S_1(\epsilon) = \frac{2g-1}{\delta}\left(\frac{2}{\delta}\right)^{g-1/2}|e_{2g+1}(\epsilon)-e_{2g}(\epsilon)| \ge \sup\{\left|\frac{F_1'(\epsilon,s)}{F_1(\epsilon,s)^2}\right| : s\in[0,1]\}.$$

Since $S_1(\epsilon) = O(r(\epsilon))$

$$\Delta_1(\epsilon) \le S_1(\epsilon)\int_0^1 \frac{tdt}{\sqrt{t(t-1)}} = O(r(\epsilon)).$$

Finally note that

$$\int_{\alpha_{g,\epsilon}} \frac{d\lambda}{F_1(\epsilon,0)\sqrt{(\lambda-e_{2g}(\epsilon))(\lambda-e_{2g+1}(\epsilon))}} = \frac{-i\pi}{F_1(\epsilon,0)}$$

which tends to $-i\pi/F_1(0,0)$ as ϵ tends to zero. This proves (6) and (9).

The only integrals left to consider are the integrals $B_{j,g}$ which involve the curve $\tilde\gamma_{2g,\epsilon}$. If $1 \le j \le g-1$, we will estimate the difference $\int_{\tilde\gamma_{2g,\epsilon}} \omega_j(\epsilon) - \int_{\tilde\gamma_{2g,0}} \omega_j(0)$. Introduce

$$
\begin{aligned}
F_2(\epsilon,t)^2 &= \prod_{\ell=1}^{2g-1} (\tilde\gamma_{2g,\epsilon}(t) - e_\ell(\epsilon)),\\
r_1(\epsilon) &= \frac{e_{2g}(0)-e_{2g}(\epsilon)}{\eta_{2g}-e_{2g}(0)},\\
r_2(\epsilon,t) &= \frac{F_2(\epsilon,t)}{F_2(0,t)} - 1 = \frac{F_2(\epsilon,t)-F_2(0,t)}{F_2(0,t)}.
\end{aligned}
$$

Note that $|F_2(\epsilon,t)| \ge (\delta/2)^{g-1/2}$. Now we obtain

$$
\begin{aligned}
&\left| \int_{\tilde\gamma_{2g,\epsilon}} \omega_j(\epsilon) - \int_{\tilde\gamma_{2g,0}} \omega_j(0) \right|\\
&\le \int_0^1 \left| \frac{(\eta_{2g}-e_{2g}(\epsilon))^{g-j} t F_2(0,t) - (\eta_{2g}-e_{2g}(0))^{g-j} F_2(\epsilon,t)\sqrt{t(t+r(\epsilon))}}{\sqrt{t(t+r(\epsilon))}F_2(\epsilon,t)tF_2(0,t)} \right| t^{g-j} dt\\
&\le \left(\frac{2}{\delta}\right)^{g-1/2} |\eta_{2g}-e_{2g}(0)|^{g-j} \int_0^1 \left| \frac{\sqrt{t}(1+r_1(\epsilon))^{g-j} - \sqrt{t+r(\epsilon)}(1+r_2(\epsilon,t))}{\sqrt{t+r(\epsilon)}} \right| dt.
\end{aligned}
$$

Given any positive number ε' we obtain that

$$
|r(\epsilon)|^{1/2}, |(1+r_1(\epsilon))^{g-j} - 1|, |r_2(\epsilon,t)| \le \varepsilon'
$$

for all $t \in [0,1]$ and all suitably small ϵ since $F_2(\epsilon,t)$ converges uniformly on $[0,1]$ to $F_2(0,t)$ and $e_{2g}(\epsilon)$ converges to $e_{2g}(0)$ as ϵ tends to zero. Hence

$$
\left| \int_{\tilde\gamma_{2g,\epsilon}} \omega_j(\epsilon) - \int_{\tilde\gamma_{2g,0}} \omega_j(0) \right| \le C \int_0^1 \left| \frac{\sqrt{t}-\sqrt{t+r(\epsilon)}}{\sqrt{t+r(\epsilon)}} \right| dt + C\varepsilon' \int_0^1 \frac{dt}{\sqrt{|t+r(\epsilon)|}}
$$

for a suitable constant C. The latter integral is bounded. To estimate the former integral note that

$$
\int_0^{|r|} \left| \frac{\sqrt{t}-\sqrt{t+r}}{\sqrt{t+r}} \right| dt \le 3|r|^{1/2} \int_0^{|r|} \frac{dt}{\sqrt{|r|-t}}
$$

and that

$$
\int_{|r|}^1 \left| \frac{\sqrt{t}-\sqrt{t+r}}{\sqrt{t+r}} \right| dt \le \int_{|r|}^1 \sqrt{\frac{t}{t-|r|}} \left| \sqrt{1+\frac{r}{t}} - 1 \right| dt \le 3|r|^{1/2} \int_{|r|}^1 \frac{dt}{\sqrt{t-|r|}}
$$

since $|\sqrt{1+x}-1| \le 3x$. These estimates show that the difference $\int_{\tilde\gamma_{2g,\epsilon}} \omega_j(\epsilon) - \int_{\tilde\gamma_{2g,0}} \omega_j(0)$ becomes arbitrarily small when ϵ becomes small provided $1 \le j \le g-1$. This completes the proof of (7).

At last we turn to (10),i.e., to the integral $\int_{\tilde\gamma_{2g,\epsilon}} \omega_g(\epsilon)$ which, as will be shown now, tends to infinity as ϵ goes to zero. To see this consider

$$
\begin{aligned}
\Delta_2(\epsilon) &= \left| \int_{\tilde\gamma_{2g,\epsilon}} \omega_g(\epsilon) - \int_{\tilde\gamma_{2g,\epsilon}} \frac{d\lambda}{F_2(\epsilon,0)\sqrt{(\lambda-e_{2g}(\epsilon))(\lambda-e_{2g+1}(\epsilon))}} \right|\\
&\le \int_0^1 \left| \frac{\eta_{2g}-e_{2g}(\epsilon)}{(\tilde\gamma_{2g,\epsilon}(t)-e_{2g}(\epsilon))(\tilde\gamma_{2g,\epsilon}(t)-e_{2g+1}(\epsilon))} \left(\frac{1}{F_2(\epsilon,t)} - \frac{1}{F_2(\epsilon,0)} \right) \right| dt.
\end{aligned}
$$

Again, the function $1/F_2(\epsilon,\cdot)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ implying

$$\left|\frac{1}{F_2(\epsilon,t)}-\frac{1}{F_2(\epsilon,0)}\right|\le tS_2$$

where

$$S_2=2(2g-1)\left(\frac{2}{\delta}\right)^{g-1/2}\ge\sup\{\left|\frac{F_2'(\epsilon,s)}{F_2(\epsilon,s)^2}\right|:s\in[0,1]\}.$$

Hence

$$\Delta_2(\epsilon)\le S_2\int_0^1\frac{tdt}{\sqrt{t|r(\epsilon)+t|}},$$

i.e., $\Delta_2(\epsilon)$ is bounded by a constant. Finally, we remark that

$$\begin{aligned}\int_{\tilde\gamma_{2g,\epsilon}}\frac{d\lambda}{F_2(\epsilon,0)\sqrt{(\lambda-e_{2g}(\epsilon))(\lambda-e_{2g+1}(\epsilon))}}&=\frac{1}{F_2(\epsilon,0)}\int_0^1\frac{dt}{t(t+r(\epsilon))}\\&=\frac{2\log(1+\sqrt{1+r(\epsilon)})-\log r(\epsilon)}{F_2(\epsilon,0)}\end{aligned}$$

which tends to infinity as ϵ tends to zero. □

Introduce the notation

$$A(\epsilon)=\begin{pmatrix}\hat A(\epsilon)&\vec a_1(\epsilon)\\\vec a_2(\epsilon)^T&A_{g,g}(\epsilon)\end{pmatrix}\quad\text{and}\quad B(\epsilon)=\begin{pmatrix}\hat B(\epsilon)&\vec b_1(\epsilon)\\\vec b_2(\epsilon)^T&B_{g,g}(\epsilon)\end{pmatrix}$$

where $\hat A(\epsilon)$ and $\hat B(\epsilon)$ are $(g-1)\times(g-1)$ matrices and where $\vec a_1$, $\vec a_2$, $\vec b_1$, and $\vec b_2$ are columns in $\mathbb{C}^{g-1}$. Because $\vec a_1(\epsilon)=O(r(\epsilon))$ and $\hat A(0)$ is invertible as a period matrix of the holomorphic differentials $\hat\omega_j$ on the Riemann surface $\hat M$ we find that

$$\det A(\epsilon)=A_{g,g}(\epsilon)\det\hat A(\epsilon)+O(r(\epsilon))=A_{g,g}(\epsilon)\det\hat A(\epsilon)(1+O(r(\epsilon))).$$

This implies the existence of $A(0)^{-1}$. Let

$$A(\epsilon)^{-1}=\begin{pmatrix}Z(\epsilon)&\vec z_1(\epsilon)\\\vec z_2(\epsilon)^T&Z_{g,g}(\epsilon)\end{pmatrix}$$

where again $Z(\epsilon)$ is a $(g-1)\times(g-1)$ matrix while $\vec z_1(\epsilon)$ and $\vec z_2(\epsilon)$ are a columns in $\mathbb{C}^{g-1}$. Then

$$Z_{g,g}(\epsilon)=\frac{\det\hat A_{g,g}(\epsilon)}{\det A_{g,g}(\epsilon)}=\frac{1}{A_{g,g}(\epsilon)}+O(r(\epsilon))$$

and

$$\vec z_1(\epsilon)=O(r(\epsilon)).$$

The normalized holomorphic differentials ζ_1, ..., ζ_g which are determined by the normalization conditions

$$\int_{a_{k,\epsilon}}\zeta_j(\epsilon)=\delta_{k,j}$$

are given by

$$(\zeta_1(\epsilon),...,\zeta_g(\epsilon))^T=A(\epsilon)^{-1}(\omega_1(\epsilon),...,\omega_g(\epsilon))^T.$$

The b-periods of the normalized holomorphic differentials are then given by $\tau_{j,k}(\epsilon)=\int_{b_{k,\epsilon}}\zeta_j(\epsilon)$. These comprise a matrix $\tau(\epsilon)$ in the g-dimensional Siegel upper half space, i.e., a symmetric $g \times g$ matrix with positive definite imaginary part. The matrix $tau(\epsilon)$ is given in terms of $A(\epsilon)$ and $B(\epsilon)$ by

$$\begin{aligned}\tau = A(\epsilon)^{-1}B(\epsilon) &= \begin{pmatrix} Z(\epsilon) & \vec{z}_1(\epsilon) \\ \vec{z}_2(\epsilon) & Z_{g,g}(\epsilon) \end{pmatrix} \begin{pmatrix} \hat{B}(\epsilon) & \vec{b}_1(\epsilon) \\ \vec{b}_2(\epsilon) & B_{g,g}(\epsilon) \end{pmatrix} \\ &= \begin{pmatrix} Z(\epsilon)\hat{B}(\epsilon) + \vec{z}_1(\epsilon)\vec{b}_2(\epsilon)^T & Z(\epsilon)\vec{b}_1(\epsilon) + \vec{z}_1(\epsilon)B_{g,g}(\epsilon) \\ \vec{z}_2(\epsilon)^T\hat{B}(\epsilon) + Z_{g,g}(\epsilon)\vec{b}_2(\epsilon)^T & \vec{z}_2(\epsilon)^T\vec{b}_1(\epsilon) + Z_{g,g}(\epsilon)B_{g,g}(\epsilon) \end{pmatrix}.\end{aligned}$$

This shows that the entries in the first $g-1$ columns of $\tau(\epsilon)$ all have finite limits as ϵ tends to zero. Since $\tau(\epsilon)$ is symmetric the elements in the top $g-1$ rows of the last column of $\tau(\epsilon)$ are also finite even though they involve $B_{g,g}(\epsilon)$ which becomes infinite when ϵ becomes zero (recall that $\vec{z}_1(0) = 0$). For $\tau_{g,g}(\epsilon)$ we obtain

$$\tau_{g,g}(\epsilon) = \vec{z}_2(\epsilon)^T\vec{b}_1(\epsilon) + Z_{g,g}(\epsilon)B_{g,g}(\epsilon) = \vec{z}_2(\epsilon)^T\vec{b}_1(\epsilon) + \left(\frac{1}{A_{g,g}(\epsilon)} + O(r(\epsilon))\right) B_{g,g}(\epsilon).$$

Since $r(\epsilon)B_{g,g}(\epsilon)$ tends to zero when ϵ goes to zero and

$$\frac{B_{g,g}(\epsilon)}{A_{g,g}(\epsilon)} = \frac{-2\log r(\epsilon) + O(1)}{-2\pi i + O(r(\epsilon))} = i\frac{-\log|r(\epsilon)|}{\pi} + \frac{\arg r(\epsilon)}{\pi} + O(1)$$

We have shown the following

Theorem 2. *The entries $\tau_{j,k}(\epsilon) = \int_{b_{k,\epsilon}} \zeta_j(\epsilon)$ of the matrix of the b-periods of normalized holomorphic differentials on M_ϵ have finite limits as ϵ tends to zero with the exception of $\tau_{g,g}(\epsilon)$ whose imaginary part behaves like*

$$\operatorname{Im}\tau_{g,g}(\epsilon) = \frac{-\log|e_{2g+1}(\epsilon) - e_{2g}(\epsilon)|}{\pi} + O(1).$$

Moreover, the $(g-1)\times(g-1)$ matrix $\hat{\tau}(0) = \hat{A}(0)^{-1}\hat{B}(0)$ which is the upper left corner of $\tau(0)$ is the matrix of b-periods of normalized holomorphic differentials on $\hat{M}$.

3. Theta functions

Let τ be an element of the g-dimensional Siegel upper half space $\mathcal{S}_g$, i.e., τ is a symmetric $g \times g$ matrix whose imaginary part is positive definite. For any such τ and any $n, n' \in \mathbb{R}^g$ the series

$$\theta_g[n,n'](z,\tau) = \sum_{k\in\mathbb{Z}^g} \exp\left(\pi i(k+\frac{n}{2})^T\tau(k+\frac{n}{2})\right)\exp\left(2\pi i(k+\frac{n}{2})^T(z+\frac{n'}{2})\right) \tag{11}$$

converges absolutely and uniformly for (z,τ) in compact subsets of $\mathbb{C}^g \times \mathcal{S}_g$. The function $\theta_g[n,n']$ was introduced by Riemann and is called the first order theta function with characteristic $[n,n']$. The function $\theta_g[0,0]$ is often abbreviated as θ_g.

We will henceforth consider the case of integer characteristics, i.e., we will suppose that $n, n' \in \mathbb{Z}^g$. In this case the θ_g has the following properties. Firstly,

$$\begin{aligned}\theta_g[n,n'](z+m,\tau) &= \exp(\pi i n^T m)\theta_g[n,n'](z,\tau), \\ \theta_g[n,n'](z+\tau m,\tau) &= \exp(-\pi i(2m^T z + m^T\tau m + m^T n')\theta_g[n,n'](z,\tau),\end{aligned}$$

when $m \in \mathbb{Z}^g$. Secondly,

$$\begin{aligned}&\theta_g[n+m, n'+m'](z,\tau)\\ =&\exp(\frac{\pi i}{4}(m^T\tau m + 2m^T(n'+m') + 4m^T z))\theta_g[n,n'](z+\frac{m'}{2}+\tau\frac{m}{2},\tau),\end{aligned}$$

and in particular,

$$\theta_g[n+2m, n'+2m'](z,\tau) = \exp(\pi i n^T m')\theta_g[n,n'](z,\tau).$$

Because of this last equality it is sufficient to assign only the values zero and one to the components of n and n'. Therefore there are precisely 2^{2g} different first order theta functions to consider. Finally, since

$$\theta_g[n,n'](-z,\tau) = \exp(\pi n^T n')\theta_g(z,\tau)$$

the theta functions with integer characteristic may be classified as even or odd depending on whether $n^T n'$ is an even or an odd integer. These and other properties may be found in any standard reference on the Riemann theta function, e.g., Krazer [9], Baker [2], or Farkas and Kra [5].

The matrix of normalized b-periods associated with any algebraic curve is an element of the Siegel upper half space. Hence theta functions $\theta_g[n,n']$ are associated with each such curve. In particular, this is true for the hyperelliptic curves M_ϵ introduced in the previous section. We want to study the behavior of these theta functions as ϵ tends to zero. Introduce the symmetric $(g-1)\times(g-1)$ matrix $\hat\tau(\epsilon)$ and the vector $\gamma(\epsilon) \in \mathbb{C}^{g-1}$ by the equation

$$\tau(\epsilon) = \begin{pmatrix}\hat\tau(\epsilon) & \gamma(\epsilon)\\ \gamma(\epsilon)^T & \tau_{g,g}(\epsilon)\end{pmatrix}.$$

When $\hat k = (k_1, ..., k_{g-1})^T$ and $\hat n = (n_1, ..., n_{g-1})^T$ then

$$\begin{aligned}&(k+\frac{n}{2})^T\tau(\epsilon)(k+\frac{n}{2})\\ =&(\hat k+\frac{\hat n}{2})^T\hat\tau(\epsilon)(\hat k+\frac{\hat n}{2}) + 2(\hat k+\frac{\hat n}{2})^T\gamma(\epsilon)(k_g+\frac{n_g}{2}) + (k_g+\frac{n_g}{2})^2\tau_{g,g}(\epsilon).\end{aligned}$$

Since $\operatorname{Im}\tau_{g,g}(\epsilon)$ tends to infinity we have that

$$\exp(\pi i(k+\frac{n}{2})^T\tau(\epsilon)(k+\frac{n}{2}))$$

tends to zero as ϵ tends to zero unless $k_g + n_g/2 = 0$. Hence, if $n_g = 0$ all summands in (11) where $k_g \neq 0$ vanish and hence $\theta_g[n,n'](z,\tau(\epsilon))$ converges to

$$\begin{aligned}&\sum_{\hat k\in\mathbb{Z}^{g-1}} \exp\left(\pi i(\hat k+\frac{\hat n}{2})^T\tau(\epsilon)(\hat k+\frac{\hat n}{2})\right)\exp\left(2\pi i(\hat k+\frac{\hat n}{2})^T(\hat z+\frac{\hat n'}{2})\right)\\ =&\theta_{g-1}[\hat n,\hat n'](\hat z,\hat\tau(0))\end{aligned}$$

where, of course, $\hat z = (z_1, ..., z_{g-1})^T$ and $\hat n' = (n'_1, ..., n'_{g-1})^T$.

If however, $n_g = 1$ then $\theta_g[n,n']$ tends to zero everywhere when ϵ approaches zero. In this case we will therefore consider the renormalized theta function

$$\exp(-\frac{\pi i}{4}\tau_{g,g}(\epsilon))\theta_g[n,n'](z,\tau(\epsilon)).$$

Note that $(k_g+1/2)^2-1/4=k_g(k_g+1)$ and hence

$$\begin{aligned}&(k+\frac{n}{2})^T\tau(\epsilon)(k+\frac{n}{2})-\frac{1}{4}\tau_{g,g}(\epsilon)\\ =&(\hat{k}+\frac{\hat{n}}{2})^T\hat{\tau}(\epsilon)(\hat{k}+\frac{\hat{n}}{2})+2(\hat{k}+\frac{\hat{n}}{2})^T\gamma(\epsilon)(k_g+\frac{n_g}{2})+k_g(k_g+1)\tau_{g,g}(\epsilon).\end{aligned}$$

Now, only those summands in (11) where $k_g\neq 0,-1$ vanish when ϵ goes to zero. Therefore

$$\begin{aligned}&\lim_{\epsilon\to 0}\exp(-\frac{\pi i}{4}\tau_{g,g}(\epsilon))\theta_g[n,n'](z,\tau)\\ =&\sum_{\hat{k}\in\mathbb{Z}^{g-1}}\exp\left(\pi i[(\hat{k}+\frac{\hat{n}}{2})^T\hat{\tau}(0)(\hat{k}+\frac{\hat{n}}{2})]\right)\exp\left(2\pi i(\hat{k}+\frac{\hat{n}}{2})^T(\hat{z}+\frac{\hat{n}'}{2})\right)\\ &\times\left\{\exp\left(\pi i[(\hat{k}+\frac{\hat{n}}{2})^T\gamma(0)]\right)\exp\left(\pi i(z_g+\frac{n'_g}{2})\right)\right.\\ &\quad\left.+\exp\left(-\pi i(\hat{k}+\frac{\hat{n}}{2})^T\gamma(0)\right)\exp\left(-\pi i(z_g+\frac{n'_g}{2})\right)\right\}.\end{aligned}$$

Putting everything together we have proved the following

Theorem 3. *The theta functions $\theta_g[n,n'](\cdot,\tau(\epsilon))$ with integer characteristics associated with the hyperelliptic Riemann surface M_ϵ and the canonical homology basis given by* (3) *satisfies*

$$\begin{aligned}&\lim_{\epsilon\to 0}\exp(-\frac{\pi i n_g^2\tau_{g,g}(\epsilon)}{4})\theta_g[n,n'](z,\tau)\\ =&\sum_{\hat{k}\in\mathbb{Z}^{g-1}}\exp\left(\pi i[(\hat{k}+\frac{\hat{n}}{2})^T\hat{\tau}(0)(\hat{k}+\frac{\hat{n}}{2})]\right)\exp\left(2\pi i(\hat{k}+\frac{\hat{n}}{2})^T(\hat{z}+\frac{\hat{n}'}{2})\right)\\ &\times 2\cos\left(\pi[(\hat{k}+\frac{\hat{n}}{2})^T\gamma(0)+z_g+\frac{n'_g}{2}]\right).\end{aligned}$$

In particular, if $g=1$, then

$$\begin{aligned}\lim_{\epsilon\to 0}\theta_1[0,0](z,\tau(\epsilon))=\lim_{\epsilon\to 0}\theta_1[0,1](z,\tau(\epsilon))&=1,\\ \lim_{\epsilon\to 0}\exp(-\pi i\tau(\epsilon)/4)\theta_1[1,0](z,\tau(\epsilon))&=2\cos(\pi z),\\ \lim_{\epsilon\to 0}\exp(-\pi i\tau(\epsilon)/4)\theta_1[1,1](z,\tau(\epsilon))&=-2\sin(\pi z).\end{aligned}$$

We finally remark that multiplying $\theta_g[n,n'](z,\tau(\epsilon))$ by the z-independent quantity $\exp(-\pi i n_g^2\tau(\epsilon)/4)$ does not affect the logarithmic derivative of the function or the ratio of two theta functions with different arguments, i.e, the function

$$\exp(-\pi i n_g^2\tau(\epsilon)/4)\theta_g[1,0](z,\tau(\epsilon))$$

is just as suitable as the theta function itself when used in the formulae (1) and (2).

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